Two questions about extracted models

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in which i presented Jensen's proof. I got two quite good questions:

> (1) Is every model of T$\mathbb{Z}$TU extracted from a model of TZT?
>
> (2) Why not define the membership of the extracted model by $x \in' y$ iff $y = \iota^n(z) \wedge x \in z$?

I now think i have two helpful answers.

Question 1

The answer to (1) is: no. One thing that is clear is that, if a model $\mathfrak{M}$ extracted from a model of T$\mathbb{Z}$T is actually a model of T$\mathbb{Z}$TU, then each each level $l+1$ of the extracted model is of size $\beth_n$ of the size of level l, for some concrete n. For each n this is a first-order condition expressible in $\mathcal{L}$(T$\mathbb{Z}$T).

Definition 1
• *Let $\phi_{l,m}$ be the formula of $\mathcal{L}$(T$\mathbb{Z}$T) that says that the cardinality of level $l+1$ is $\beth_n$ of the cardinality of level l.*
• *For each $l \in \mathbb{Z}$, let Σ_l be the type $\{\neg\phi_{l,m} : m \in \mathbb{N}\}$.*
• *Let us give the name 'T$\mathbb{Z}$T(Omit)' to the smallest theory that locally omits all the Σ_l.*

T$\mathbb{Z}$T(Omit) is axiomatisable but (presumably) not recursively axiomatisable. (It's obviously not vacuous!) Let us minute the following fact.

Remark 1
A model of T$\mathbb{Z}$TU is a model extracted from a model of T$\mathbb{Z}$T iff, for each $l \in \mathbb{Z}$, it omits Σ_l.

Proof:

One direction is obvious: clearly an extracted model omits all the Σ_l.

For the other direction, consider a model $\mathfrak{M} \models$ T$\mathbb{Z}$TU that omits all the Σ_i. Such a model knows (for each l) that the size of its level $l+1$ is $\beth_n$ of the size of its level l, for some concrete n (depending on l). But then it knows about sets of sizes of all these intermediate $\beth$ numbers, and can use these sets to fake up a model in which no levels between l and $l+1$ have been discarded. Evidently all models obtained from $\mathfrak{M}$ in this way are isomorphic.

Let us call this process **backfilling**.

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So the extracted model knows whence it came. One might have thought that all information in the discarded levels is irretrievably lost, since the atoms cannot be distinguished, but—as we have seen—the information is retained. This may be something to do with the fact that all the information is stratified[1]. This is probably worth flagging.

Remark 2 *Any model of T$\mathbb{Z}$T can be recovered from any model extracted from it.*

Clearly by compactness there are models of T$\mathbb{Z}$T that realize plenty of these types, so the answer to question (2) is 'no'.

I suspect that this fact—that there is a first-order way of detecting whether or not a model of T$\mathbb{Z}$TU is an extracted model—is a new observation, elementary tho' it be. Let's see if it can be put to any use.

Let's retrace Jensen's original proof (or—perhaps i should say—my recollection of it!). We start with a model $\mathfrak{M}$ of T$\mathbb{Z}$T, and successively extract models $\mathfrak{M}_i : i \in \mathbb{N}$ from it, with the $\mathfrak{M}_i$ satisfying ambiguity for ever more expressions as i increases. All the $\mathfrak{M}_i$ are models of T$\mathbb{Z}$T(Omit). We then take an ultraproduct, $\mathfrak{M}_\infty$, which will be ambiguous. (That was the point, indeed). However it will also be a model of T$\mathbb{Z}$T(Omit). Now, although it is a model of T$\mathbb{Z}$T(Omit) it obviously realizes all the Σ_l, and therefore cannot be an extracted model. Observe, too, that altho' Th($\mathfrak{M}_\infty$) extends T$\mathbb{Z}$T(Omit), it does not itself locally omit the Σ_l. If it did, it would have a model that omitted the Σ_l, and that would be an extracted model, and the model from which it was extracted would be an ambiguous model of T$\mathbb{Z}$T. Finite extensions of theories that locally omit a type will locally omit that type, but infinite extensions might not, and the example to hand is a useful illustration.

If $\mathfrak{M}^*$ is a model of T$\mathbb{Z}$TU extracted from $\mathfrak{M}$ a model of T$\mathbb{Z}$T, and $\mathfrak{M}^*$ is ambiguous then, for some n, it satisfies $\phi_{l,n}$ for every l. This means that when we backfill to recover $\mathfrak{M}$ we find that it is a model of Amb^n. And clearly if $\mathfrak{M} \models$ Amb^n then we can extract an ambiguous model of T$\mathbb{Z}$TU + the scheme $\phi_{l,n}$ for all l.

