5.1 The Hyperbolic Plane.

Lemma 5.1.1 Schwarz' lemma

If $f : \mathbb{D} \to \mathbb{D}$ is an analytic map with $f(0) = 0$ then

$$|f(w)| \leqslant |w| \quad \textit{for} \quad w \in \mathbb{D} \setminus \{0\} \qquad \textit{and} \qquad |f'(0)| \leqslant 1.$$

Moreover, if equality holds at any point then f must be the map $z \mapsto \omega z$ for some ω of modulus 1.

Proof:

The map

$$g : \mathbb{D} \to \mathbb{C} \qquad ; \qquad w \mapsto \begin{cases} f(w)/w & \text{for } w \in \mathbb{D} \setminus \{0\} \\ f'(0) & \text{for } w = 0 \end{cases}$$

is analytic on $\mathbb{D} \setminus \{0\}$ and continuous at 0. So it has a removable singularity at 0 and hence is analytic on all of $\mathbb{D}$. The maximum modulus principle shows that, for $r < 1$,

$$|g(w)| \;\leqslant\; \sup(|g(z)| : |z| = r) \;=\; \sup(|f(z)|/r : |z| = r) \quad \text{for} \quad |w| \leqslant r.$$

Hence,

$$|g(w)| \leqslant 1 \quad \text{for} \quad w \in \mathbb{D}$$

which is the first part of the lemma. Moreover, if there is equality at any point of $\mathbb{D}$ then the maximum modulus principle implies that g is constant. The constant must be of modulus 1. □

This lemma will enable us to prove that the only conformal maps $f : \mathbb{D} \to \mathbb{D}$ are the Möbius transformations which do map $\mathbb{D}$ to itself. To discover which Möbius transformations these are, consider the map $J : \mathbb{C}_\infty \to \mathbb{C}_\infty; z \mapsto \overline{z}^{-1}$. This is inversion in the unit circle, so $J(z) = z$ if, and only if, $z \in \partial\mathbb{D}$. Hence, a Möbius transformation $T : z \mapsto (az + b)/(cz + d) \quad (ad - bc = 1)$ will map the unit circle onto itself if, and only if,

$$\begin{aligned} & JT = TJ \\ \Leftrightarrow \quad & JTJ(z) = \frac{\overline{d}z + \overline{c}}{\overline{b}z + \overline{a}} = \frac{az + b}{cz + d} = T(z) \quad \text{for} \quad z \in \mathbb{C}_\infty \\ \Leftrightarrow \quad & d = \pm\overline{a} \quad ; \quad b = \pm\overline{c}. \end{aligned}$$

The + signs give the Möbius transformations

$$T : z \mapsto \frac{az + \overline{c}}{cz + \overline{a}} \qquad \text{for} \quad |a|^2 - |c|^2 = 1 \tag{$*$}$$

which map $\mathbb{D}$ onto $\mathbb{D}$. While the $-$ signs give the Möbius transformations $T : z \mapsto (az - \overline{c})/(cz - \overline{a})$ for $|a|^2 - |c|^2 = -1$ which map $\mathbb{D}$ onto $\{z \in \mathbb{C}_\infty : |z| > 1\}$. In particular we see that the maps $(*)$ form a group of conformal maps from $\mathbb{D}$ onto $\mathbb{D}$. This group of maps is transitive on $\mathbb{D}$, so for every $z \in \mathbb{D}$ there is a map T with $T(0) = z$. The following result shows that there are no other conformal maps from $\mathbb{D}$ onto itself.

Theorem 5.1.2

$$\operatorname{Aut}\mathbb{D} \quad = \quad \left\{ z \mapsto \frac{az + \overline{c}}{cz + \overline{a}} \;:\; |a|^2 - |c|^2 = 1 \right\}.$$

Proof:

Suppose that $h \in \operatorname{Aut}\mathbb{D}$. Then $h(0) \in \mathbb{D}$ so we can find $a, c \in \mathbb{C}$ with $|a|^2 - |c|^2 = 1$ and $h(0) = -\overline{c}/a$. Let T be the Möbius transformation $z \mapsto (az + \overline{c})/(cz + \overline{a})$. Then $f = Th \in \operatorname{Aut}\mathbb{D}$ and $f(0) = Th(0) = 0$. By Schwarz' lemma, $|f'(0)| \leqslant 1$. However, f^{-1} is also an automorphism of $\mathbb{D}$ so $|(f^{-1})'(0)| = |f'(0)|^{-1} \leqslant 1$. Hence equality must hold and so $f(z) = \omega z$ for some ω of modulus 1. It follows that f, and hence also h, is a Möbius transformation which maps $\mathbb{D}$ onto itself. $\square$

If $T : z \mapsto (az+\overline{c})/(cz+\overline{a})$ (with $|a|^2-|c|^2 = -1$) is in $\operatorname{Aut}\mathbb{D}$ then $\tau(T) = \operatorname{tr}(T)^2/4 = (\Re a)^2 \in [0, \infty)$. So T can be the identity, or elliptic, or hyperbolic, or parabolic, but not loxodromic.

