7 7 CONSTRUCTING HARMONIC FUNCTIONS

7.1 Subharmonic Functions

Let $u : R \to \mathbb{R}$ be a continuous function on the Riemann surface R. A *disc* Δ in R is $\phi^{-1}\mathbb{D}$ when $\phi : U \to \phi U \supset \overline{\mathbb{D}}$ is a chart for R. Then

$$\partial\mathbb{D} \to \mathbb{R} \quad ; \quad z \mapsto u(\phi^{-1}z)$$

is a continuous function so its Poisson integral gives a continuous function $v : \overline{\mathbb{D}} \to \mathbb{R}$ which is harmonic on $\mathbb{D}$. Define $u_\Delta : R \to \mathbb{R}$ by

$$u_\Delta(z) = \begin{cases} u(z) & \text{if } z \notin \Delta \\ v(\phi z) & \text{if } z \in \Delta \ . \end{cases}$$

Then u_Δ is continuous on R and harmonic on Δ.

The function u is clearly harmonic if, and only if, $u_\Delta = u$ for every disc. It is difficult to construct harmonic functions directly so we introduce a weaker condition which is easier to work with. The **continuous** function $u : R \to \mathbb{R}$ is *subharmonic* if $u_\Delta \geqslant u$ for every disc Δ in R. Similarly, u is *superharmonic* if $u_\Delta \leqslant u$ for every disc Δ. We will usually prove results for subharmonic functions and tacitly assume the corresponding results for superharmonic functions. The following properties are elementary:

Proposition 7.1.1

A continuous function $u : R \to \mathbb{R}$ is subharmonic if, and only if, $-u$ is superharmonic.
A continuous function $u : R \to \mathbb{R}$ is harmonic if, and only if, it is both subharmonic and superharmonic.
If $f : S \to R$ is conformal and $u : R \to \mathbb{R}$ is continuous and subharmonic then $uf : S \to \mathbb{R}$ is subharmonic.
Let (u_n) be continuous subharmonic functions on R.
- *$\lambda_1 u_1 + \lambda_2 u_2$ is subharmonic for $\lambda_1, \lambda_2 > 0$.*
- *$\max(u_1, u_2)$ is subharmonic.*
- *If $u_n \to u$ locally uniformly then u is subharmonic.*

□

To obtain further examples of subharmonic functions we prove:

Proposition 7.1.2

A C^2 function $u : R \to \mathbb{R}$ is subharmonic if, and only if, the Laplacian $\triangle u_\alpha$ of $u_\alpha = u\phi_\alpha^{-1} : \phi_\alpha U_\alpha \to \mathbb{R}$ is non-negative for every chart $\phi_\alpha : U_\alpha \to \phi_\alpha U_\alpha \subset \mathbb{C}$.

Note that $u_\beta = u_\alpha t_{\alpha\beta}$ for the analytic transition map $t_{\alpha\beta}$, so

$$\triangle u_\beta(z) = \triangle u_\alpha(t_{\alpha\beta}(z))|t'_{\alpha\beta}(z)|^2.$$

Hence, $\triangle u_\alpha(\phi_\alpha z) \geqslant 0$ at a point $z \in U_\alpha \cap U_\beta$ if, and only if, $\triangle u_\beta(\phi_\beta z) \geqslant 0$.

Proof:

It is sufficient to prove this when R is a domain in $\mathbb{C}$.

Suppose first that $\triangle u \geqslant \varepsilon > 0$ on R and Δ is a disc in R. The continuous function $u - u_\Delta$ attains its supremum s at some point z_o in the compact set $\overline{\Delta}$. If $s > 0$ then $z_o \in \Delta$. Now $u - u_\Delta$ has a Taylor expansion

$$(u - u_\Delta)(z_o + z) = A + Bz + \overline{B}\overline{z} + Cz^2 + \overline{C}\overline{z}^2 + Dz\overline{z} + o(|z|^2).$$

If this has a local maximum at z_o then we must have $B = C = 0$ and $D \leqslant 0$. Since $D = \partial^2 u(z_o)/\partial\overline{z}\partial z = \frac{1}{4}\triangle u(z_o)$ we see that $\triangle u(z_o) \leqslant 0$ which is a contradiction. Therefore, $s = 0$ and so $u \leqslant u_\Delta$.

