The Union of a Strongly Cantorian set of Strongly Cantorian Sets is Strongly Cantorian

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This is not the *five minute argument* version for Randall; it's rather the *full half hour* version for Alice and Adam.

Let's abbreviate this proposition to 'SCU'. That was Alice Vidrine's suggestion. It's pronounced *scum* or *screw* or perhaps *skew*.

SCU crops up naturally in the analysis of NF from a category theoretic perspective. Consider the (conjectural) category of small sets and small maps, where 'small' means 'strongly cantorian', and a small map is a map whose every fibre is strongly cantorian. For this gadget to be a category we need a composition of small maps to be small and this is equivalent to SCU.

The status of SCU is unclear at the time of writing. As Holmes points out, it's a theorem of NFU+AC. Let X be a strongly cantorian set of strongly cantorian sets. AC implies that every strongly cantorian set is the same size as an initial segment of the ordinals (and all the ordinals in that inital segment will be cantorian). Use AC to pick one such bijection for each $x \in X$ and fix such a bijection for X itself. Thus everything in $\bigcup X$ has an address that is an ordered pair of cantorian ordinals, so $\bigcup X$ now injects into a set of ordered pairs of cantorian ordinals. Any such set is strongly cantorian, so $\bigcup X$ must be strongly cantorian too. SCU doesn't appear to be a theorem of NF, but nor does it appear to be strong: i tried to deduce the Axiom of Counting from it but without success. It's clearly unstratified, so one might hope that it could be proved consistent relative to NF by means of Rieger-Bernays permutation models, but i have found what i hope is a correct proof that SCU is invariant.

REMARK 1 *SCU is invariant*

Proof:
SCU$^\sigma$ is

$$(\forall x)(stcan(x) \wedge (\forall y)(y \in x \to stcan(y)). \to (\forall z)(z = \bigcup x \to stcan(z)))^\sigma$$

Now $(stcan(x))^\sigma$ is $stcan(\sigma(x))$ or equivalently $stcan(\sigma``\sigma(x))$, and $(z = \bigcup x)^\sigma$ is $\sigma(z) = \bigcup \sigma``\sigma(x)$ giving

$$(\forall x)(stcan(\sigma(x)) \wedge (\forall y)(y \in \sigma(x) \to stcan(\sigma(y))). \to (\forall z)(\sigma(z) = \bigcup \sigma``\sigma(x) \to stcan(\sigma(z))))$$

reletter 'z'

$$(\forall x)(stcan(\sigma(x)) \wedge (\forall y)(y \in \sigma(x) \to stcan(\sigma(y))). \to (\forall z)(z = \bigcup \sigma``\sigma(x) \to stcan(z)))$$

and simplify

$$(\forall x)(stcan(\sigma(x)) \wedge (\forall y)(y \in \sigma(x) \to stcan(\sigma(y))). \to stcan(\bigcup \sigma``\sigma(x))))$$

reletter 'x'

$$(\forall x)(stcan(x) \wedge (\forall y)(y \in x \to stcan(\sigma(y))). \to stcan(\bigcup \sigma``x)))$$

Now this last is equivalent to

$$(\forall x)(stcan(x) \wedge (\forall y)(y \in x \to stcan(y)). \to stcan(\bigcup x)))$$

... beco's we can just substitute '$\sigma_2^{-1}(x)$' for 'x'.

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Early in the morning of 6/xi/2014 i had a flash of insight with which i will now regale you.

We start with a banal observation. Let F_1 be the function that sends each strongly cantorian set x to $\iota\restriction x$. F_1 cannot be a set: if it were then $F_1``(\iota``V) = \{\iota\restriction\{x\} : x \in V\}$ would be a set (beco's the image of a set in a set is a set) and $\bigcup F_1``(\iota``V)$ would be the graph of the singleton function, and that cannot be a set. However this line of talk leaves open the possibility that $F_1\restriction x$ might be a set whenever x is strongly cantorian. In fact we have the following.

Remark 2
SCU is equivalent to the assertion that, for all strongly cantorian sets x of strongly cantorian sets, $F_1\restriction x$ is a set.

Proof:

L → R

Assume SCU and let X be a strongly cantorian set of strongly cantorian sets. Then $\iota\restriction\bigcup X$ is a set. Let's call it F. Consider now the function that sends each $x \in X$ to $F\restriction x$. This is a set, since it is the extension of a stratified set abstract. But X was an arbitrary strongly cantorian set of strongly cantorian sets. So SCU implies that F_1 is locally a set, in the sense that, for any strongly cantorian set X [the graph of] its restriction to X is a set.

R → L

Let X be a strongly cantorian set of strongly cantorian sets. Then $F_1 \restriction X = \lambda x \in X.\iota \restriction x$ is a set and so too is the image of X in it, namely $\{\iota \restriction x : x \in X\}$. But then $\bigcup\{\iota \restriction x : x \in X\}$ is a set, and is $\iota \restriction \bigcup X$ making $\bigcup X$ strongly cantorian as desired.

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Very well. So SCU tells us that, for every strongly cantorian set X of strongly cantorian sets, [the graph of] $F_1 \restriction x$ is a set. Consider now the function $F_2 : X \mapsto F_1 \restriction X$ for every strongly cantorian set X of strongly cantorian sets. Can the graph of F_2 be a set? Clearly not: ι^2"V is a strongly cantorian set of strongly cantorian sets, and its image in this function would be the set $\{\iota \restriction \{x\} : \{x\} \in \iota``V\}$, which is $\{\iota \restriction \{x\} : x \in V\}$, whose sumset is simply the graph of ι. However, there seems to be no obvious objection to the existence of [the graph of] the restriction of F_2 to any *strongly cantorian* set.

