GEOMETRY — Example Sheet 1

TKC Lent 2009

-1. Let R be a rotation of the Euclidean plane $\mathbb{E}^2$ through an angle θ about a point $\boldsymbol{C}$. Describe the conjugate TRT^{-1} of R by any isometry T. (T may be orientation preserving or orientation reversing.)

2. A *hyperplane* in $\mathbb{E}^N$ is a set of the form $\Pi = \{\boldsymbol{x} \in \mathbb{E}^N : \boldsymbol{x} \cdot \boldsymbol{u} = k\}$ for $\boldsymbol{u}$ a unit vector in $\mathbb{R}^N$ and $k \in \mathbb{R}$. Show that the map $R : \boldsymbol{x} \mapsto \boldsymbol{x} - 2(\boldsymbol{x} \cdot \boldsymbol{u} - k)\boldsymbol{u}$ is an isometry of $\mathbb{E}^N$. It is called *reflection in the hyperplane* Π.

   Let $\boldsymbol{P}$, $\boldsymbol{Q}$ be two distinct points in the Euclidean N-space $\mathbb{E}^N$. Show that there is a reflection in some hyperplane that maps $\boldsymbol{P}$ to $\boldsymbol{Q}$.

   By applying this result when $\boldsymbol{P}$ is either $\boldsymbol{0}$ or one of the standard basis vectors of $\mathbb{R}^N$, deduce that any isometry T of $\mathbb{E}^N$ can be written as the composition of at most $N+1$ reflections.

   For each isometry T of the Euclidean plane, find the minimum number of reflections we need to compose to produce T. Do the same for isometries of Euclidean 3-space.

3. Show that any isometry of $\mathbb{E}^3$ that fixes the origin is one of:

(a) The identity;

(b) A rotation about an axis through $\boldsymbol{0}$;

(c) A reflection in a plane through $\boldsymbol{0}$;

(d) A rotatory reflection, that is reflection in a plane through $\boldsymbol{0}$ followed by rotation about the axis through $\boldsymbol{0}$ perpendicular to this plane.

   For a cube centred on the origin, show that there are 48 symmetries and identify which of the classes (a), (b), (c) and (d) they belong to.

4. Let $\boldsymbol{A}, \boldsymbol{B}$ be two distinct points in the Euclidean plane. Show that the points equidistant from $\boldsymbol{A}$ and $\boldsymbol{B}$ form a straight line (the perpendicular bisector of AB).

   Let $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ be three vertices of a triangle in the plane $\mathbb{E}^2$ that does not lie on a line. Show that perpendicular bisectors of the three sides meet at a point. Deduce that there is a unique circle that passes through the three vertices. This is the circumcircle with radius R.

   Prove that the angles α, β, γ and the side lengths of the triangle satisfy

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R\ .$$

5. Let $\boldsymbol{C}$ be a point on the unit sphere and r a fixed number with $0 < r < \pi$. The spherical circle with centre $\boldsymbol{C}$ and radius r is the set of all points on the sphere at distance r from $\boldsymbol{C}$. Show that this is a circle in $\mathbb{R}^3$ and find its length. Find a formula for the area of the spherical cap

$$\{\boldsymbol{x} \in S^2 : d(\boldsymbol{C}, \boldsymbol{x}) < r\}$$

   enclosed by the circle.

6. Given a point $\boldsymbol{P}$ on the sphere S^2 and a spherical line ℓ, show that there is a spherical line ℓ' through $\boldsymbol{P}$ that crosses ℓ at right-angles. Prove that the minimum distance from $\boldsymbol{P}$ to a point $\boldsymbol{Q}$ on ℓ is attained at one of the two points where ℓ intersects ℓ'. Show that ℓ' is unique unless this minimum distance is $\frac{1}{2}\pi$.

-7. Let Δ be a spherical triangle with sides of length a, b, c and angles α, β, γ. Extend the sides of Δ to form complete great circles. Show that this divides the sphere into 8 triangles and find the side lengths and angles for each.

8. In the spherical triangle Δ show that $b = c$ if and only if $\beta = \gamma$. Show that this occurs if and only if there is a reflection M with $M(\boldsymbol{A}) = \boldsymbol{A}$, $M(\boldsymbol{B}) = \boldsymbol{C}$ and $M(\boldsymbol{C}) = \boldsymbol{B}$.

   Are there equilateral spherical triangles? Are they all isometric to one another?