If $\triangle u \geqslant 0$ then $u_\varepsilon(z) = u(z) + \varepsilon z\overline{z}$ satisfies $\triangle u_\varepsilon \geqslant 4\varepsilon > 0$. So $u_\varepsilon \leqslant u_{\varepsilon\ \Delta}$. As $\varepsilon \to 0$ the functions u_ε converge locally uniformly to u, so $u \leqslant u_\Delta$ and u is subharmonic.

Conversely, suppose that u is subharmonic and $z_o \in R \subset \mathbb{C}$. Let Δ be the disc $\{z : |z - z_o| < r\}$. Then

$$u_\Delta(z_o) = \int_0^{2\pi} u(z_o + re^{i\theta})\,\frac{d\theta}{2\pi}.$$

If u has a Taylor expansion

$$u(z_o + z) = u(z_o) + Bz + \overline{B}\overline{z} + Cz^2 + \overline{C}\overline{z}^2 + Dz\overline{z} + o(|z|^2)$$

then

$$u_\Delta(z_o) = u(z_o) + Dr^2 + o(r^2).$$

By hypothesis $u_\Delta(z_o) \geqslant u(z_o)$ so $\frac{1}{4}\triangle u(z_o) = D \geqslant 0$. □

Proposition 7.1.3 The maximum principle for subharmonic functions

If $u : R \to \mathbb{R}$ is a continuous subharmonic function which attains its maximum value then it is constant.

Proof:

Let $S = \sup(u(z) : z \in R)$ and $A = u^{-1}(S)$. Since u is continuous, A is closed. We will show that A is also open.

Suppose that $z_o \in A$ and let $\phi : U \to V$ be a chart with $\phi(z_o) = 0$. Then $v = u \circ \phi^{-1} : V \to \mathbb{R}$ is continuous and subharmonic with a maximum value S attained at 0. Hence, for each $r > 0$ with $\{z : |z| \leqslant r\} \subset V$, we have

$$S = v(0) \leqslant \int_0^{2\pi} v(re^{i\theta})\,\frac{d\theta}{2\pi} \leqslant \int_0^{2\pi} S\,\frac{d\theta}{2\pi} = S$$

by Poisson's formula. Thus equality must hold throughout. Since v is continuous, $v(re^{i\theta}) = S$ for all θ. This shows that v is constant on a neighbourhood of 0, so u is constant on a neighbourhood of z_o.

Thus A is both open and closed. Since R is connected we must have $A = \emptyset$ or $A = R$. □

Note that the Proposition actually gives a little more. A function $u : R \to \mathbb{R}$ is *locally subharmonic* if it is subharmonic on a neighbourhood of each point. The proof shows that locally subharmonic functions satisfy the maximum principle. Now suppose that Δ is any disc in R and consider $u - u_\Delta$. This is locally subharmonic and 0 outside Δ. So it must attain its maximum value and this must be 0. Consequently, $u \leqslant u_\Delta$. So we have shown that every locally subharmonic function is subharmonic.

Exercises

1. A continuous function $u : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{C}$ has the property S if, for each $z_o \in \Omega$, there is a $\rho > 0$ such that

$$\int_0^{2\pi} u(z_o + re^{i\theta})\,\frac{d\theta}{2\pi} \geqslant u(z_o) \quad \text{for } 0 \leqslant r < \rho.$$

   Prove that
   (a) if u has property S and attains its maximum value then it is constant.
   (b) u has the property S if, and only if, u is subharmonic.

2. Let $u : \Omega \to \mathbb{R}$ be a continuous subharmonic function on a domain $\Omega \subset \mathbb{C}$. Show that, for $r \geqslant 0$, the function $\int_0^{2\pi} u(z_o + re^{i\theta})\, d\theta/2\pi$ is continuous and increasing.

   Let $\phi : \mathbb{C} \to [0, \infty)$ be a smooth function with

   $\phi(w) = \phi(|w|)$ for all $w \in \mathbb{C}$.