Exercises

1. Show that $z \mapsto \omega(z - z_0)/(1 - \overline{z_0}z)$ is in $\operatorname{Aut}\mathbb{D}$ for ω with $|\omega| = 1$ and $z_o \in \mathbb{D}$. Conversely every map in $\operatorname{Aut}\mathbb{D}$ is of this form.
2. Find $\operatorname{Aut}\mathbf{H}^+$ for the upper half plane $\mathbf{H}^+ = \{z \in \mathbb{C} : \Im z > 0\}$.
3. Consider $\mathbb{D}$ as the subset
$$\{[z_1 : z_2] : |z_1|^2 - |z_2|^2 < 0\} \quad \text{of} \quad \mathbb{P}(\mathbb{C}^2).$$
Show that an invertible linear map $T : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ with determinant 1 induces a conformal map $\mathcal{T} : \mathbb{D} \to \mathbb{D}$ if, and only if,
   (a) T preserves the indefinite form
$$\beta : \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \mapsto \overline{z_1}w_1 - \overline{z_2}w_2.$$
   (That is $\beta(T\mathbf{z}, T\mathbf{w}) = \beta(\mathbf{z}, \mathbf{w})$.)

   and

   (b) $\mathcal{T}(0) \in \mathbb{D}$. (That is $|b| < |d|$.)
4. Let $T \in \operatorname{Aut}\mathbb{D}$. Show that
   (a) if T is elliptic, it has exactly one fixed point in $\mathbb{D}$.
   (b) if T is hyperbolic, it has two fixed points both on $\partial\mathbb{D}$.
   (c) if T is parabolic, it has one fixed point on $\partial\mathbb{D}$.

   Find the conjugacy classes in $\operatorname{Aut}\mathbb{D}$.
5. Prove directly that a loxodromic Möbius transformation cannot map any disc in $\mathbb{C}_\infty$ onto itself.

There is a metric on $\mathbb{D}$, called the *hyperbolic metric* ρ, which is invariant under all of the maps in $\operatorname{Aut}\mathbb{D}$. To define it, let $\gamma : I \to \mathbb{D}$ be a smooth curve and let its length be

$$L_\rho(\gamma) = \int_\gamma \left(\frac{2}{1 - |z|^2} \right) |dz| = \int_0^1 \left(\frac{2}{1 - |\gamma(t)|^2} \right) |\gamma'(t)| \, dt.$$

Then define $\rho(z_0, z_1)$ to be the infimum of the lengths $L_\rho(\gamma)$ for all smooth paths in $\mathbb{D}$ from z_0 to z_1. This is certainly symmetric and satisfies the triangle inequality. We will see shortly that it is 0 if, and only if, $z_0 = z_1$ and then we will know that it is a metric.

If $T : z \mapsto (az + \overline{c})/(cz + \overline{a})$ (with $|a|^2 - |c|^2 = -1$), then

$$T'(z) = \frac{1}{(cz + \overline{a})^2} \qquad \text{and} \qquad 1 - |T(z)|^2 = \frac{1 - |z|^2}{|cz + \overline{a}|^2}$$

so $L_\rho(T\gamma) = L_\rho(\gamma)$ and hence T is an isometry. Suppose that $x \in [0, 1)$ and γ is a path in $\mathbb{D}$ from 0 to x. Then the path $\Re\gamma$ is also a path in $\mathbb{D}$ from 0 to x and, since,

$$\left(\frac{2}{1 - |\Re\gamma(t)|^2}\right) |(\Re\gamma)'(t)| \quad \leqslant \quad \left(\frac{2}{1 - |\gamma(t)|^2}\right) |\gamma'(t)|$$

it has a shorter length than γ. So the straight line path from 0 to x is the unique shortest path between these points and hence

$$\rho(0, x) = \int_0^x \left(\frac{2}{1 - |t|^2}\right) \, dt = \log\left(\frac{1 + x}{1 - x}\right).$$

