2.1 Definitions

Let R be a connected, Hausdorff space. A *chart* is a homeomorphism $\phi : U \to \phi U$ from an open subset U of R onto a domain $\phi U \subset \mathbb{C}$. A collection of such charts $\phi_\alpha : U_\alpha \to \phi_\alpha U_\alpha$ for $\alpha \in A$ whose domains U_α cover R is an *atlas* for R with *transition maps*

$$t_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta).$$

If all the transition maps are analytic, then we say that R, with this atlas, is a *Riemann surface*. Since $t_{\alpha\beta}^{-1} = t_{\beta\alpha}$ the transition maps for a Riemann surface are conformal. The collection of all charts $\phi : U \to \phi U$ which have

$$\phi\phi_\alpha^{-1} : \phi_\alpha(U \cap U_\beta) \to \phi(U \cap U_\alpha)$$

analytic for every $\alpha \in A$ also forms an atlas with analytic transition maps and is the unique maximal such atlas containing $\{\phi_\alpha : \alpha \in A\}$. We will identify two Riemann surfaces which have the same maximal atlases.

Let R, S be two Riemann surfaces and $f : R \to S$ a map between them. For charts $\phi : U \to \phi U$ for R and $\psi : V \to \psi V$ for S we obtain a map

$$\psi f \phi^{-1} : \phi(U \cap f^{-1}V) \to \psi(V)$$

between domains in $\mathbb{C}$. We will say that the map f is *smooth* if each of these maps is smooth, and that it is *analytic* if each of these maps is analytic. It is apparent that if these definitions hold for any atlases then they also hold for the associated maximal atlases.

Note that, since each point of a Riemann surface has a neighbourhood homeomorphic to a domain in $\mathbb{C}$, the Riemann surface is path connected as well as being connected. Also, any local properties of analytic maps of domains in $\mathbb{C}$ can be transferred to analytic maps on a Riemann surface. We will do this without further remark.

As a first example, any domain $\Omega \subset \mathbb{C}$ becomes a Riemann surface with the atlas consisting of the single chart $I : \Omega \to \Omega \subset \mathbb{C}$ which is the identity. The definition of analytic maps between such Riemann surfaces obviously agrees with our earlier definition of analytic maps between domains in $\mathbb{C}$. In particular, the complex plane $\mathbb{C}$ is a Riemann surface. The collection of analytic maps $f : R \to \mathbb{C}$ from a Riemann surface R into $\mathbb{C}$ forms a vector space $\mathcal{O}(R)$. The constant functions are always analytic but there may be no others. For example, if R is a compact Riemann surface and $f : R \to \mathbb{C}$ is an analytic map which is not constant, then $f(R)$ is compact and also open (by the open mapping theorem for analytic maps). This contradicts the connectedness of R, so $\mathcal{O}(R)$ must consist of the constant functions alone.

Let $\mathbb{C}_\infty$ be the one-point compactification of $\mathbb{C}$. So $\mathbb{C}_\infty$ consists of the points of $\mathbb{C}$ together with an additional point denoted by ∞. The open sets in $\mathbb{C}_\infty$ are those open sets in $\mathbb{C}$ together with the complements in $\mathbb{C}_\infty$ of the compact subsets of $\mathbb{C}$. It is easy to see that $\mathbb{C}_\infty$ is a compact, connected, Hausdorff topological space. The two maps

$$\begin{aligned} i : \mathbb{C}_\infty \setminus \{\infty\} &\to \mathbb{C} \quad ; \quad z \mapsto z \\ j : \mathbb{C}_\infty \setminus \{0\} &\to \mathbb{C} \quad ; \quad \begin{cases} z \mapsto z^{-1} \\ \infty \mapsto 0 \end{cases} \end{aligned}$$

form an atlas for $\mathbb{C}_\infty$ which makes it a Riemann surface called the *extended complex plane* or *Riemann sphere*. An analytic function $f : R \to \mathbb{C}_\infty$ from a Riemann surface R into $\mathbb{C}_\infty$ is called a *meromorphic* function on R provided that it is not identically ∞. The collection of all such functions is denoted by $\mathcal{M}(R)$. It is clear that $\mathcal{M}(R)$ contains the constant functions but it is not obvious that there are any other meromorphic functions. We shall prove later that there are. The set of all meromorphic functions on R forms a field $\mathcal{M}(R)$ which is called the *function field* of R.

Proposition 2.1.1

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be an analytic function and R one component of $f^{-1}(0)$. If $f'(z_1, z_2) \neq \mathbf{0}$ at every point of R then R can be made into a Riemann surface with the projection maps $pr_k : (z_1, z_2) \mapsto z_k$ being analytic.

Proof:

Suppose that the conditions are satisfied and $(w_1, w_2) \in R$. Then either $\frac{\partial f}{\partial z_1}$ or $\frac{\partial f}{\partial z_2}$ (or both) are non-zero at (w_1, w_2). Suppose it is the former. Then the map

$$F : \mathbb{C}^2 \to \mathbb{C}^2 \quad ; \quad (z_1, z_2) \mapsto (f(z_1, z_2), z_2)$$

has $F'(w_1, w_2)$ invertible. The inverse function theorem implies that F is invertible when restricted to some open neighbourhood N of (w_1, w_2) in $\mathbb{C}^2$. Then $U = \{(z_1, z_2) \in N : f(z_1, z_2) = 0\}$ is a neighbourhood of (w_1, w_2) in R and F acts on this as the second co- ordinate projection

$$pr_2 : U \to \mathbb{C} \quad ; \quad (z_1, z_2) \mapsto z_2.$$

Take this as a chart for R.

Similarly, if $\frac{\partial f}{\partial z_2} \neq 0$ at (w_1, w_2), then we take the restriction of pr_1 to some neighbourhood of (w_1, w_2) as a chart for R. Since the mappings F which we constructed were (complex) differentiable with differentiable inverses, it follows that the transition maps for these charts are differentiable. Hence R is a Riemann surface. $\square$

For example, suppose that p is a polynomial in two complex variables. If $p'(z_1, z_2) \neq 0$ on the set $\{(z_1, z_2) : p(z_1, z_2) = 0\}$, then each component of this set is a Riemann surface.

It is more natural to consider homogeneous polynomials p in three variables. Then $\{[z_1 : z_2 : z_3] \in \mathbf{P}(\mathbb{C}^3) : p(z_1, z_2, z_3) = 0\}$ is a subset of the complex projective space $\mathbf{P}(\mathbb{C}^3)$. Let R be a component of this set and suppose that $p'(z_1, z_2, z_3)$ is never 0 on R. Then we may apply the above result on polynomials in two variables to the polynomial

$$(z_1, z_2) \mapsto p(z_1, z_2, 1)$$

and the two similar polynomials with z_1 or z_2 set to 0. This shows that R is a Riemann surface with the maps

$$[z_1 : z_2 : z_3] \mapsto \frac{z_j}{z_k} \qquad \text{for } j \neq k$$

meromorphic on R. Since $\mathbf{P}(\mathbb{C}^3)$ is compact, we see that R is a compact Riemann surface. (In fact every compact Riemann surface is conformally equivalent to a Riemann surface embedded in this way in $\mathbb{P}(\mathbb{C}^4)$.)