   $\phi(w) = 0$ for $|w| > 1$.

   $\int_{\mathbb{C}} \phi(w)\, du \wedge dv = \int_0^\infty \phi(r) 2\pi\, r dr = 1$ where $w = u + iv$.

   For $\varepsilon > 0$ set $\phi_\varepsilon(w) = \varepsilon^2 \phi(w/\varepsilon)$, and $\Omega_\varepsilon = \{z \in \Omega : \mathbb{D}(z, \varepsilon) \subset \Omega\}$. Define $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$u_\varepsilon(z) = \int_{|w| \leqslant \varepsilon} u(z + w) \phi_\varepsilon(w)\, du \wedge dv.$$

   Prove that

   (a)
$$\frac{\partial u_\varepsilon}{\partial z}(z) = \frac{\partial}{\partial z} \int_{\mathbb{C}} u(w) \phi_\varepsilon(w - z)\, du \wedge dv = -\int_{\mathbb{C}} u(w) \frac{\partial \phi_\varepsilon}{\partial z}(w - z)\, du \wedge dv.$$

   (b) u_ε is a smooth function on Ω_ε.

   (c) Each u_ε is subharmonic and $u_\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.

   Thus each continuous subharmonic function is the locally uniform limit of smooth (and hence C^2) subharmonic functions.

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* The most useful definition of subharmonic functions is that they are **distributions** θ on R with $\triangle\theta \geqslant 0$ in the sense that

$$\langle \triangle\theta, f \rangle = \langle \theta, \triangle f \rangle \geqslant 0$$

for every *test* function f which is smooth and has compact support in $\mathbb{R}$. It can then be proved that there is an upper semi-continuous function $u : R \to [-\infty, \infty)$ with $\langle \theta, f \rangle = \int_R uf$. The function u need not be continuous. Hence the most common definition of subharmonic functions only requires that they are upper semi-continuous. Because we will only use them to construct harmonic functions from their limits, we never need this extra generality. *

A *Perron family* on R is a family $\mathcal{F}$ of subharmonic functions on R which satisfy:

(a) if $u \in \mathcal{F}$ and Δ is a disc in R then $u_\Delta \in \mathcal{F}$.

(b) if $u_1, u_2 \in \mathcal{F}$ then $\max(u_1, u_2) \in \mathcal{F}$.

The purpose of the definition is:

Theorem 7.1.4 Perron families.

If $\mathcal{F}$ is a Perron family on R then $\sup \mathcal{F}$ *is either $+\infty$ on all of R or else it is a harmonic function on R.*

Proof:

Let $h(z) = \sup(u(z) : u \in \mathcal{F})$ and choose $z_o \in R$. Let Δ be a disc containing z_o. Then there exist $a_n \in \mathcal{F}$ with $a_n(z_o) \nearrow h(z_o)$. So $u_n = \max(a_1, a_2, \ldots, a_n)$ form an increasing sequence in $\mathcal{F}$ with $u_n(z_o) \nearrow h(z_o)$. Also, $v_n = u_{n\,\Delta}$ will be an increasing sequence in $\mathcal{F}$ with $v_n(z_o) \nearrow h(z_o)$. Each v_n is harmonic on Δ so Harnack's theorem shows that $v = \sup(v_n : n \in \mathbb{N})$ is either identically $+\infty$ on Δ or else it is harmonic on Δ. Clearly $v \leqslant h$ with equality at z_o.

Suppose that $v(z) < h(z)$ at some $z \in \Delta$. Then we can find $b_n \in \mathcal{F}$ with $b_n(z) \nearrow h(z)$. Set $\tilde{a}_n = \max(a_n, b_n) \in \mathcal{F}$ and define $\tilde{u}_n, \tilde{v}_n$ and $\tilde{v}$ as above using $\tilde{a}_n$ in place of a_n. Then $\tilde{v}$ is harmonic on Δ with $\tilde{v} \geqslant v$ and

$$\tilde{v}(z_o) = h(z_o) = v(z_o) \quad ; \quad \tilde{v}(z) = h(z) > v(z).$$

So the function $\tilde{v} - v$ is harmonic on Δ, not constant and has a minimum at z_o. This is impossible, so $v(z) = h(z)$ throughout Δ. In particular, h is harmonic on Δ.