The invariance of ρ under $\operatorname{Aut}\mathbb{D}$ enables us to deduce that

$$\rho(z_0, z_1) = \log\left(\frac{1 + \left|\frac{z_0 - z_1}{1 - \overline{z_0}z_1}\right|}{1 - \left|\frac{z_0 - z_1}{1 - \overline{z_0}z_1}\right|}\right)$$

and that the unique path from z_0 to z_1 with shortest length is the arc of a circle orthogonal to $\partial\mathbb{D}$. In particular, $\rho(z_0, z_1) = 0$ if, and only if, $z_0 = z_1$ so ρ is indeed a metric.

Theorem 5.1.3 Schwarz - Pick theorem

Every analytic function $f : \mathbb{D} \to \mathbb{D}$ is a contraction for the hyperbolic metric, so

$$\rho(f(z_0), f(z_1)) \quad \leqslant \quad \rho(z_0, z_1) \qquad \text{for } z_0, z_1 \in \mathbb{D}.$$

Proof:

Since each $T \in \operatorname{Aut}\mathbb{D}$ is an isometry and the group acts transitively on $\mathbb{D}$ we can assume that $z_0 = f(z_0) = 0$. Then the result is Schwarz' lemma. □

Exercises

6. Show that the hyperbolic metric on $\mathbb{D}$ is complete.
7. (This requires the definition of a Riemannian metric.) Show that the only Riemannian metrics on $\mathbb{D}$ which are invariant under $\operatorname{Aut}\mathbb{D}$ are the multiples of the hyperbolic metric. Show that there are no Riemannian metrics on $\mathbb{C}_\infty$ or $\mathbb{C}$ which are invariant under $\operatorname{Aut}\mathbb{C}_\infty$ or $\operatorname{Aut}\mathbb{C}$.
8. Find the hyperbolic metric on the upper half plane $\mathbf{H}$ for which any Möbius transformation mapping $\mathbb{D}$ onto $\mathbf{H}$ is an isometry.
9. Let z_1, z_2, w_1 and w_2 be four points in $\mathbb{D}$. Show that there is an analytic function $f : \mathbb{D} \to \mathbb{D}$ with $f(z_1) = w_1$ and $f(z_2) = w_2$ if, and only if, $\rho(w_1, w_2) \leqslant \rho(z_1, z_2)$.
10. Let C be the unique circle through the two points $z_0, z_1 \in \mathbb{D}$ which is orthogonal to $\partial\mathbb{D}$. Then C meets $\partial\mathbb{D}$ at the points w_0, w_1 with w_0, z_0, z_1, w_1 in that order on C. Express the cross ratios of w_0, z_0, z_1, w_1 and of $J(z_0), z_0, z_1, J(z_1)$ in terms of $\rho(z_0, z_1)$.
11. Prove that
$$\left|\frac{z_0 - z_1}{1 - \overline{z_0}z_1}\right| = \tanh \tfrac{1}{2}\rho(z_0, z_1).$$
Does the left side of this equation define a metric on $\mathbb{D}$? Find similar formulae for $\sinh\rho(z_0, z_1)$ and $\cosh\rho(z_0, z_1)$.

5.2 The Poisson - Jensen Formula.

Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous map which is analytic on the open disc $\mathbb{D}$ and never 0 on $\partial\mathbb{D}$. Then f can only have finitely many zeros in $\mathbb{D}$, say $z_1, z_2, \ldots, z_N$ each repeated according to its multiplicity. Now, for $z_1 \in \mathbb{D}$, the Möbius transformation $T : z \mapsto (z - z_1)/(1 - \overline{z_1}z)$ maps $\overline{\mathbb{D}}$ continuously onto itself and its only zero is z_1. Hence, $f_1(z) = f(z)/T(z)$ is continuous on $\overline{\mathbb{D}}$, analytic on $\mathbb{D}$ and has $|f_1(z)| = |f(z)|$ for $z \in \partial\mathbb{D}$. Also f_1 has one less zero in $\mathbb{D}$ than f. Repeating this argument we find that

$$f(z) = \left\{ \prod_{n=1}^{N} \left(\frac{z - z_n}{1 - \overline{z_n}z} \right) \right\} f_N(z)$$

where $|f_N(z)| = |f(z)|$ for $z \in \partial\mathbb{D}$. Since f_N has no zeros in $\mathbb{D}$ we can write $f_N = \exp g$ for a continuous map $g : \overline{\mathbb{D}} \to \mathbb{C}$ which is analytic on $\mathbb{D}$.