Exercises

1. Let p be a polynomial in one complex variable which has no repeated zeros. Show that

$$\{(w, z) : w^2 = p(z)\}$$

   is a Riemann surface. What happens if p does have repeated zeros?

2. Show that

$$R = \{(w, z) \in \mathbb{C}^2 : w^2 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)\}$$

   is a Riemann surface provided that the four complex numbers are distinct. Prove that it may be made into a compact Riemann surface by adjoining two points. Prove that this compact surface is homeomorphic to a torus (i.e. $S^1 \times S^1$).

3. Extend the results of the previous question to surfaces defined by
$$w^2 = (z - z_1)(z - z_2)\dots(z - z_N).$$

2.2 Analytic functions.

Proposition 2.2.1

Let $f : R \to S$ be an analytic function between two Riemann surfaces. Then either f is constant on R or else $f^{-1}(w)$ consists of isolated points for each $w \in S$.

Proof:

Suppose that z is a non-isolated point of $f^{-1}(w)$. Then there are charts $\phi : U \to \phi U$ for R and $\psi : V \to \psi V$ for S with $z \in U$, $w \in V$ and $fU \subset V$. The composite $F = \psi f \phi^{-1} : \phi U \to \mathbb{C}$ is an analytic function on the plane domain ϕU and $\phi(z)$ is a non-isolated point of $F^{-1}(\psi w)$. Consequently, F must be constant and so f is constant on the neighbourhood U of z. This shows that set of non-isolated points of $f^{-1}(w)$ is open in R. It is obviously closed so, since R is connected, it must be either empty or else all of R. □

The following analogue of Proposition 1.1.1 is straightforward.

Proposition 2.2.2

Let $f : R \to S$ be an analytic function between two Riemann surfaces and $z_o \in R$. Either f is constant or there is a natural number $N \in \{1, 2, 3, \dots\}$ and charts $\phi : U \to \phi U = \mathbb{D}$ and $\psi : V \to \psi V = \mathbb{D}$ for R and S respectively with $\phi(z_o) = 0$, $\psi(f(z_o)) = 0$ and
$$\psi(f(z)) = \phi(z)^N \qquad \text{for } z \in U.$$

The number N is unique and is called the *degree* $\deg f(z_o)$ of f. A point z_o is a *critical point* for f if $\deg f(z_o) > 1$ and $f(z_o)$ is then a *critical value*. The theorem shows that each critical point is isolated.

Proof:

Suppose that f is not constant. We can certainly find charts $\phi_o : U_o \to \phi_o U_o$ for R and $\psi_o : V_o \to \psi_o V_o$ for S with $\phi_o(z_o) = 0$, $\psi_o f(z_o) = 0$ and $fU_o \subset V_o$. Then $F = \psi_o f \phi_o^{-1} : \phi_o U_o \to \mathbb{C}$ is an analytic map with $F(0) = 0$. Take N as the degree of F at 0. Proposition 1.1.1 now shows that there is a neighbourhood D of 0 in $\phi_o U_o$ and a conformal map $g : D \to D'$ onto a neighbourhood of 0 with $F(\zeta) = g(\zeta)^N$ for $\zeta \in D$. By reducing the size of D we may ensure that $D' = \{\zeta \in \mathbb{C} : |\zeta| < r\}$ for some $r > 0$. Then $FD = {D'}^N = \{\zeta \in \mathbb{C} : |\zeta| < r^N\}$.

Set $U = \phi_o^{-1} D$ and
$$\phi : U \to \mathbb{C} \quad ; \quad z \mapsto \frac{g\phi_o(z)}{r}.$$
So ϕ is a chart for R with $\phi U = \mathbb{D}$. Similarly, set $V = fU$ and
$$\psi : V \to \mathbb{C} \quad ; \quad w \mapsto \frac{\psi_o(w)}{r^N}$$
so that ψ is a chart for S with $\psi V = \mathbb{D}$. Then, for $z \in U$ we have
$$\psi f(z) = \frac{\psi_o f(z)}{r^N} = \frac{F\phi_o(z)}{r^N} = \left(\frac{g\phi_o(z)}{r}\right)^N = \phi(z)^N$$
as required. □

This proposition certainly implies that, for each $z\_o \in R$, there is a neighbourhood U of z in R and a neighbourhood V of $f(z_o)$ so that every value $w \in V \setminus \{f(z_o)\}$ is taken exactly $\deg f(z_o)$ times in U. The value $f(z_o)$ is only taken at the point z_o of U but it has multiplicity $\deg f(z_o)$.

Theorem 2.2.3

Let $f : R \to S$ be an analytic map from a compact Riemann surface R into another Riemann surface S. If f is not constant then S is also compact and

$$\sum (\deg f(z) : f(z) = w)$$

is a natural number $N \in \{1, 2, 3, \ldots\}$ independent of the point $w \in S$.

The number N is called the *degree* of f. The critical points of f form a discrete subset of the compact set R and are therefore finite in number. Consequently the set of critical values is finite. If $w \in S$ is not one of these critical values then there are exactly N points in $f^{-1}(w)$. We often call the sum $\sum(\deg f(z) : f(z) = w)$ the number of solutions of $f(z) = w$ *counting their multiplicity*. The theorem then shows that this number is independent of w.

Proof:

If f is not constant then the open mapping theorem shows that the image $f(R)$ is open in S. Since f is continuous, $f(R)$ is compact and hence closed. Therefore $f(R) = S$ and S is compact.

Suppose that f is not constant and set

$$N(w) = \sum (\deg f(z) : f(z) = w)$$

for $w \in S$. Let w_o be a point of S. The previous proposition shows that each $z_o \in f^{-1}(w_o)$ is contained in a neighbourhood U which is mapped $\deg f(z_o)$-to-1 onto a neighbourhood V of $f(z_o)$. By taking W equal to the intersection of the neighbourhoods V for each $z_o \in f^{-1}(w_o)$ we find that there is an open set T in R containing $f^{-1}(w_o)$ which is mapped $N(w_o)$-to-1 onto an open set W in S containing w_o. This certainly implies that $N(w) \geqslant N(w_o)$ for $w \in W$. Moreover, the set $R \setminus T$ is compact so its image under f will be a compact subset of S which will not meet w_o. Hence $S \setminus f(R \setminus T)$ is a neighbourhood of w_o. Clearly $N(w) \leqslant N(w_o)$ on this neighbourhood. Therefore $N(w)$ will be constant on some neighbourhood of each point w_o. Since S is connected this means that $N(w)$ is constant on S. □

2.3 Covering Surfaces.

Throughout this section R will be a Riemann surface with a specified base point $z_o \in R$. A *path* in R is a continuous map $\gamma : I \to R$ from the unit interval I. It starts at $\gamma(0)$ and ends at $\gamma(1)$. Let $C(z_1, z_2)$ be the set of all paths in R which start at z_1 and end at z_2. A *homotopy* (relative to $\{0, 1\}$) is a family of paths $\gamma_s \in C(z_1, z_2)$ for $s \in I$ with

$$h : I \times I \to R \quad ; \quad (s, t) \mapsto \gamma_s(t)$$

continuous. When such a homotopy exists we say that γ_0 and γ_1 are *homotopic* and write $\gamma_0 \sim \gamma_1$. Then $\sim$ is an equivalence relation on $C(z_1, z_2)$.

