Stratification modulo n

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I am very grateful to Nathan Bowler for supplying a crucial little *aperçu* that enables one to dispense with the axiom of choice for pairs in the proof of one of the principal results (corollary 3). And also for helpful critical comments (in particular spotting an embarrassingly large *lacuna* in the proof of lemma 8) that have sharpened up the presentation greatly. By rights he should be a co-author, but he has asserted his moral right to *not* be identified as the (or *an*) author of this work: his reward will be in heavan ... and all mistakes are mine.

1 Introduction and Summary

Recently Zuhair Abdul Ghafoor Al-Johar [17] has directed our attention to a syntactic constraint that is—on the face of it—*tighter* than NF's device of stratification[1]; in this little essay I consider a *weakening*, namely the generalisation of stratification to *stratification modulo n*. So far the coterie of NFistes has considered neither the possibility that the class of unstratified formulæ in the language of set theory might admit any structure or gradation, nor the possibility that failure-of-stratification (which perhaps we can call *dysstratification*) might come in degrees, let alone the possibility that recognition of such degrees might allow one to gain understanding and prove useful facts.

So stratification-mod-n opens a new vein, and the purpose of this note is to advertise some nuggets and prepare the ground for future. It has to be admitted that stratification-mod-n comes across as a highly artificial notion, of interest only to those whose critical faculties have been awakened by prior exposure to the idea of stratification. However, as we shall see below, there are familiar set-theoretic notions that are stratifiable-mod-n so the concept is not vacuous in practice. Further, there is a nontrivial result that makes essential use of this notion, and we will see it in section 8 where we show (theorem 6) that—for NF—duality for formulæ that are stratifiable-mod-2 is consistent relative to NF. Altho' we do not believe that this result is best possible it is nevertheless worth mentioning beco's it is a significant improvement on what has so far been known about duality. We still believe that duality for *all* formulæ is consistent relative to NF. If we achieve that, stratification-mod-n can perhaps go back to the shades whence it came. But perhaps by then it will have thrown useful light on other ideas: we shall see.

2 Stratification

Even readers who are familiar with the idea of stratification should probably read this section, since the treatment here is slightly more abstract than the usual one, and is tailored to the developments that follow.

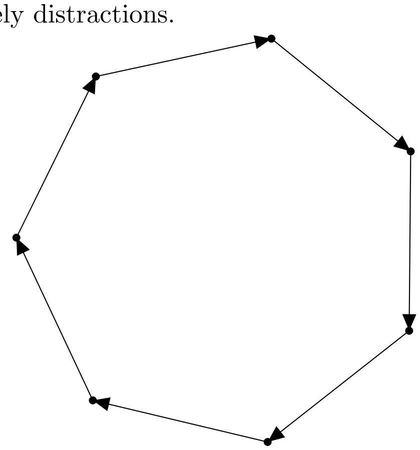
[1]Tho' recent work of Nathan Bowler seems to establish (modulo some very minor set-theoretic assumptions) that every stratifiable formula is equivalent to an acyclic formula. I do not yet understand his proof, and he hasn't published it. However I see no reason to doubt it.

Let $\mathcal{L} = \mathcal{L}(\in, =)$ be the language of set theory. We associate to every formula $\phi \in \mathcal{L}$ a digraph as follows. First we identify two variables 'v' and 'v'' if ϕ contains either of the atomic subformulæ '$v = v'$' or '$v' = v$', and so on, recursively. The vertices of the digraph are the equivalence classes of variables in ϕ, and we place a directed edge from one vertex v to another vertex v' if the atomic formula '$v \in v'$' is a subformula of ϕ.

We call this graph the *derived graph* of ϕ, and write it G_ϕ.