Theorem 5.2.1 The Poisson - Jensen Formula

If $f : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous, analytic on $\mathbb{D}$ and never 0 on $\partial\mathbb{D}$ then the zeros $z_1, z_2, \ldots, z_N$ of f satisfy

$$\log|f(z)| = \sum_{n=1}^{N} \log\left| \frac{z - z_n}{1 - \overline{z_n}z} \right| + \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}$$

for $z \in \mathbb{D}$ with $|f(z)| \neq 0$.

Proof:

As shown above, we can write

$$f(z) = \left\{ \prod_{n=1}^{N} \left(\frac{z - z_n}{1 - \overline{z_n}z} \right) \right\} \exp g(z)$$

with $g : \overline{\mathbb{D}} \to \mathbb{C}$ continuous and analytic on $\mathbb{D}$. Hence,

$$\log|f(z)| = \sum_{n=1}^{N} \log\left| \frac{z - z_n}{1 - \overline{z_n}z} \right| + \Re g(z).$$

However, $\Re g \in \mathcal{H}(\mathbb{D})$ so Poisson's formula gives

$$\Re g(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \Re g(e^{i\theta}) \frac{d\theta}{2\pi} = \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}.$$

□

Corollary 5.2.2

Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function with $f(0) \neq 0$ and let (z_n) be the sequence of zeros of f each repeated according to its multiplicity. For $0 \leqslant r < 1$ we have

$$\log|f(0)| = \sum \left(\log \frac{|z_n|}{r} \ : \ |z_n| < r \right) + \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi}.$$

Proof:

If f has no zeros on $\{z : |z| = r\}$ then we may obtain the desired result by applying the theorem to the function $z \mapsto f(rz)$. The two sides of the equation are clearly continuous functions of r, so equality must persist even when f has a zero on $\{z : |z| = r\}$. $\square$

A particularly important case is when $f : \mathbb{D} \to \mathbb{C}$ is a **bounded** analytic function. Then

$$-\sum \log |z_n| \quad \leqslant \quad \log ||f||_\infty - \log |f(0)|$$

so $\sum \log |z_n|$ converges provided that $f(0) \neq 0$. If f has a zero of order k at 0 then we may apply this result to $f(z)/z^k$ to find that the series $\sum \log |z_n|$ is still convergent when we sum over all of the zeros z_n of f in $\mathbb{D} \setminus \{0\}$.

The converse of this is also true: if (z_n) is a sequence of points in $\mathbb{D} \setminus \{0\}$ with $\sum \log |z_n|$ convergent, then there is a bounded analytic function with zeros precisely at the points (z_n). We will prove this by constructing a product

$$B(z) \quad = \quad \prod \omega_n \left(\frac{z - z_n}{1 - \overline{z_n} z} \right)$$

where $|\omega_n| = 1$. For this to converge we must have each term converging to 1 as $n \to \infty$ and $|z_n| \to 1$. Hence we must take $\omega_n = -|z_n|/z_n$ when $z_n \neq 0$ (and ω_n is arbitrary when $z_n = 0$). For this reason we define a *Blaschke product* B for a discrete sequence (z_n) in $\mathbb{D}$ to be

$$B(z) \quad = \quad \omega z^k \prod \frac{-|z_n|}{z_n} \left(\frac{z - z_n}{1 - \overline{z_n} z} \right)$$

where $\omega \in \mathbb{C}$ is of modulus 1, 0 occurs k times in the sequence (z_n), and the product is over the non-zero elements of the sequence. We often set $\omega = 1$ and call B *the Blaschke product* for (z_n).

Lemma 5.2.3 Blaschke products

For each discrete sequence (z_n) in $\mathbb{D}$ for which $\sum 1 - |z_n|$ is convergent, the Blaschke product B converges to an analytic function $B : \mathbb{D} \to \mathbb{D}$ with zeros at the points of the sequence and nowhere else.