A pattern is beginning to emerge! Let us write 'stcan' for the class of strongly cantorian sets, 'stcan^2' for the class of strongly cantorian sets of strongly cantorian sets, and so on.

Let F_0 be the function ι;
F_0 cannot exist globally but $F_0 \restriction x$ exists for any x in stcan.

Let F_1 be the function $\lambda x \in$ stcan.$F_0 \restriction x$;
F_1 cannot exist globally but $F_1 \restriction X$ can exist for any set in stcan^2.

Let F_2 be the function $\lambda x \in$ stcan^2.$F_1 \restriction x$;
F_2 cannot exist globally but $F_2 \restriction X$ can exist for any set in stcan^3.

$\vdots$

Let F_{n+1} be the function $\lambda x \in$ stcan^n.$F_n \restriction x$;
F_n cannot exist globally but $F_n \restriction X$ can exist for any X in stcan^{n+1}.

and so on.

Let SCU$_n$ be the assertion that restrictions of F_n exist locally, so that $F_n \restriction X$ is a set whenever $X \in$ stcan^n. SCU$_1$ is of course SCU.

We record for later use the trivial observation that SCU implies that if $x \in$ stcan^{n+1} then $\bigcup x \in$ stcan^n.

The following remark answers the question that was on the Reader's lips.

make sure these superscripts are correct

Remark 3 *All SCU$_n$ for $n \in \mathbb{N}$ are equivalent.*

Proof:

SCU$_{n+1}$ implies SCU$_n$.

Suppose $x \in$ stcan^n; we will show that $F_n \restriction x$ exists. Since $x \in$ stcan^n we have $\iota``x \in$ stcan$^{n+1}$. So, by SCU$_{n+1}$, $F_{n+1} \restriction \iota``x$ exists. This is the function that, on being given $\{y\} \in \iota``x$, returns $F_n \restriction \{y\}$. This value is the singleton $\{\langle y, F_n(y)\rangle\}$. So $F_{n+1}``(\iota``x)$ (which is a set) is $\{\{\langle y, F_n(y)\rangle\} : y \in x\}$, and the sumset of this last object is precisely $F_n \restriction x$, as desired.

For the other direction we assume SCU_n, and suppose x to be an arbitrary member of stcan^{n+1}; we will show that $F_{n+1} \restriction x$ is a set.

Clearly $\mathrm{stcan}^{n+1} \subseteq \mathrm{stcan}^n$ so $x \in \mathrm{stcan}^n$, whence—by SCU—$\bigcup x \in \mathrm{stcan}^n$. SCU_n now tells us that $F_n \restriction \bigcup x$ is a set. Let's call this function H for the moment. But then the function that takes subsets S of $\bigcup x$ and returns the restriction $H \restriction S$ is also a set. H is defined on $\mathcal{P}(\bigcup x)$ which is a superset of x. So the restriction of this function to x is a set.

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SCU implies that a strongly cantorian product of strongly cantorian sets is strongly cantorian

Theorem 1
If SCU then, for all I, if stcan(I) and $(\forall i \in I)$(stcan(A_i)) then stcan$(\prod_{i \in I} A_i)$

Proof:
This is easy. The product is a subset of $\mathcal{P}((\bigcup_{i \in I} A_i \times I))$. Assuming SCU the union $\bigcup_{i \in I} A_i$ is strongly cantorian because I is and all the A_i are. The cartesian product of two strongly cantorian sets is strongly cantorian, a power set of a strongly cantorian set is strongly cantorian, and every subset of a strongly cantoarian set is strongly cantorian. ■

We can now prove

Theorem 2 *(SCU)*
Let $\langle I, \leq_I \rangle$ be a directed poset with I strongly cantorian, and let $\{A_i : i \in I\}$ be a family of sets with surjections $\pi_{i,j} : A_i \twoheadrightarrow A_j$ whenever $i >_I j$, and the surjections all commute. Suppose further that, for every i and j, the fibres of $\pi_{i,j}$ are strongly cantorian. Naturally there is a limit object A_I, a least thing that maps onto all the A_i—with maps $\pi_{I,i} : A_I \twoheadrightarrow A_i$ for each $i \in I$.
Then all the fibres of $f_{I,i}$ are strongly cantorian.

Proof:
The inverse (projective) limit A_I is

$$\{f \in \prod_{i \in I} : (\forall j >_I i)(\pi_{j,i}(f(j)) = f(i)\}$$

For $x \in A_i$, the fibre $\pi_{I,i}^{-1}$“$\{x\}$ is

$$\{f \in \prod_{j > i \in I} : (\forall j >_I i)(\pi_{j,i}(f(j)) = x)\}$$

So a fibre for $x \in A_i$ is set of functions f that, for each $j > i \in I$, pick something that $\pi_{j,i}$ sends to x. So it's a subset of the product of all the subsets $\pi_{j,i}^{-1}$“$\{x\}$ of A_j ... and, by assumption, all those sets are strongly cantorian. So the fibre is a subset of a direct product of a strongly cantorian family of strongly cantorian sets, and accordingly by theorem 1 is strongly cantorian.

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In plain language, SCU implies that the inverse (projective) limit of a strongly cantorian family of strongly cantorian structures is strongly cantorian.