Our digraphs are allowed to have loops at vertices, and may have multiple edges in the restricted sense that there could be a directed edge from v to v' as well as a directed edge from v' to v—but only one in each direction. In a digraph we can have a special notion of a path from v_1 to v_2 which allows us to "go the wrong way". The **length** of such a path is computed by adding 1 every time you follow an arrow the right way, and subtracting 1 every time you go the wrong way. To keep things tidy we will regard loops-at-vertices as paths of length 0.

For $n \leq \aleph_0$ the n-**gon** G_n is the unique connected digraph with precisely n vertices where every vertex has indegree 1 and outdegree 1. It is a reduct of the integers mod n, in that it has successor-mod-n but does not have addition or multiplication. Despite this document bearing the title "stratification mod n" arithmetic mod n plays essentially no rôle in what follows: if we are to sensibly describe the circular stratification that is of interest to us here then it is the n-gon G_n that we need—rather than $\mathbb{Z}/n\mathbb{Z}$—because the additive and multiplicative structures of $\mathbb{Z}/n\mathbb{Z}$ do nothing for us when computing stratifications; they are merely distractions.

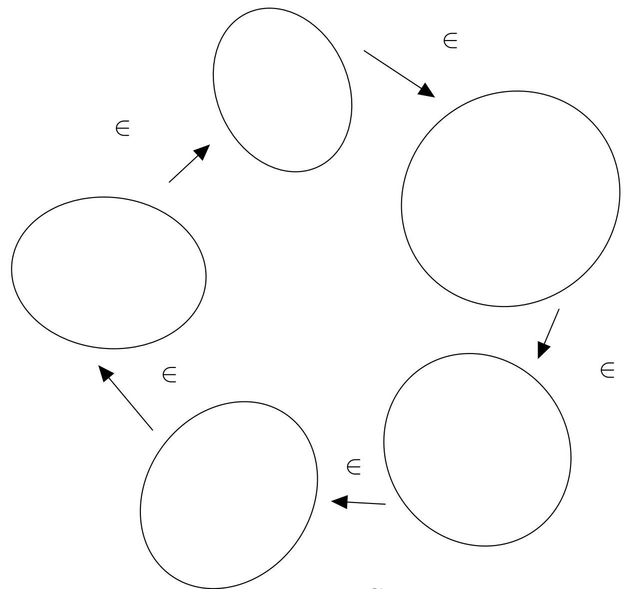


G_7

Unlike the integers-mod-n the n-gon G_n is not rigid: its automorphism group is the cyclic group[2] C_n. This matters because the set of stratifications-mod-n

[2]Dana Scott points out that thinking of G_n as a polygon isn't *entirely* correct either, since polygons have reflections and reflections have no meaning in this context.

of a formula ϕ are "closed under rotation" so that if there is one there are n.



TC_5T

There is a slight problem when $n = 2$, since digraphs cannot normally have multiple edges, but we will tough this one out. And we still entertain hopes that the $\aleph_0$-gon will turn out to have a name already. For the moment let's call it the $\mathbb{Z}$-gon.

The theory of n-gons is Horn, so the class of n-gons is closed under products and homomorphisms. In particular there is a homomorphism $G_m \twoheadrightarrow G_n$ whenever n divides m, and we will exploit this fact, for example in the proof of remark 1.

DEFINITION 1
A **stratification graph** *is one where*

$$(\forall v_1)(\forall v_2)(\textit{all paths from } v_1 \textit{ to } v_2 \textit{ are the same length}).$$

A **stratification-mod-**n *graph is one with a homomorphism onto the n-gon.*

If we don't want to mention the 'n' we will say that a graph that is stratified-mod-n is **circularly stratified**.

Equivalently a graph is a stratification-mod-n graph iff, for any two vertices v_1 and v_2, all paths from v_1 to v_2 have the same length modulo n.

Observe that, for each n, the theory of stratification-mod-n graphs is a first-order theory, indeed a universal theory.

A '*moiety*' is a set x such that $|x| = |V| = |V \setminus x|$.