If $\beta \in C(z_1, z_2)$, $\gamma \in C(z_2, z_3)$ are two paths in R their *product* is the path

$$\gamma \cdot \beta : I \to R \quad ; \quad \begin{cases} \beta(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

in $C(z_1, z_3)$. The product respects the equivalence relation of homotopy. In particular, the quotient $C(z_o, z_o)/\sim$ is a group $\pi_1(R, z_o)$ called the *fundamental* or *first homotopy* group of R. We say that R is *simply connected* if the fundamental group of R is trivial.

Exercises

4. Prove that $\pi_1(R, z_0)$ is a group. Show that $\pi_1(R, z)$ is isomorphic to $\pi_1(R, z_o)$ for any $z \in R$. (The isomorphism is not natural.) Calculate $\pi_1(R, z_o)$ for the following Riemann surfaces: (a) $\mathbb{D}$, (b) an annulus, (c) a torus, (d) $\mathbb{C} \setminus \{0, 1\}$.

If S is another Riemann surface with a base point w_o then we will write $f : (S, w_o) \to (R, z_o)$ to denote a map $f : S \to R$ which maps w_o to z_o. A *regular covering* of (R, z_o) is a map $p : (S, w_o) \to (R, z_o)$ from a Riemann surface S which satisfies

(a) p is locally conformal, so each $w \in S$ has an open neighbourhood U with the restriction $p| : U \to pU$ conformal.

(b) for each path γ in R which starts at z_o there is a path Γ in S which starts at w_o and satisfies $p\Gamma = \gamma$.

The path Γ is called a *lift* of γ. It is unique, for if Γ' were another lift then $T = \{t \in I : \Gamma'(t) = \Gamma(t)\}$ is closed, because Γ' and Γ are continuous, and open, because p is a local homeomorphism. Since $0 \in T$ and I is connected we must have $T = I$ and so $\Gamma' = \Gamma$.

The choice of base points is largely irrelevant, as the following proposition shows.

Proposition 2.3.1

If $p : (S, w_o) \to (R, z_o)$ is a regular covering then so is $p : (S, w) \to (R, f(w))$ for every $w \in S$.

Proof:

Let γ be a path in R starting at $f(w)$. Since S is connected there is a path B from w_o to w. Its image under p is a path β in R from z_o to $f(w)$. Since $p : (S, w_o) \to (R, z_o)$ is a regular covering the path $\gamma \cdot \beta$ lifts to a path A which starts at w_o. The first part of A is a lift of β and so agrees with B. The remainder is a path Γ which starts at w and is a lift of γ. □

Recall (or prove) the:

Theorem 2.3.2 The monodromy theorem

Let $p : (S, w_o) \to (R, z_o)$ be a regular covering of the Riemann surface R. If (γ_s) is a homotopy of curves in R which start at z_o and end at some point $z \in R$, then the lifts (Γ_s) form a homotopy of curves in S which start at w_o and end at some point $w \in S$.

□

A regular covering $p : (S, w_o) \to (R, z_o)$ is a *universal* covering if S is simply connected. The name is derived from the following "universal property".

Proposition 2.3.3

Let $p : (S, w_o) \to (R, z_o)$ be a regular covering and $q : (\hat{R}, \hat{z}_o) \to (R, z_o)$ a universal covering. Then there is a map $f : (\hat{R}, \hat{z}_o) \to (S, w_o)$ and $pf = q$. Moreover, f is unique and is itself a universal covering of S.

Proof:

For each $\zeta \in \hat{R}$ there is a path Γ in $\hat{R}$ starting from $\hat{z}_o$ and ending at ζ. Then $\beta = q\Gamma$ is a path in R starting at z_o. Because p is a regular covering, β lifts to a path B in S starting at w_o and ending at a point $w \in S$.

Suppose that Γ' were another path from $\hat{z}_o$ to ζ and that $\beta' = q\Gamma'$ has a lift B'. Then Γ and Γ' are homotopic because $\hat{R}$ is simply connected. The image of this homotopy under q gives a homotopy between β and β'. Now the monodromy theorem 2.3.2 implies that B and B' are homotopic. In particular the endpoint of B' is w. Therefore we can define a map $f : \hat{R} \to S$ by $f : \zeta \mapsto w$. It is clear that $pf = q$.

If $F : (\hat{R}, \hat{z}_o) \to (S, w_o)$ satisfies $pF = q$ then $F\Gamma$ is a path in S with $p(F\Gamma) = q\Gamma = \beta$. Hence $F\Gamma$ is a lift of β and must therefore be B. In particular, $f(\zeta) = B(1) = F(\zeta)$. So f is unique, and paths in S lift to $\hat{R}$. Since p and q are locally conformal, f must be locally conformal. Hence f is a regular covering of S. Since $\hat{R}$ is simply connected f is a universal covering. □

This proposition shows that if $q : (\hat{R}, \hat{z}_o) \to (R, z_o)$ and $q' : (\hat{R}', \hat{z}'_o) \to (R, z_o)$ are two universal coverings, then there is an unique conformal map $f : (\hat{R}', \hat{z}'_o) \to (\hat{R}, \hat{z}'_o)$ with $qf = q'$. So the universal covering of R is determined up to a conformal map.

Theorem 2.3.4 Universal coverings

Every Riemann surface has a universal covering.

Proof:

Let (R, z_o) be the Riemann surface and C the set of all paths in R which start at z_o. Set $\hat{R}$ equal to the quotient $C/\sim$ and define a map

$$q : \hat{R} \to R \quad \text{by} \quad [\gamma] \to \gamma(1).$$

We will prove that this is a universal covering of R.

The constant map $e : I \to R\ ;\ t \mapsto z_o$ will be the base point of $\hat{R}$. Clearly $q(e) = z_o$. Let $\gamma \in C$ be a path in R which ends at z and $\phi : U \to \phi U = \mathbb{D}$ be a chart for R with $z \in U$. Then set

$$[\gamma, U] = \{[\beta \cdot \gamma] : \beta \text{ is a path in } U \text{ starting at } z\}.$$

This is a subset of $\hat{R}$ and the restriction $q| : [\gamma, U] \to U$ is bijective. We will show that these sets form the basis for a Hausdorff topology on $\hat{R}$.

For suppose that $[\alpha] \in [\gamma_1, U_1] \cap [\gamma_2, U_2]$. So there are paths β_k in U_k with $\beta_1 \cdot \gamma_1 \sim \alpha \sim \beta_2 \cdot \gamma$. Since α ends at a point $z \in U_1 \cap U_2$ we can find a chart $\phi : U \to \phi U = \mathbb{D}$ with $z \in U \subset U_1 \cap U_2$. Any path in U starting from z lies entirely within U_1 and U_2 so

$$[\alpha, U] \subset [\gamma_1, U_1] \cap [\gamma_2, U_2].$$

Therefore the sets $[\gamma, U]$ do from the basis for a topology on $\hat{R}$. Note that when $\hat{R}$ is given this topology each of the restrictions $q| : [\gamma, U] \to U$ is a homeomorphism.