Proof:

It is clear that the Blaschke product B, provided that it converges, has the desired properties. We can assume that the sequence (z_n) is infinite and does not contain 0. The condition $\sum 1 - |z_n| < \infty$ certainly implies that the product $B(0) = \prod |z_n|$ converges. Hence, it will suffice to show that

$$\prod \frac{-1}{z_n} \left(\frac{z - z_n}{1 - \overline{z_n} z} \right)$$

converges (to $B(z)/B(0)$) locally uniformly on $\mathbb{D}$. This would certainly be implied by the locally uniform convergence of the series

$$\begin{aligned} & \sum \left| 1 - \frac{-1}{z_n} \left(\frac{z - z_n}{1 - \overline{z_n} z} \right) \right| \\ = \ & \sum \left| \frac{z(1 - |z_n|^2)}{z_n (1 - \overline{z_n} z)} \right| \\ = \ & \sum \frac{(1 - |z_n|^2)|z|}{|z_n||1 - \overline{z_n} z|}. \end{aligned}$$

This last series clearly converges locally uniformly on $\mathbb{D}$ by comparison with $\sum 1 - |z_n|$. $\square$

Theorem 5.2.4

The sequence of zeros (z_n) *of any bounded analytic function* $f : \mathbb{D} \to \mathbb{C}$*, repeated according to their multiplicity, is discrete and has* $\sum 1 - |z_n|$ *convergent. Conversely any sequence with these properties is the set of zeros of a bounded analytic function.*

$\square$

Note that if $f : \mathbb{D} \to \mathbb{C}$ is bounded and B is the Blaschke product on the sequence of zeros of f, then f/B is an analytic function $g : \mathbb{D} \to \mathbb{C} \setminus \{0\}$. Also, since all the partial products of the Blaschke product have modulus 1 on $\partial\mathbb{D}$, we have $||g||_\infty \leqslant ||f||_\infty$.

Exercises

12. Let (z_n) be a discrete sequence of points in $\mathbb{D}$. For each $w \in \mathbb{D}$ show that following conditions are equivalent.
    (a) The series $\sum 1 - |z_n|$ converges.
    (b) The series $\sum \exp -\rho(w, z_n)$ converges.
    (c) The Blaschke product for (z_n) converges at w.
13. A Blaschke product on a finite set of points in $\mathbb{D}$ is called a *finite Blaschke product*. (This includes the constant maps $z \mapsto \omega$ for $|\omega| = 1$.) Prove that a continuous function $f : \overline{\mathbb{D}} \to \mathbb{C}$ is a finite Blashke product if, and only if, it is analytic on $\mathbb{D}$ and maps $\partial\mathbb{D}$ into itself.

    What are the continuous maps $f : \mathbb{D} \to \mathbb{C}_\infty$ which are meromorphic on $\mathbb{D}$ and map $\partial\mathbb{D}$ into itself?
14. Let B be the Blaschke product for a sequence (z_n) in $\mathbb{D}$ which satisfies $\sum 1 - |z_n| < \infty$. Show that the Blaschke product converges not only on $\mathbb{D}$ but also on $\{z \in \mathbb{C}_\infty : |z| > 1\}$ giving a meromorphic function with poles at the points $(J(z_n))$. Prove that $JB(z) = BJ(z)$ for $z \in \mathbb{D}$.

    If $z \in \partial\mathbb{D}$ is not the limit point of a sequence (z_n) then prove that the Blaschke product converges at z, is analytic on a neighbourhood, and satisfies $|B(z)| = 1$.

5.3 Models for the Hyperbolic Plane

We have defined the hyperbolic plane as the unit disc $\mathbb{D}$ with the hyperbolic metric ρ. Set

$$\kappa(w,z) = 2\sinh \tfrac{1}{2}\rho(w,z)\ .$$

Note that

$$\tau = \tanh \tfrac{1}{2}\rho(w,z) = \left|\frac{w-z}{1-\overline{w}z}\right|$$

satisfies

$$\sinh \tfrac{1}{2}\rho(w,z) = \frac{\tau}{\sqrt{(1-\tau^2)}}\ ; \qquad \cosh \tfrac{1}{2}\rho(w,z) = \frac{1}{\sqrt{(1-\tau^2)}}\ .$$

Therefore,

$$\kappa(w,z)^2 = \frac{4\tau^2}{1-\tau^2} = \frac{4|w-z|^2}{|1-\overline{w}z|^2-|w-z|^2} = \frac{4|w-z|^2}{(1-|w|^2)(1-|z|^2)}$$

and consequently:

$$\kappa(w,z) = \frac{2|w-z|}{\sqrt{(1-|w|^2)}\sqrt{(1-|z|^2)}}\ .$$

This is the *chordal distance*. It should be compared to the chordal metric on the Riemann sphere.