[1] There are echoes here of an old question about how discernible the atoms in a model of NFU can be.

If we have a model of TℤTU + ambiguity obtained by extracting, then, for some n, we retained every nth level. By backfilling we recover a model of TℤT, and that model satisfies Amb^n. So you get a model of $\mathrm{TC}_n\mathrm{T}$[2]! See [1].

So

Remark 3 *The following are equiconsistent, for any concrete n:*

- $TC_n T$;
- $NFU + |V| = \beth_n|sets|$;
- $T\mathbb{Z}T$+ *the scheme* $\{\phi_{l,n} : l \in \mathbb{Z}\}$.

Question 2

The answer to (2) is that, no, it doesn't make any difference.

If we are to extract level X and the level $\mathcal{P}^n(X)$ above it ($n > 1$ obviously) then we discard[3] the intermediate levels. We fix an injection $i : \mathcal{P}(X) \hookrightarrow \mathcal{P}^n(X)$, and then say that:

> x (a member of X) is a "member of" Y (a member of $\mathcal{P}^n(X)$) iff $x \in i^{-1}(Y)$.

Notice that things not in the range of i are empty, just as they should be.

A fundamental requirement is that this new membership relation should support axioms of comprehension (it clearly supports extensionality for nonempty sets) in the extracted model, and for this it is necessary that the expression "x is a member of y in the new sense" should be a formula of $\mathcal{L}$(TℤT). We now show that any two injections which are definable in this sense give rise to the same model (up to isomorphism).

Remark 4 *The model obtained by extracting some chosen levels depends only on the levels chosen and not on the manner in which the extraction is performed.*

Proof:

Key fact: all injections satisfying the above condition are *conjugate*. Roughly this is beco's $\mathcal{P}^n(X)$ is *so* much bigger than $\mathcal{P}(X)$ that if i and j are two injections from $\mathcal{P}(X)$ into $\mathcal{P}^n(X)$ then the two complements (in $\mathcal{P}^n(X)$) of their ranges are the same size. Then reflect that, in general, if $X, Y \subseteq V$ satisfy $|X| = |Y|$ and $|V \setminus X| = |V \setminus Y|$, then there is a permutation of V mapping X onto Y ... so there is a permutation π of $\mathcal{P}^n(X)$ such that $i = \pi \cdot j$. This relies on the model knowing that $i``\mathcal{P}(X)$ and $j``\mathcal{P}(X)$ are the same size. And the model will know this, because i and j are both definable in the original model.

That's the idea. Mind you, a bit of detail will not go amiss.

Suppose we are working in TℤT. If we discard a single level between two levels that we are extracting we need to know the following. Let α be the size

[2] I am not sure where this fact is written up!

[3] For obvious reasons we don't want to use the word 'omit'!

of the level we are discarding; then we want that whenever $\alpha + \beta = 2^\alpha$, then $\beta = 2^\alpha$. This is an immediate consequence of Bernstein's lemma, since (and this is where we exploit the fact that we are in T$\mathbb{Z}$T) we have $2^\alpha \cdot 2^\alpha = 2^\alpha$. So, whatever we take i to be, the complement of its range is of size 2^α. Thus, whatever i amd j are, there is a bijection between the complements of their ranges.

Discarding more than one intermediate level is essentially the same; if anything, it's even easier.

However, for the sake of completeness, let us consider TST as well. If we are working in TST then either (i) the bottom level is inductively finite (internally) in which case everything is easy, or (ii) the bottom level is not inductively finite, in which case—after a small finite amount of grinding of gears—everything works as in the T$\mathbb{Z}$T case. But since we are interested in infinitely many levels, a finite amount of gear grinding costs us nothing: we can always discard an initial segment of badly behaved levels.

Notice that all the reasoning in either case (TST or T$\mathbb{Z}$T) can be carried out inside the model and makes no use of AC.

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I think this makes for a nicer way of presenting Jensen's extracted models than the usual method: we don't need to know what the injection is; all we need to know is that there are injections and that it matters not which one we use. I think one should just say: if we want to discard levels $n \cdots m - 1$, then one calls to mind any internally definable injection from level n to level m, and feeds it into the above construction.

It may be worth making the point that this presentation does a better job of making it clear that a composition of two extractions is an extraction. Compose the two injections with ι in the middle: $i \cdot j \iota \cdot j$.

References

[1] Thomas Forster "Stratification-mod-n and the Cylindrical Theory of Types"`www.dpmms.cam.ac.uk/~tf/stratificationmodn.pdf`