Suppose that $[\gamma_1], [\gamma_2]$ are two different points of $\hat{R}$. If $\gamma_1(1) \neq \gamma_2(1)$ then we can certainly find charts $\phi_k : U_k \to \phi_k U_k = \mathbb{D}$ with $\gamma_k(1) \in U_k$ and U_1, U_2 disjoint. Then we claim that $[\gamma_k, U_k]$ are disjoint open sets containing $[\gamma_k]$. Otherwise, $\gamma_1(1) = \gamma_2(1)$ but γ_1 and γ_2 are not homotopic. Then we can find a chart $\phi : U \to \phi U = \mathbb{D}$ with $\gamma_1(1) = \gamma_2(1) \in U$. Suppose that $[\alpha] \in [\gamma_1, U] \cap [\gamma_2, U]$ so that $\beta_1 \cdot \gamma_1 \sim \alpha \sim \beta_2 \cdot \gamma_2$ for some curves β_k in U. The curves β_k have the same endpoints and lie in U which is homeomorphic to $\mathbb{D}$. Hence $\beta_1 \sim \beta_2$ and consequently $\gamma_1 \sim \gamma_2$, which was forbidden. Thus we have shown that $\hat{R}$ is Hausdorff.

For each chart $\phi : U \to \phi U = \mathbb{D}$ the restriction $q| : [\gamma, U] \to U$ is a homeomorphism so we can take the composites

$$[\gamma, U] \xrightarrow{q} U \xrightarrow{\phi} \phi U$$

as charts for $\hat{R}$. The transition maps for these charts are the same as those for R, so $\hat{R}$ becomes a Riemann surface and q a locally conformal map.

Let γ be a path in R starting from z_o. For each $s \in I$ define

$$\gamma_s : I \to R \quad ; \quad t \mapsto \gamma(st).$$

Then $s \mapsto [\gamma_s]$ is a lift of γ to $\hat{R}$ starting from $\gamma_0 = e$. Hence $q : \hat{R} \to R$ is a regular covering.

Now suppose that Γ is a path in $\hat{R}$ with both endpoints at e. Then $\gamma = q\Gamma$ is a path in R with both endpoints at z_o. Both Γ and $s \mapsto [\gamma_s]$ are lifts of γ so they are equal. In particular, $[\gamma_1] = \Gamma(1) = [e]$ so $\gamma = \gamma_1 \sim \gamma_0 = e$. The monodromy theorem 2.3.2 shows that the homotopy from γ to e lifts to a homotopy in $\hat{R}$ from Γ to the path which is constantly e. So Γ is null-homotopic and $\hat{R}$ is simply connected. $\square$

An *automorphism* of the regular covering $p : (S, w_o) \to (R, z_o)$ is a homeomorphism $f : S \to S$ with $pf = p$. The collection of these form a group $\mathrm{Aut}(p)$ called the *automorphism group* of p (or the *deck transformation group*. Since p is locally conformal, each automorphism f must also be conformal. The set $\{w \in S : f(w) = w\}$ is then closed, since f is continuous, and open, since f is locally conformal. So it is either empty or all of S. Thus the the only automorphism of p which fixes a point is the identity.

Theorem 2.3.5

Let $q : \hat{R} \to R$ be a universal covering for the Riemann surface R. Then $q(w') = q(w)$ if, and only if, there is an automorphism f of q with $f(w') = w$. Thus R is the quotient of $\hat{R}$ by the action of the group $\mathrm{Aut}(q)$.

Proof:

If there is an automorphism f with $f(w') = w$ then $q(w) = qf(w') = q(w')$. Conversely, suppose that there are two points $w, w' \in \hat{R}$ with $q(w') = z = q(w)$. By Proposition 2.3.1 both $p : (\hat{R}, w') \to (R, z)$ and $p : (\hat{R}, w) \to (R, z)$ are universal coverings. Hence Proposition 2.3.3 shows that there is a map $f : (\hat{R}, w') \to (\hat{R}, w)$ with $qf = q$. This map is an automorphism of q with $f(w') = w$. $\square$

This result shows that every Riemann surface R is the quotient $\hat{R}/G$ of a simply connected Riemann surface $\hat{R}$ by the action of a subgroup G of the group $\mathrm{Aut}\,\hat{R}$ of all conformal automorphisms of $\hat{R}$.

Theorem 2.3.6

Let G be a subgroup of the group $Aut(S)$ of conformal maps from a Riemann surface S to itself. Suppose that, for each $w \in S$, there is a neighbourhood U of w with U and $T(U)$ disjoint for every $T \in G \setminus \{I\}$. Then S/G is a Riemann surface. Moreover, every Riemann surface arises in this way from a simply connected Riemann surface S.

Note that the condition $U \cap T(U) = \emptyset$ certainly implies that the only element of G with a fixed point is the identity.

Proof:

Consider those charts $\phi : U \to \phi U \subset \mathbb{C}$ for S which have $U \cap T(U) = \emptyset$ for every $T \in G \setminus \{I\}$. The hypothesis on G ensures that these form an atlas. Let $[w]$ denote the equivalence class $\{Tw : T \in G\}$ in S/G. For each of the charts the map

$$\tilde{\phi} : \{[w] : w \in U\} \to \phi U \quad ; \quad [w] \mapsto \phi(w)$$

is well defined. We take these maps as charts for S/G. It is easy to check that S/G with the quotient topology is Hausdorff and that these charts form an atlas. The transition maps are analytic because each ϕ and each $T \in G$ is analytic. Hence they make S/G into a Riemann surface. The quotient map $S \to S/G$ is a regular covering.

Conversely, if R is any Riemann surface the previous theorem shows that $R = \hat{R}/G$ for G a subgroup of $\mathrm{Aut}\,\hat{R}$. If $w \in \hat{R}$ then there is a neighbourhood U of w with $\pi| : U \to \pi(U)$ conformal, because the universal covering $\pi : \hat{R} \to R$ is locally conformal. If $z \in U \cap T(U)$ then z and $T^{-1}(z)$ are in U and have $\pi(z) = \pi(T^{-1}(z))$. Therefore $z = T^{-1}(z)$. The only element of G which fixes any point is the identity, so $T = I$. □

The Riemann Mapping Theorem states that the only simply connected Riemann surfaces are $\mathbb{C}_\infty, \mathbb{C}$ and $\mathbb{D}$ (up to conformal equivalence). Hence any Riemann surface is the quotient of one of these by a group of automorphisms. We will study these three surfaces in the following chapters.

Exercises

5. Let $\psi : (M, w_o) \to (R, z_o)$ be a regular covering of R and $\pi : (\hat{R}, \hat{z}_o) \to (R, z_o)$ a universal covering. Then there is a covering $f : (\hat{R}, \hat{z}_o) \to (M, w_o)$ by Proposition 2.3.3. Prove that the following two conditions are equivalent.
   (a) If $T \in \mathrm{Aut}\,\pi$ then there is an unique $S \in \mathrm{Aut}\,\psi$ with $Sf = fT$.
   (b) $\mathrm{Aut}\,f = \{T \in \mathrm{Aut}\,\pi : fT = f\}$ is a normal subgroup of $\mathrm{Aut}\,\pi$ and the quotient $\mathrm{Aut}\,\pi/\mathrm{Aut}\,f$ is isomorphic to $\mathrm{Aut}\,\psi$.
6. Show that $\mathbb{C}_\infty, \mathbb{C}$ and $\mathbb{D}$ are all simply connected and that no two of them are conformally equivalent.
7. Exhibit explicitly a universal covering $\pi : \mathbb{D} \to \{z \in \mathbb{C} : r < |z| < 1\}$ for each $0 \leqslant r < 1$. Identify the group $\mathrm{Aut}\,\pi$. [Hint: exp.]
8. Exhibit explicitly a universal covering $\pi : \mathbb{C} \to \{z \in \mathbb{C} : 0 < |z| < \infty\}$. Identify the group $\mathrm{Aut}\,\pi$.
9. Let G be the subgroup of $\mathrm{Aut}\,\mathbb{C}$ which consists of the maps $z \mapsto z + n + mi$ for $n, m \in \mathbb{Z}$. Show that $\mathbb{C}/G$ is a Riemann surface. Is this still true when i is replaced by an arbitrary complex number τ?