Hence the set $h^{-1}(+\infty)$ is both open and closed in R so it is either R or else $\emptyset$. In the first case h is identically $+\infty$. In the second h is finite and so is harmonic. $\square$

A *harmonic majorant* for a family $\mathcal{F}$ of functions on R is a harmonic function $h : R \to \mathbb{R}$ with $h \geqslant u$ for each $u \in \mathcal{F}$.

Corollary 7.1.5

If $\mathcal{F}$ is a family of continuous, subharmonic functions on R which has a harmonic majorant, then there is an unique least harmonic majorant.

Proof:

Let

$$\mathcal{H} = \{h : R \to \mathbb{R} : h \text{ is harmonic and } h \geqslant u \text{ for all } u \in \mathcal{F}\}$$

and

$$\overline{\mathcal{F}} = \{u : R \to \mathbb{R} : u \text{ is continuous, subharmonic and } h \geqslant u \text{ for all } h \in \mathcal{H}\}.$$

Then $\mathcal{F} \subset \overline{\mathcal{F}}$. It is apparent that $\overline{\mathcal{F}}$ is a Perron family so $k = \sup \overline{\mathcal{F}}$ is either $+\infty$ or else harmonic. It can not be $+\infty$ since $\mathcal{H}$ is not empty. So k is harmonic. Then $k \in \mathcal{H} \cap \overline{\mathcal{F}}$ so k must be the unique least element of $\mathcal{H}$. $\square$

Exercises

3. Let $u : \mathbb{D} \to \mathbb{R}$ be continuous and subharmonic. Show that the least harmonic majorant of u is given by
$$\lim_{r \to 1-} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \frac{d\theta}{2\pi} = \sup_{r<1} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \frac{d\theta}{2\pi}.$$

7.2 Dirichlet's Problem

We will now show how to construct harmonic functions with specified properties by taking the suprema of suitable Perron families. As a first example we will try to solve Dirichlet's problem:

Let Ω be a relatively compact domain in R and $b : \partial\Omega \to \mathbb{R}$ a continuous function. Find a continuous function $B : \overline{\Omega} \to \mathbb{R}$ which is harmonic on Ω and has $B|\partial\Omega = b$.

There need not be a solution to the Dirichlet problem. For example, if $\Omega = \{z : 0 < |z| < 1\} \subset \mathbb{C}$ and

$$b : \partial\Omega \to \mathbb{R} \quad ; \quad z \mapsto \begin{cases} 1 & \text{if } z = 0 \\ 0 & \text{if } |z| = 1 \end{cases}$$

then any solution B would have a removable singularity at 0. Then B would be harmonic on $\mathbb{D}$ and 0 on $\partial\mathbb{D}$ so it would be identically 0, which was forbidden. If there is a solution it is unique because of the maximum principle. The domain Ω is *regular for the Dirichlet problem* if there is a solution to the Dirichlet problem for every continuous function $b : \partial\Omega \to \mathbb{R}$.

Let $\mathcal{F} = \{u : \overline{\Omega} \to \mathbb{R} : u \text{ is continuous, subharmonic on } \Omega \text{ and } u|\partial\Omega \leqslant b\}$. This is a Perron family on Ω which contains the constant function $\inf b$ and is bounded above by $\sup b$. So its supremum is a harmonic function $h : \Omega \to \mathbb{R}$. If there is a solution B to the Dirichlet problem, then each $u \in \mathcal{F}$ satisfies $u \leqslant B$ on Ω. So $h \leqslant B$. Also $B \in \mathcal{F}$ so $B \leqslant h$. Therefore, if there is a solution, it is h. We will show that h is indeed a solution when the boundary $\partial\Omega$ is sufficiently regular.