9. Two spherical triangles Δ_1, Δ_2 are *congruent* if there is an isometry of S^2 which maps one onto the other. Show that two triangles that have all three angles the same are congruent. Are two triangles that have one angle the same and two side lengths the same necessarily congruent?

10. For the spherical triangle Δ, show that

$$a \leqslant b + c\ ,\quad b \leqslant c + a\ ,\quad c \leqslant a + b \quad \text{and} \quad a + b + c \leqslant 2\pi\ .$$

Show, conversely, that any three positive numbers a, b, c that satisfies these inequalities also satisfy $\cos(b+c) \leqslant \cos a \leqslant \cos(b-c)$ and so do arise as the side lengths of a spherical triangle (unique up to isometry).

11. The points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ lie on the unit sphere S^2 and are positively oriented, so the scalar triple product $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) > 0$. These three points are the vertices of a spherical triangle Δ that has sides of length a, b, c and angles α, β, γ. Show that the three vectors

$$\boldsymbol{A}^* = \frac{\boldsymbol{B} \times \boldsymbol{C}}{\sin a} \ ; \quad \boldsymbol{B}^* = \frac{\boldsymbol{C} \times \boldsymbol{A}}{\sin b} \ ; \quad \boldsymbol{C}^* = \frac{\boldsymbol{A} \times \boldsymbol{B}}{\sin c} \ ;$$

are also unit vectors and are the vertices of another spherical triangle Δ^*. This is the *dual* triangle of Δ. Show that $\boldsymbol{A}^*$ is the unique unit vector orthogonal to $\boldsymbol{B}$ and $\boldsymbol{C}$ that has $\boldsymbol{A}^* \cdot \boldsymbol{A} > 0$. Deduce that the dual of the triangle Δ^* is the original triangle Δ itself.

Prove that the dual triangle Δ^* has sides of length $\pi - \alpha, \pi - \beta, \pi - \gamma$. Deduce that it has angles $\pi - a, \pi - b, \pi - c$.

Show that $\boldsymbol{A}^* \cdot \boldsymbol{A}$ is equal to both $(\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}))/\sin a$ and $(\boldsymbol{A}^* \cdot (\boldsymbol{B}^* \times \boldsymbol{C}^*))/\sin(\pi - \alpha)$. Hence prove the sine rule for spherical triangles.

By using the formula

$$(\boldsymbol{C} \times \boldsymbol{A}) \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{C} \cdot \boldsymbol{A})(\boldsymbol{A} \cdot \boldsymbol{B}) - (\boldsymbol{C} \cdot \boldsymbol{B})(\boldsymbol{A} \cdot \boldsymbol{A})$$

from the Vector Calculus course prove the cosine rule:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha \ .$$

By applying this rule to the dual triangle show that we also have a second cosine rule:

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a \ .$$

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12. Let $\mathcal{T}$ be an Euclidean triangle with angles π/p, π/q and π/r for integers $p, q, r \geqslant 2$. Find all the possible values for p, q and r. In each of these cases, show that reflections in the sides of the triangle generates a tessellation of the Euclidean plane.

13. Show that, in a regular pentagon drawn in the plane $\mathbb{E}^2$ with sides of unit length, each diagonal has length equal to the Golden Ratio.

    [*The Golden Ratio* $\tau > 1$ *is defined by:* $\tau/1 = 1/(\tau - 1)$. *Hence* $\tau = \frac{1}{2}(1 + \sqrt{5})$.

    Hint: Let the vertices of the regular pentagon be A, B, C, D, E *in order. The diagonals* AC *and* BE *meet at a point* P. *Show that the triangles* APB *and* BAE *are similar and that* $d(P, E) = d(A, E)$.]

14. Let τ be the Golden ratio. Show that the 12 points

$$(\pm 1, \pm \tau, 0) \ , \quad (0, \pm 1, \pm \tau) \ , \quad (\pm \tau, 0, \pm 1) \ ,$$

obtained by taking all possible choices of sign, form the vertices of a regular icosahedron with edges of length 2.

Construct a regular icosahedron by joining together three postcards with sides in the Golden ratio and taking their corners as vertices.

Show that the points $(0, \pm\tau, 0)$, $(0, 0, \pm\tau)$, $(\pm\tau, 0, 0)$ form the vertices of a octahedron embedded in the icosahedron with each vertex at the midpoint of an edge of the icosahedron. How many such embedded octahedra are there?

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Supervisors can obtain an annotated version of this example sheet from DPMMS.