2.4 Differential forms

Let $f : R \to \mathbb{C}$ be a smooth function on the Riemann surface R. Then, for each chart $\phi_\alpha : U_\alpha \to \phi_\alpha U_\alpha \subset \mathbb{C}$ we have smooth maps

$$f_\alpha = f\phi_\alpha^{-1} : \phi_\alpha U_\alpha \to \mathbb{C}$$

which satisfy

$$f_\beta(z) = f_\alpha(t_{\alpha\beta}(z))$$

for the transition maps $t_{\alpha\beta} = \phi_\alpha\phi_\beta^{-1}$. Each of these maps f_α is smooth. Conversely, a collection of smooth maps $f_\alpha : \phi_\alpha U_\alpha \to \mathbb{C}$ which satisfy $f_\beta(z) = f_\alpha(t_{\alpha\beta}(z))$ clearly determines a smooth function $f : R \to \mathbb{C}$. We will call a smooth function $f : R \to \mathbb{C}$, or the associated collection $f_\alpha : \phi_\alpha U_\alpha \to \mathbb{C}$, a 0-*form* on R. The set of all 0-forms on R forms a vector space $\mathcal{E}^0(R)$. The analytic functions $f : R \to \mathbb{C}$ form a vector subspace $\mathcal{O}(R)$ of $\mathcal{E}(R)$. The analytic functions $f : R \to \mathbb{C}_\infty$, excluding the constant function ∞ are called the *meromorphic functions on* R and denoted by $\mathcal{M}(R)$. The vector spaces $\mathcal{E}(R), \mathcal{O}(R)$ and $\mathcal{M}(R)$ are all closed under multiplication (and so form algebras). The meromorphic functions form a field, often called the *function field of* R.

However, the derivative of a 0-form f does not give another function on the Riemann surface. Indeed, the derivatives of the functions f_α satisfy

$$f'_\beta(z) = f'_\alpha(t_{\alpha\beta}(z))t'_{\alpha\beta}(z)$$

rather than $f'_\beta(z) = f'_\alpha(t_{\alpha\beta}(z))$. This leads us to the idea of a 1-*form* (or *Abelian differential*) on R as a collection of smooth maps

$$\omega_\alpha : \phi_\alpha U_\alpha \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$$

which satisfy

$$\omega_\beta(z) = \omega_\alpha(t_{\alpha\beta}z)t'_{\alpha\beta}(z) \qquad \text{for} \quad z \in \phi_\alpha(U_\alpha \cap U_\beta).$$

(It is easy to check that a collection of maps which satisfy this condition for one atlas on R can be extended, uniquely, to a collection which satisfies the corresponding condition for a maximal atlas.) It is more appealing to give a definition of 1-forms which is geometrical, so we will define them in terms of tangent vectors.

If $\gamma : [a, b] \to V$ is a smooth curve in V with $\gamma(0) = z_o \in V$, then the derivative $\gamma'(0)$ is often thought of as an arrow from z_o tangent to γ. The set of all such tangent vectors at z_o forms the tangent plane at z_o. To deal with tangent vectors to a Riemann surface, we use the curves to define the vectors.

Let R be a Riemann surface and $z_o \in R$. Let Γ be the set of all smooth curves

$$\gamma : \mathbb{R} \to R \ \text{ with } \ \gamma(0) = z_o \ .$$

Two curves $\gamma_1, \gamma_2 \in \Gamma$ agree to first order at z_o (written $\gamma_1 \sim \gamma_2$) if, for any chart $\phi : U \to V$ at z_o, we have

$$\phi(\gamma_1(t)) - \phi(\gamma_2(t)) = o(t) \ \text{ as } \ t \to 0 \ .$$

If this is true for one chart, then it is true for all analytically compatible charts at z_o. The *tangent plane* at z_o is the set of equivalence classes: $T_{z_o}(R) = \Gamma/\sim$ and an equivalence class $[\gamma]_{z_o}$ is a *tangent vector* at z_o.

If $\phi : U \to V \subset \mathbb{C}$ is a chart at z_o, then the map

$$T_{z_o}(R) \to \mathbb{C} \ ; \quad [\gamma]_{z_o} \mapsto (\phi \circ \gamma)'(0)$$

is a bijection. So we may use it to make the tangent plane into a one-dimensional complex vector space. Since the transition maps are conformal, this vector space structure does not depend on the chart we use.

Now let $f : R \to S$ be a smooth mapping between two Riemann surfaces with $f(z_o) = w_o$. Then the derivative of f at z_o is the **real** linear map

$$f'(z_o) : T_{z_o}(R) \to T_{w_o}(S) \; ; \quad [\gamma]_{z_o} \mapsto [f \circ \gamma]_{w_o} \; .$$

It is straightforward to check that this agrees with the usual definition when R and S are domains in $\mathbb{C}$ and that it obeys the chain rule. If f is analytic, then $f'(z_o)$ is a **complex** linear map.

A 1-*form* ω on R gives a **real** linear map $\omega(z) : T_z(R) \to \mathbb{C}$ at each point $z \in R$ which should vary smoothly with z. This means that for each chart $\phi : U \to V$ the maps

$$\omega(z) \circ \phi'(z)^{-1} : \mathbb{C} \to \mathbb{C}$$

vary smoothly with $z \in U$. Obviously this is independent of the charts chosen. The set of all 1-forms on R is the complex vector space $\mathcal{E}^1(R)$. Let $f : S \to R$ be a smooth mapping and ω a 1-form on R. Then we can define a 1-form $f^*\omega$ on S by

$$f^*\omega(z) = \omega(f(z)) \circ f'(z) : T_{f(z)}(S) \to \mathbb{C} \; .$$

This is called the *pull-back* of ω by f.

To give examples of 1-forms, consider $f \in \mathcal{E}^0(R)$. The *differential* of f at z is the map

$$df(z) : T_z(R) \to \mathbb{C} \; ; \quad [\gamma]_z \mapsto (f \circ \gamma)'(0) \; .$$

This clearly varies smoothly with z and so $df \in \mathcal{E}^1(R)$.