A *barrier* at $\zeta \in \partial\Omega$ is a continuous function $\beta : \overline{\Omega} \to \mathbb{R}$ which is subharmonic on Ω and has $\beta(z) \leqslant 0$ on $\overline{\Omega}$ with equality if, and only if, $z = \zeta$. If β is a barrier at ζ and U is an open neighbourhood of ζ then $\beta|\overline{\Omega \cap U}$ is obviously a barrier at ζ for $\Omega \cap U$. Conversely, suppose that γ is a barrier at ζ for $\Omega \cap U$. Then γ is bounded on $\overline{\Omega} \cap \partial U$, say $\gamma(z) < -m < 0$ for $z \in \overline{\Omega} \cap \partial U$. Set

$$\beta(z) = \begin{cases} \max(\gamma(z), -m) & \text{for } z \in \overline{\Omega} \cap U \\ -m & \text{for } z \in \overline{\Omega} \setminus U \ . \end{cases}$$

Then β is easily seen to be continuous on $\overline{\Omega}$ and subharmonic on Ω, so it is a barrier at ζ for Ω. Thus the existence of a barrier depends only on the shape of Ω in the neighbourhood of ζ.

Theorem 7.2.1

A relatively compact domain Ω in R is regular for the Dirichlet problem if, and only if, there is a barrier at each point of $\partial\Omega$.

Proof:

Suppose that Ω is regular. For $\zeta \in \partial\Omega$ we can certainly find a continuous function $b : \partial\Omega \to \mathbb{R}$ with $b(z) \leqslant 0$ and equality if, and only if, $z = \zeta$. Let $B : \overline{\Omega} \to \mathbb{R}$ be the corresponding solution to the Dirichlet problem. Then B is continuous on $\overline{\Omega}$ and harmonic on Ω. Since b is not constant, neither is B, and so the maximum principle shows that $B(z) < 0$ for every $z \in \overline{\Omega}$ except ζ. Thus B is a barrier at ζ.

For the converse, suppose that $b : \partial\Omega \to \mathbb{R}$ is continuous and let $h = \sup \mathcal{F}$ as above. We will show that $h(z) \to b(\zeta)$ as $z \in \Omega$ tends to any boundary point $\zeta \in \partial\Omega$, so h will be the desired solution of the Dirichlet problem.

Let γ be a barrier at $\zeta \in \partial\Omega$. Consider c_1, c_2 with $c_1 < b(\zeta) < c_2$. There is an open neighbourhood U of ζ with $c_1 < b(z) < c_2$ for all $z \in U \cap \partial\Omega$. Since $\gamma \leqslant 0$ we have $c_1 + K\gamma(z) < b(z) < c_2 - K\gamma(z)$ for each $z \in \partial\Omega \cap U$ and any positive K. Since γ is bounded away from 0 on $\overline{\Omega} \setminus U$, we can choose K so large that

$$c_1 + K\gamma(z) < b(z) < c_2 - K\gamma(z) \qquad \text{for } z \in \partial\Omega.$$

This implies that $z \mapsto c_1 + K\gamma(z)$ is a member of $\mathcal{F}$ so

$$c_1 + K\gamma(z) \leqslant h(z) \qquad \text{for } z \in \Omega.$$

Taking the limit as $z \to \zeta$ gives $c_1 \leqslant \liminf_{z\to\zeta} h(z)$. Similarly, $s : z \mapsto c_2 - K\gamma(z)$ is superharmonic on Ω. If $u \in \mathcal{F}$ then $u - s$ is subharmonic on Ω, continuous on $\overline{\Omega}$ and negative on $\partial\Omega$. The maximum principle shows that $u - s$ is negative on all of $\overline{\Omega}$ so $u \leqslant s$. Therefore

$$h(z) \leqslant c_2 - K\gamma(z) \qquad \text{for } z \in \Omega$$

and so $\limsup_{z\to\zeta} h(z) \leqslant c_2$.

Together these inequalities show that $h(z)$ tends to $b(\zeta)$ as $z \to \zeta$ from within Ω. So the function which is b on $\partial\Omega$ and h on Ω is the desired solution for the Dirichlet problem. □

For domains with suitably smooth boundaries it is easy to construct barriers. As a simple but useful example, suppose that there is a chart $\phi : U \to V \subset \mathbb{C}$ at $\zeta \in \partial\Omega$ with $\phi(\zeta) = 0$ and $\phi(\overline{\Omega})$ disjoint from the strictly negative real axis. Then the principle branch σ of the square root is continuous on $\overline{\Omega} \cap U$ and analytic on $\Omega \cap U$. Its real part β is 0 at ζ and strictly positive at every other point of $\overline{\Omega} \cap U$. So β is a barrier at ζ for $\Omega \cap U$. If $\partial\Omega$ is piecewise C^1 then we can certainly find such charts at each boundary point.