Note that the chordal distance is not a metric on the disc. For, suppose that z_1, z_2, z_3 are three points in order on a hyperbolic geodesic in $\mathbb{D}$. Then $\rho(z_1,z_3) = \rho(z_1,z_2)+\rho(z_2,z_3)$, so

$$\kappa(z_1,z_3) = 2\sinh \tfrac{1}{2}\left(\rho(z_1,z_2)+\rho(z_2,z_3)\right)$$

while

$$\kappa(z_1,z_2)+\kappa(z_2,z_3) = \sinh \tfrac{1}{2}\rho(z_1,z_2) + \sinh \tfrac{1}{2}\rho(z_2,z_3)\ .$$

Hence we can not have $\kappa(z_1,z_3) \leqslant \kappa(z_1,z_2)+\kappa(z_2,z_3)$ for all z_1,z_2,z_3. However, the chordal distance is a strictly increasing function of the hyperbolic distance and so the sets

$$\{z \in \mathbb{D} : \kappa(w,z) < r\} \qquad \text{for } 0 < r < \infty$$

are a basis for the neighbourhoods of w.

The group Möb($\mathbb{D}$) is the group of orientation preserving isometries for the hyperbolic metric ρ or for the chordal distance κ.

There are other models for the hyperbolic plane. Let $\langle\ ,\ \rangle$ be the indefinite inner product (or sesquilinear form)

$$\langle \boldsymbol{w}, \boldsymbol{z}\rangle = \overline{w_0}z_0 - \overline{w_1}z_1\ .$$

Set

$$\begin{aligned}\mathcal{D} &= \{[\boldsymbol{z}] \in \mathbb{P}(\mathbb{C}^2) : \langle \boldsymbol{z},\boldsymbol{z}\rangle > 0\} \\ &= \{[z_o : z_1] \in \mathbb{P}(\mathbb{C}^2) : |z_0|^2 - |z_1|^2 > 0\}\ .\end{aligned}$$

Then the map

$$\begin{aligned}\alpha : \mathcal{D} \to \mathbb{D}\ ;&\quad [z_0 : z_1] \mapsto \frac{z_1}{z_0} \\ \alpha^{-1} : \mathbb{D} \to \mathcal{D}\ ;&\quad z \mapsto [1 : z]\end{aligned}$$

is a bijection. So we may take $\mathcal{D}$ as a model for the hyperbolic plane. The *Study distance* on $\mathcal{D}$ is:

$$d([\boldsymbol{w}],[\boldsymbol{z}]) = 2\sqrt{\left(-1+\frac{|\langle \boldsymbol{w},\boldsymbol{z}\rangle|^2}{\langle \boldsymbol{w},\boldsymbol{w}\rangle\langle \boldsymbol{z},\boldsymbol{z}\rangle}\right)}\ .$$

Note that

$$\frac{|\langle \boldsymbol{w}, \boldsymbol{z}\rangle|^2}{\langle \boldsymbol{w}, \boldsymbol{w}\rangle\langle \boldsymbol{z}, \boldsymbol{z}\rangle} = \frac{|1-\overline{\alpha(\boldsymbol{w})}\alpha(\boldsymbol{z})|^2}{(1-|\alpha(\boldsymbol{w})|^2)(1-|\alpha(\boldsymbol{z})|^2)} = 1+\frac{|\alpha(\boldsymbol{w})-\alpha(\boldsymbol{z})|^2}{(1-|\alpha(\boldsymbol{w})|^2)(1-|\alpha(\boldsymbol{z})|^2)} = 1+\sinh^2 \tfrac{1}{2}\rho(\alpha(\boldsymbol{w}), \alpha(\boldsymbol{z}))\ .$$

Hence,

$$d([\boldsymbol{w}], [\boldsymbol{z}]) = \kappa(\alpha(\boldsymbol{w}), \alpha(\boldsymbol{z}))$$

so the chordal distance on $\mathbb{D}$ corresponds to the Study distance on $\mathcal{D}$.

Let $[\boldsymbol{w}]$ be a point in $\mathbb{P}(\mathbb{C}^2)$ with $\langle \boldsymbol{w}, \boldsymbol{w}\rangle \neq 0$. Then $[\boldsymbol{w}]$ is a complex 1-dimensional subspace of $\mathbb{C}^2$ and its orthogonal complement:

$$[\boldsymbol{w}]^\perp = \{\boldsymbol{z} \in \mathbb{C}^2 : \langle \boldsymbol{w}, \boldsymbol{z}\rangle = 0\}$$

is another point in $\mathbb{P}(\mathbb{C}^2)$.