In particular, if Ω is a domain in $\mathbb{C}$ then we have smooth functions

$$\begin{array}{cccc} x : \Omega \to \mathbb{C} & y : \Omega \to \mathbb{C} & z : \Omega \to \mathbb{C} & \overline{z} : \Omega \to \mathbb{C} \\ z \mapsto \Re z & z \mapsto \Im z & z \mapsto z & z \mapsto \overline{z} \end{array}$$

and their differentials $dx, dy, dz, d\overline{z}$ which map each point of Ω to the real linear maps

$$\Re : w \mapsto \Re w \quad \Im : w \mapsto \Im w \quad I : w \mapsto w \quad C : w \mapsto \overline{w}$$

respectively. Any 1-form ω on Ω can be written as $adx + bdy$ or $\lambda dz + \mu d\overline{z}$ for some 0-forms a, b, λ, μ. In particular, for $f : \Omega \to \mathbb{C}$ we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} \; .$$

The 1-forms are defined precisely so that we can integrate them along paths. Let ω be a 1-form on R and $\gamma : I = [a, b] \to R$ a smooth path. Then we define the integral of ω along γ by:

$$\int_\gamma \omega = \int_I \omega(\gamma(t))[\gamma]_{\gamma(t)} \; dt \; .$$

Here $\omega(\gamma(t))$ is a real linear map $T_{\gamma(t)}(R) \to \mathbb{C}$ which is applied to the tangent vector $[\gamma]_{\gamma(t)}$ at $\gamma(t)$. For the particular case where $\omega = \alpha dz + \beta d\overline{z}$ on a domain $\Omega \subset \mathbb{C}$ we have

$$\int_\gamma \omega = \int_I \alpha(\gamma(t))\gamma'(t) + \beta(\gamma(t))\gamma'(t) \; dt$$

which agrees with the usual definition of $\int_\gamma \alpha dz + \beta d\overline{z}$. (We can also regard $\int_\gamma \omega$ as the integral of pull-back $\gamma^*\omega$ along I.)

Proposition 2.4.1

Let $\gamma : I = [a,b] \to R$ be a smooth path in a Riemann surface R and f a 0-form on R. Then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

Proof:

$$\int_\gamma df = \int_I f'(\gamma(t))[\gamma]_{\gamma(t)}\, dt = \int_I (f \circ \gamma)'(t)\, dt = f(\gamma(b)) - f(\gamma(a))$$

by the fundamental theorem of calculus. □

If γ is a closed curve in R, then $\int_\gamma df = 0$. However, the integral of 1-forms around γ need not give 0. For example, consider the 1-form:

$$\omega = \frac{1}{z} dz \qquad \text{on } \mathbb{C} \setminus \{0\}\ .$$

The integral of this around the unit circle: $\gamma : t \mapsto \exp(2\pi i t)$ is $2\pi i$. This shows that not every 1-form ω is of the type df for some function $f \in \mathcal{E}(R)$. When $\omega = df$ for some $f \in \mathcal{E}(R)$, we say that ω is *exact*. A 1-*cycle* in R is a finite collection of closed curves $\Gamma = (\gamma_n)_{n=1}^N$. The integral of $\omega \in \mathcal{E}^1(R)$ around Γ is defined to be

$$\int_\Gamma \omega = \sum_{n=1}^N \int_{\gamma_n} \omega\ .$$

This will be 0 if ω is exact.

Any **real** linear map $\lambda : T_z(R) \to \mathbb{C}$ can be decomposed uniquely as the sum of a **complex** linear map $\alpha : T_z(R) \to \mathbb{C}$ and a conjugate linear map $\beta : T_z(R) \to \mathbb{C}$ (with $w \mapsto \overline{\beta(w)}$ complex linear). Therefore, each 1-form $\omega \in \mathcal{E}^1(R)$ can be written as the sum $\alpha + \beta$ where each $\alpha(z) : T_z(R) \to \mathbb{C}$ is complex linear and each $\beta(z) : T_z(R) \to \mathbb{C}$ is conjugate linear. We write $\mathcal{E}^{1,0}$ for the space of 1-forms α with each $\alpha(z)$ complex linear and $\mathcal{E}^{0,1}$ for the space of 1-forms β with each $\beta(z)$ conjugate linear. Then $\mathcal{E}(R)$ is the direct sum $\mathcal{E}^{1,0}(R) \oplus \mathcal{E}^{0,1}(R)$. On a domain $\Omega \subset \mathbb{C}$ we can write a 1-form ω as $\alpha dz + \beta d\overline{z}$. Then $\alpha dz \in \mathcal{E}^{1,0}(\Omega)$ and $\beta dz \in \mathcal{E}^{0,1}(\Omega)$. For any 0-form f we can decompose df as $\partial f + \overline{\partial} f$ with $\partial f \in \mathcal{E}^{1,0}(R)$ and $\overline{\partial} f \in \mathcal{E}^{0,1}(R)$. This defines ∂f and $\overline{\partial} f$. When f is defined an a domain in $\mathbb{C}$ we have

$$\partial f = \frac{\partial f}{\partial z} dz \qquad ; \qquad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}\ .$$

We can also define analytic and meromorphic 1-forms. Let $\omega \in \mathcal{E}^1(R)$ and let $\phi : U \to V$ be a chart for R. Then $\tilde{\omega} = (\phi^{-1})^* \omega$ is a 1-form on V. If, for each chart ϕ, $\tilde{\omega} = \alpha dz$ for some **analytic** function $\alpha : V \to \mathbb{C}$, then we say that ω is an analytic differential and write $\omega \in \mathcal{O}^1(R)$. Clearly $\mathcal{O}^1(R)$ is a vector subspace of $\mathcal{E}^{1,0}(R)$. Similarly, suppose that S is a discrete subset of R and ω a 1-form on $R \setminus S$. Then ω is a *meromorphic* 1-form on R if $\tilde{\omega} = (\phi^{-1})^* \omega$ can always be expressed as αdz with $\alpha : V \to \mathbb{C}_\infty$ meromorphic. The points of S are the possible positions for poles in R. The set of meromorphic 1-forms forms a complex vector space.

Proposition 2.4.2

If ω and η are both meromorphic 1-forms on a Riemann surface R and η is not identically zero, then there is a meromorphic function $f : R \to \mathbb{C}_\infty$ with $\omega = f.\eta$.

Proof:

On the image of a chart $\phi : U \to V \subset \mathbb{C}$ both the 1-forms ω and η define meromorphic functions, say $(\phi^{-1})^*\omega = \alpha dz$ and $(\phi^{-1})^*\eta = \beta dz$ with α, β meromorphic. Since η is not identically zero, the function β cannot be identically zero either. Thus $f = \alpha/\beta$ is a meromorphic function on V. The definition of f does not depend on which charts we choose, so we obtain a meromorphic function on all of R. Clearly we have $\omega = f.\eta$ on each chart and hence on all of R. □

A 2-form θ on R should be an object which we can integrate over areas. So $\theta(z) : T_z(R)\times T_z(R) \to \mathbb{C}$ should send a pair of tangent vectors (w_1, w_2) to some complex multiple of the area spanned by w_1 and w_2. Hence $\theta(z)$ should be **real**-bilinear and alternating (skew-symmetric). Therefore, a 2-form θ on R gives, for each $z \in R$, a real-bilinear, alternating map $\theta(z) : T_z(R) \times T_z(R) \to \mathbb{C}$ which varies smoothly with z. This means that for a chart $\phi : U \to V \subset \mathbb{C}$ the bilinear maps

$$\mathbb{C} \times \mathbb{C} \to \mathbb{C} \ ; \quad (w_1, w_2) \mapsto \theta(\phi^{-1}(z))[\phi'(z)^{-1}(w_1), \phi'(z)^{-1}(w_2)]$$

vary smoothly with $z \in U$. The collection of all 2-forms on R forms a vector space $\mathcal{E}^2(R)$.