We can readily adapt these results to domains $\Omega \subset R$ which are not relatively compact provided the boundary values are bounded, continuous functions. We say that Ω is regular for the Dirichlet problem if, for each bounded, continuous function $b : \partial\Omega \to \mathbb{R}$ there is a bounded, continuous function $B : \overline{\Omega} \to \mathbb{R}$ which is harmonic on Ω and agrees with b on the boundary $\partial\Omega$. Then

Theorem 7.2.1'

A domain Ω in R is regular for the Dirichlet problem if, and only if, there is a barrier at each point of $\partial\Omega$.

□

Note that the function B need no longer be uniquely determined by b.

Exercises

4. Prove that a relatively compact domain with piecewise smooth boundary is regular for the Dirichlet problem.

-5. Show that there cannot be a barrier at an isolated boundary point.

6. Show that every relatively compact domain Ω in R is contained within another relatively compact domain Ω' which is regular for the Dirichlet problem.

7.3 Green's Functions

We will now consider the related problem of constructing the Green's function for a Riemann surface R.

Let $u : R \setminus \{z_o\} \to \mathbb{R}$ be a harmonic function and $\phi : U \to \phi(U) = \mathbb{D}$ a chart with $\phi(z_o) = 0$. We want $u(z)$ to behave like $c \log|\phi(z)|$ for z near z_o. More formally, we will say that u has a *logarithmic singularity* with coefficient c at z_o if there is a harmonic function h on a neighbourhood of z_o with

$$u(z) = c \log|\phi(z)| + h(z) \ .$$

The crucial part of this is that h is harmonic at z_o. The value of c does not depend on which chart we choose. (Indeed,

$$c = \frac{1}{\pi i} \int_\gamma \partial u$$

for a small closed curve γ winding once around z_o.) We will sometimes write "$u(z) \sim c \log|\phi(z)|$ near z_o" as an abbreviation for "u has a logarithmic singularity with coefficient c at z_o".

Let $\mathcal{G}(z_o)$ be the set of all positive harmonic functions $g : R \setminus \{z_o\} \to \mathbb{R}^+$ which have a logarithmic singularity at z_o with coefficient -1. The least element of $\mathcal{G}(z_o)$ is called the *Green's function* for R with a *pole* at z_o. For this definition to make sense we need:

Proposition 7.3.1

If $\mathcal{G}(z_o)$ is non-empty, then it has a least element.

Proof:

Let $\mathcal{F}(z_o)$ be the set of continuous subharmonic functions $u : R \setminus \{z_o\} \to \mathbb{R}$ which have compact support and

$$z \mapsto u(z) + \log|\phi z| \qquad (z \in U)$$

continuous and subharmonic on U. This is clearly a Perron family on $R \setminus \{z_o\}$. It contains

$$z \mapsto \begin{cases} -\log|\phi(z)| & \text{if } z \in U \\ 0 & \text{if } z \notin U \ . \end{cases}$$

so it is non-empty.

Let $u \in \mathcal{F}(z_o)$ and $g \in \mathcal{G}(z_o)$. Then u is zero outside some compact set K. The difference $u - g$ is continuous and subharmonic on all of R, including z_o. It is negative outside K and, by the maximum principle (Proposition 7.1.3), it must be negative within K. Therefore $u \leqslant g$. The Perron family $\mathcal{F}(z_o)$ is thus bounded above and so its supremum $s : R \setminus \{z_o\} \to \mathbb{R}$ is harmonic. We also have

$$s(z) + \log|\phi z| \quad = \quad \sup(u(z) + \log|\phi z| : u \in \mathcal{F}(z_o))$$

so s has a logarithmic singularity at z_o with coefficient -1. Thus $s \in \mathcal{G}(z_o)$ and $s \leqslant g$. □

We will denote the Green's function for R with a pole at z_o by g_{z_o} (or g if there is no possibility of confusion). If $\inf g = m > 0$, then $g - m$ would be a smaller element of $\mathcal{G}(z_o)$. So $\inf g$ must be 0. The following result shows that, if the Green's function with a pole at z_o exists, then so does the Green's function with pole at any other point of R.

Theorem 7.3.2

The following conditions on the Riemann surface R are equivalent.

(a) *There is a function $s : R \to [0, \infty)$ which is non-constant, positive, continuous and superharmonic .*

(b) *For $z_o \in R$ there is a Green's function g on R with pole at z_o.*

Proof:

If (b) holds, then set $s(z) = \min(g(z), 1)$. Since $g(z)$ takes values close to 0 and ∞, the function s is not constant. It is certainly continuous, superharmonic and positive, so (a) holds.

Now suppose that (a) holds. Let $\Delta = \phi^{-1}\mathbb{D}$ and $\Delta' = \phi^{-1}\{z : |z| < \frac{1}{2}\}$. We will first construct the *harmonic measure* of Δ', that is a continuous function $\omega : R \to \mathbb{R}$ which is 1 on Δ', harmonic on $R \setminus \Delta'$ and not constant. The collection

$$\mathcal{A} = \{u : R \to \mathbb{R} : u \text{ continuous with compact support, subharmonic on } R \setminus \overline{\Delta'} \text{ and } u \leqslant 1\}$$

is a Perron family on $R \setminus \overline{\Delta'}$. The function

$$u_o(z) = \begin{cases} 1 & \text{if } z \in \Delta' \\ -\log|\phi(z)|/\log 2 & \text{if } z \in \Delta \setminus \Delta' \\ 0 & \text{if } z \in R \setminus \Delta \end{cases}$$

is in $\mathcal{A}$. Also, every $u \in \mathcal{A}$ satisfies $u \leqslant 1$. So the supremum $\omega = \sup \mathcal{A}$ is harmonic on $R \setminus \Delta'$ with $u_o \leqslant \omega \leqslant 1$. It follows that $\omega(z) \to 1$ as $z \to \partial\Delta'$ so we can extend ω continuously to all of R by setting $\omega(z) = 1$ for $z \in \overline{\Delta'}$. We may assume that $\inf s = 0$ for otherwise we could replace s by $s - \inf s$. Since s is superharmonic it cannot attain this infimum, so $\inf(s(z) : z \in \overline{\Delta'}) = m > 0$. The maximum principle for subharmonic functions shows that each $u \in \mathcal{A}$ satisfies $u \leqslant s/m$ so $\omega \leqslant s/m$. In particular, $\inf(\omega(z) : z \in R) = 0$. Hence ω is not constant. By the maximum principle for harmonic functions, $0 < \omega(z) < 1$ for $z \in R \setminus \overline{\Delta'}$.

Now that we have the harmonic measure we can construct a positive superharmonic function v on $R \setminus \{z_o\}$ with $v(z) \sim -\log|\phi(z)|$ as $z \to z_o$. We can choose c with $\sup(\omega(z) : z \in \partial\Delta) < c < 1$. Define v by

$$v(z) = \begin{cases} Ac - \log|\phi(z)| & \text{if } z \in \Delta' \\ \min(Ac - \log|\phi(z)|, A\omega(z)) & \text{if } z \in \Delta \setminus \Delta' \\ A\omega(z) & \text{if } z \in R \setminus \Delta \end{cases}$$

for some $A > \log 2/(1-c)$. It is apparent that v is continuous and superharmonic on the interiors of the three regions $\Delta', \Delta \setminus \Delta',$ and $R \setminus \Delta$. The only difficulty is at the boundaries. If $z \in \partial\Delta'$ then

$$Ac - \log|\phi(z)| = Ac + \log 2 < A = A\omega(z)$$

so v is continuous and superharmonic at z because $Ac - \log|\phi(z)|$ is. If $z \in \partial\Delta$ then

$$Ac - \log|\phi(z)| = Ac > A\sup(\omega(w) : w \in \partial\Delta) \geqslant A\omega(z)$$

so v is continuous and superharmonic at z because $A\omega(z)$ is.