If $f : S \to R$ is a smooth mapping, then the pull-back $f^*\theta$ of a 2-form $\theta \in \mathcal{E}^2(R)$ is given by

$$f^*\theta : z \mapsto \theta(f(z)) \circ (f'(z) \times f'(z)) \ .$$

So $f^* : \mathcal{E}^2(R) \to \mathcal{E}^2(S)$ is a real linear map.

We can construct 2-forms from two 1-forms as follows. Let $\omega_1, \omega_2 \in \mathcal{E}^1(R)$. Define a 2-form $\omega_1 \wedge \omega_2$ by

$$\omega_1 \wedge \omega_2(z) : (v_1, v_2) \mapsto \omega_1(v_1)\omega_2(v_2) - \omega_1(v_2)\omega_2(v_1) \ .$$

It is easy to check that this is a 2-form and that the wedge product is distributive over addition:

$$\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1 \ ;$$
$$\omega \wedge \omega = 0 \ ;$$
$$\nu \wedge (\omega_1 + \omega_2) = \nu \wedge \omega_1 + \nu \wedge \omega_2 \ .$$

For example, on a domain $\Omega \subset \mathbb{C}$ the 2-form $dx \wedge dy$ is given by

$$dx \wedge dy(z) : (w_1, w_2) \mapsto u_1v_2 - u_2v_1$$

where $w_j = u_j + iv_j$. Then

$$dx \wedge dx = dy \wedge dy = 0 \ ; dy \wedge dx = -dx \wedge dy \ ;$$

and the forms $dz, d\overline{z}$ satisfy

$$dz \wedge d\overline{z} = -d\overline{z} \wedge dz = 2idx \wedge dy \ .$$

Any real-bilinear, alternating form on $\mathbb{C}$ is a complex multiple of the determinant: $(w_1, w_2) \mapsto u_1v_2 - u_2v_1$. So every 2-form on Ω is $fdx \wedge dy = gdz \wedge d\overline{z}$ for some smooth functions $f, g : \Omega \to \mathbb{C}$.

Let $\sigma : \Delta \to R$ be a smooth mapping on a compact subset Δ of $\mathbb{R} \times \mathbb{R}$ with a (piecewise) smooth boundary. Then the integral of $\theta \in \mathcal{E}^2(R)$ over σ is defined by:

$$\int_\sigma \theta = \int\int_{(s,t)\in\Delta} \theta(\sigma(s,t)) \left(\frac{\partial\sigma}{\partial s}(s,t), \frac{\partial\sigma}{\partial t}(s,t)\right) \, ds\, dt \ .$$

(Note that the orientation of Δ is important. Reversing it changes the sign of the integral.) The boundary $\partial\Delta$ of Δ is oriented so that the interior of Δ lies to its left. Then the image of $\partial\Delta$ under σ is a 1-cycle in R, which we denote by $\partial\sigma$.

We now wish to define the differential $d\omega$ of a 1-form. First consider the case of forms on a domain $\Omega \subset \mathbb{C}$. Then $\omega = \alpha dz + \beta d\overline{z}$ and we define

$$d\omega = d\alpha \wedge dz + d\beta \wedge d\overline{z} \ .$$

This definition transforms properly under conformal maps. For suppose that $f : \Omega_1 \to \Omega_2$ is conformal and $\omega = \alpha dz_2 + \beta d\overline{z}_2 \in \mathcal{E}^1(\Omega_2)$. Then $f^*\omega(z_1) = \alpha(f(z_1))f'(z_1)dz_1 + \beta(f(z_1))\overline{f'(z_1)}d\overline{z}_1$ and

$$df^*\omega(z_1) = d\alpha(f(z_1)) \wedge f'(z_1)dz_1 + d\beta(f(z_1)) \wedge \overline{f'(z_1)}d\overline{z}_1 = f^*d\omega(z_1) \ .$$

This means that we can define the differential $d\omega$ of a 1-form $\omega \in \mathcal{E}^1(R)$ locally on charts for any Riemann surface R. It is now simple to prove the elementary properties of d:

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2 \ ; \quad d(f\omega) = df \wedge \omega + f d\omega \ ; \quad d(df) = 0$$

for $f \in \mathcal{E}(R), \omega, \omega_j \in \mathcal{E}^1(R)$.

We can define the operators ∂ and $\overline{\partial}$ on 1-forms in a similar way. On a domain $\Omega \subset \mathbb{C}$ the 1-form $\omega = \alpha dz + \beta d\overline{z}$ satisfies:

$$\partial\omega = \partial\alpha \wedge dz + \partial\beta \wedge d\overline{z}$$
$$\overline{\partial}\omega = \overline{\partial}\alpha \wedge dz + \overline{\partial}\beta \wedge d\overline{z}$$

Now we can check that $d = \partial + \overline{\partial}$ and $\partial(\partial f) = \overline{\partial}(\overline{\partial} f) = 0$. Also, $\partial(\overline{\partial} f) = \overline{\partial}(\partial f)$. On a plane domain we find that

$$\partial(\overline{\partial} f) = \frac{\partial^2 f}{\partial z \, \partial \overline{z}} dz \wedge d\overline{z} = \tfrac{1}{4}\Delta f \partial(\overline{\partial} f) = \overline{\partial}(\partial f)$$

where Δf is the Laplacian of f. Hence, we say that a 0-form f is *harmonic* if $\partial(\overline{\partial} f) = 0$.

Proposition 2.4.3

A 0-form $f : R \to \mathbb{C}$ on a Riemann surface R is analytic if, and only if, $\overline{\partial} f = 0$ and is harmonic if, and only if, the 1-form ∂f is analytic.

Proof:

The differential ∂f exists and is a 1-form in $\mathcal{E}^{1,0}(R)$. On a chart, we have

$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}$$

so $\overline{\partial} f = 0$ precisely when $\frac{\partial f}{\partial \overline{z}} = 0$. This is when f is analytic.

Similarly, on each chart

$$\partial f = \frac{\partial f}{\partial z} dz$$

and this is an analytic 1-form when $u = \frac{\partial f}{\partial z}$ is analytic. Now u is analytic when $\overline{\partial} u = 0$. This corresponds to $\overline{\partial}(\partial f) = 0$. □

Exercises

10. Show that it is not true that a 1-form ω is analytic if, and only if, $\overline{\partial}\omega = 0$.
11. Show that a 0-form $f : R \to \mathbb{C}$ is harmonic if, and only if, there are two analytic 1-forms $\alpha, \beta \in \mathcal{O}^1(R)$ with

$$df = \alpha + \overline{\beta} \ .$$

Show that, when f takes only real values, we may choose $\alpha = \beta$

All of the results for harmonic functions on plane domains which we proved in Chapter 1 can now be transferred without difficulty to deal with harmonic functions on Riemann surfaces. Do so!

The fundamental theorem of calculus also extends to 1-forms, where it is traditionally called Stokes' theorem or Green's theorem.

Theorem 2.4.4 Stokes' Theorem

Let $\sigma : \Delta \to R$ be a smooth map from a compact subset Δ of $\mathbb{C}$ with piecewise smooth boundary into the Riemann surface R. Let ω be a 1-form on R. Then

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega \ .$$

Proof:

Let the pull-back of ω by σ be $\sigma^*\omega = a\,dx + b\,dy$. Then

$$d\omega = \left(-\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} \right) dx \wedge dy \ .$$

Therefore,

$$\int_\sigma d\omega = \int_\Delta \left(-\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} \right) dxdy$$

and

$$\int_{\partial\sigma} \omega = \int_{\partial\Delta} a\,dx + b\,dy \ .$$

Green's theorem in the plane shows that these two are equal. □

Let $C_0(R)$ be the free Abelian group generated by the points of R, $C_1(R)$ the free Abelian group generated by piecewise smooth curves $\gamma : I = [a,b] \to R$, and $C_2(R)$ the free Abelian group generated by smooth mappings $\sigma : \Delta \to R$ from compact subsets $\Delta \subset \mathbb{C}$ with piecewise smooth boundary. We can define boundary homomorphisms $\partial : C_{k+1}(R) \to C_k(R)$ by

$$\begin{aligned} \partial : C_1(R) \to C_0(R) \ ; &\quad \gamma \mapsto \gamma(b) - \gamma(a) \ ; \\ \partial : C_2(R) \to C_1(R) \ ; &\quad \sigma \mapsto \partial\sigma \ . \end{aligned}$$

Then the group of *k-cycles* $Z_k(R)$ is the subgroups $\ker(\partial : C_k(R) \to C_{k-1}(R))$ of $C_k(R)$ and the group of *k-boundaries* $B_k(R)$ is the subgroup $\mathrm{Im}(\partial : C_{k+1}(R) \to C_k(R))$. Since $\partial\circ\partial = 0$ we have $B_k(R) \subset Z_k(R)$. The quotient $Z_k(R)/B_k(R)$ is the *kth homology group* $H_k(R)$. Two k-cycles are *homologous* if they differ by a k-boundary, so they represent the same element of $H_k(R)$.

Since every Riemann surface is path-connected, $P - Q$ is a boundary for every pair of points $P, Q \in R$. Thus $H_0(R)$ is isomorphic to the additive group $\mathbb{Z}$ and is generated by any point of R. Every Riemann surface is orientable, since the transition maps are orientation preserving. Hence, if R is compact and triangulable, we can find a 2-cycle by mapping to each face of the triangulation. This generates $H_2(R)$ which is isomorphic to $\mathbb{Z}$. If R is not compact, every finite union of images $\sigma(\Delta)$ must be compact and so cannot be a cycle. Thus $H_2(R) = Z_2(R) = \{0\}$.

Theorem 2.4.5 Cauchy's theorem

Let $\omega \in \mathcal{O}^1(R)$ be an analytic 1-form on the Riemann surface R. Then,

$$\int_\gamma \omega = 0$$

for each 1-cycle γ which is homologous to 0 (that is , γ is a 1-boundary).

Let $u \in \mathcal{E}(R)$ be harmonic. Then

$$\int_\gamma \partial u = \int_\gamma \overline{\partial} u = 0$$

for each 1-cycle γ homologous to 0.

Proof:

On a chart we have $\omega = a\,dz$ for some analytic function a. Hence, $d\omega = da \wedge dz = \frac{\partial a}{\partial z} dz \wedge dz = 0$. Thus $d\omega = 0$ on all of R. Hence Stokes' theorem gives

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega = 0 \ .$$

For any 0-form u we have

$$\int_\gamma du = \int_\gamma \partial u + \overline{\partial} u = 0$$

by the fundamental theorem of calculus. When u is harmonic, $\partial u \in \mathcal{O}^1(R)$ by Proposition 2.4.3, so

$$\int_\gamma \partial u = 0 \ .$$

□

A k-form ω is *closed* if $d\omega = 0$. Clearly every exact form is closed since $d(df) = 0$. However, there are closed forms which are not exact, for example $(1/z)\,dz$ on $\mathbb{C} \setminus \{0\}$. The quotient

$$H^k_{dR}(R) = \frac{\{\text{closed } k\text{-forms on } R\}}{\{\text{exact } k\text{-forms on } R\}}$$

is called the kth *de Rham cohomology group of* R. Any closed k-form ω has $\int_\gamma \omega = 0$ for any k-boundary γ, because of Stokes' theorem. So the map

$$H_k(R) \to \mathbb{C} \ ; \quad [\gamma] \mapsto \int_\gamma \omega$$

is a well defined group homomorphism. If ω is exact, then $\int_\gamma \omega$ is 0 for all k-cycles γ. Thus we get a $\mathbb{Z}$-bilinear pairing

$$H^k_{dR}(R) \times H_k(R) \to \mathbb{C} \ ; \quad ([\omega], [\gamma]) \mapsto \int_\gamma \omega \ .$$

Finally, let us consider forms on the quotient of the unit disc $\mathbb{D}$ by a discrete subgroup Γ of $\mathrm{Aut}(\mathbb{D})$. Let $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$ be the quotient map. If f is a 0-form on $\mathbb{D}/\Gamma$, then $F = f \circ \pi$ is 0-form on $\mathbb{D}$ which satisfies

$$F(T(z)) = F(z) \qquad \text{for all } T \in \Gamma \text{ and } z \in \mathbb{D} \ .$$

Such a function $F : \mathbb{D} \to \mathbb{C}$ is called an *automorphic* 0*-form* for Γ. Similarly, a 1-form ω on $\mathbb{D}/\Gamma$ corresponds to a 1-form $\Omega = \pi^*\omega$ on $\mathbb{D}$ which satisfies

$$T^*\Omega = \Omega \qquad \text{for all } T \in \Gamma \ .$$

Such forms are called *automorphic* 1*-forms* for Γ. The 1-form Ω on $\mathbb{D}$ can be written as $\Omega = a\,dz + b\,d\overline{z}$ for some functions $a, b : \mathbb{D} \to \mathbb{C}$. Then

$$T^*\Omega(z) = a(T(z))T'(z)\,dz + b(T(z))\overline{T'(z)}\,d\overline{z} \ .$$

Thus $a\,dz + b\,d\overline{z}$ is an automorphic 1-form if, and only if,

$$a(T(z))T'(z) = a(z) \qquad \text{and} \qquad b(T(z))\overline{T'(z)} = b(z)$$

for $T \in \Gamma$ and $z \in \mathbb{D}$. More explicitly, if $T : z \mapsto (az+b)/(\overline{b}z+\overline{a})$, with $|a|^2 - |b|^2 = 1$, then

$$a\left(\frac{az+b}{\overline{b}z+\overline{a}}\right)\frac{1}{(\overline{b}z+\overline{a})^2} = a(z) \ .$$

In a similar way, a 2-form θ on $\mathbb{D}/\Gamma$ corresponds to a 2-form $\Theta = \pi^*\theta = c\,dz \wedge d\overline{z}$ which is automorphic in that

$$c(Tz)|T'(z)|^2 = c(z) \qquad \text{for all } T \in \Gamma \text{ and } z \in \mathbb{D} \ .$$

We can try to construct forms on $\mathbb{D}/\Gamma$ by producing automorphic forms as sums of series. If $h : \mathbb{D} \to \mathbb{C}$ is smooth, then

$$\tilde{h}(z) = \sum_{T \in \Gamma} h(T(z))$$

will be an automorphic 0-form provided that it converges suitably. Similarly,

$$\tilde{h}(z) = \sum_{T \in \Gamma} h(T(z))T'(z),$$

when it converges, gives an automorphic 1-form $\tilde{h}\,dz$. These series are called *Poincaré series*. Typically we take h to be a meromorphic or harmonic function which is 0 on the boundary $\partial\mathbb{D}$.