If $u \in \mathcal{F}(z_o)$ then $u - v$ is subharmonic on all of R (even near z_o where it is $h(z) - Ac$). Outside of the compact support of u it is negative. The maximum principle then shows that $u - v \leqslant 0$ on all of R. So $\sup \mathcal{F}(z_o) \leqslant v$. Hence $\sup \mathcal{F}(z_o)$ is finite and so it is the desired Green's function. □

We will call R *hyperbolic* if it has a Green's function. For example, the disc $\mathbb{D}$ has a Green's function

$$z \mapsto -\log|z|$$

with pole at 0, so it is hyperbolic.

We will call R *elliptic* if it is compact. An elliptic surface can not be hyperbolic, for if g were a Green's function it would attain its infimum and that would contradict the minimum principle for harmonic functions. Hence the classes of elliptic and hyperbolic surfaces are disjoint. The Riemann sphere $\mathbb{C}_\infty$ is an example of an elliptic Riemann surface.

We will call R *parabolic* if it is neither elliptic nor hyperbolic. The complex plane $\mathbb{C}$ is parabolic. For it is certainly not compact and, if g were a Green's function, it would have a removable singularity at ∞ and so give a Green's function for $\mathbb{C}_\infty$.

Corollary 7.3.3

If Ω is a domain in R and there is a barrier at one point $\zeta \in \partial\Omega$ then Ω is hyperbolic and the Green's function g satisfies $g(z) \to 0$ as $z \to \zeta$ with $z \in \Omega$.

Proof:

Let β be the barrier at ζ. Then $-\beta$ is a non-constant, positive, superharmonic function on Ω so Ω is certainly hyperbolic and has a Green's function g with pole at z_o. Recall from Proposition 7.3.1 that g is the supremum of $\mathcal{F}(z_o)$ which consists of continuous subharmonic functions u which are zero outside some compact subset of Ω. We will extend any such function u continuously to $\partial\Omega$ by setting it equal to 0 there. Let V be a compact neighbourhood of z_o contained in Ω. Then ∂V is compact so we can find K with $g(z) \leqslant K$ for $z \in \partial V$. This implies that $u(z) \leqslant K$ for $z \in \partial V$. However, u is 0 on $\partial\Omega$ so the maximum principle for subharmonic functions gives

$$u(z) \leqslant K \qquad \text{for} \quad z \in \Omega \setminus V.$$

The barrier β has $\beta(z) \leqslant 0$ for $z \in \overline{\Omega}$ and $\beta(z) = 0$ if, and only if, $z = \zeta$. Hence, we can find $m > 0$ with $\beta(z) \leqslant -m$ for $z \in \partial V$. It follows that the subharmonic function $z \mapsto u(z) + (K/m)\beta(z)$ is at most 0 on $\partial\Omega \cup \partial V$. So the maximum principle gives

$$u(z) + (K/m)\beta(z) \leqslant 0 \qquad \text{for} \quad z \in \Omega \setminus V.$$

Taking the supremum over all u gives

$$g(z) \leqslant -(K/m)\beta(z) \qquad \text{for} \quad z \in \Omega \setminus V.$$

Hence $g(z) \to 0$ as $z \to \zeta$ with $z \in \Omega$. □

Exercises

-7. If g is the Green's function for R with pole at z_o and $f : S \to R$ is a conformal equivalence, then gf is the Green's function on S with pole at $f^{-1}(z_o)$.

8. Find the Green's function on the unit disc with a pole at any specified point (and prove that it is the Green's function).

9. Let g be the Green's function on a domain $\Omega \subset \mathbb{C}$ with a pole at z_o and let f be a smooth function on Ω with compact support within Ω. Prove that

$$\int_\Omega \triangle f(z) g(z) \, d\overline{z} \wedge dz = -4\pi i g(z_o).$$

Hint: Stokes' Theorem.

(This means that g defines a distribution with $\triangle g = -4\pi i \delta_{z_o}$. In partial differential equations and applied mathematics this is often taken as the definition of the Green's function.)