A formula is **(Crabbé)-elementary** iff all its variables are related by the ancestral of the relation "v and v' occur in an atomic subformula together". We will tacitly assume in what follows that all our formulæ are Crabbé-elementary. Classically (though not constructively) every first-order formula is equivalent to a boolean combination of elementary formulæ (and every *closed* first-order formula is equivalent to a boolean combination of *closed* elementary formulæ) so there is little cost in making this simplifying assumption. Without it, some of the proofs below would become snarled up in annoying minor details, so we plead for the reader's indulgence.

Definition 2

A formula is **stratifiable** *iff its derived digraph is a stratification graph.*

A **stratification** *of a formula ϕ is a homomorphism from the derived graph G_ϕ of ϕ to the $\mathbb{Z}$-gon;*

A **stratification-mod**-*n of a formula ϕ is a homomorphism from the derived graph G_ϕ of ϕ onto the n-gon.*

A formula is **stratifiable mod** *n iff its derived digraph is a stratification-mod-n graph.*

Again, if we do not want to mention the 'n' we will say of a formula that is stratifiable-mod-n that it is **circularly stratifiable**.

Equivalently a stratification graph is one where, for all vertices v, all paths from v to v are of length 0; a stratification-mod-n graph is one where, for all vertices v and v', all paths from v to v' are of the same length mod n, or—equivalently—for all vertices v, all paths from v to v are of length 0 mod n.

Remark 1

(i) A formula that can be stratified both mod-n and mod-m can be stratified mod-LCM(m, n), and conversely.

(ii) A formula that is stratifiable-mod-n for arbitrarily large n is just plain stratifiable, and a stratifiable formula is stratifiable-mod-n for all n.

Proof:

(i) Let ϕ be such a formula, and G_ϕ its derived graph. ϕ is both stratifiable-mod-n and stratifiable-mod-m which is to say that there are homomorphisms $f : G_\phi \twoheadrightarrow G_n$ and $g : G_\phi \twoheadrightarrow G_m$. Consider now the graph $G = \{\langle f(v), g(v)\rangle : v \in G_\phi\}$ with the obvious edge relation. We want to show that G is the LCM(m, n)-gon. It is a graph of size at most $n{\cdot}m$. There is a homomorphism $\lambda v.\langle f(v), g(v)\rangle : G_\phi \twoheadrightarrow G$. Clearly every vertex in G has indegree 1 and outdegree 1, so it is either a gon (if it is connected) or a union of gons (o/w). It is also clear that if we apply the edge operation of the graph G n times to an ordered pair we reach an ordered pair with the same first component, and if we apply the edge operation m times to an ordered pair we reach an ordered pair with the same second component, so if we apply the edge operation LCM(m, n) times to an ordered pair we get back to that same ordered pair. And LCM(m, n) is the smallest number of times we can apply the edge operation of G to secure this effect. Therefore one of the connected components of G is the LCM(m, n)-gon, so G is the LCM(m, n)-gon as long as it is connected.

To establish that it is—indeed—connected, we show that, for all vertices v, v' in G, there is a path from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$. Recall that G_ϕ is a stratification graph, so there is a well-defined distance, d, from v to v'. We can now see that the distance from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$ is precisely d, so G is connected.

For the converse, if ϕ is stratifiable-mod-LCM(m, n) then there is a homomorphism $f : G_\phi \twoheadrightarrow G_{LCM(m,n)}$. We compose f with the homomorphism from $G_{LCM(m,n)}$ onto G_n, thereby showing that ϕ is stratifiable-mod-n; similarly ϕ is also stratifiable-mod-m.

(ii) If $n > \text{length}(\phi)$, then any stratification-mod-n of ϕ is (or, more correctly, can be easily modified into) a stratification. For the other direction, observe that, for every n, the $\mathbb{Z}$-gon maps onto the n-gon G_n. ∎

That is literally true, but we have to be very careful how we state things like this, because we will see later (in section 3) examples of expressions that are, for each n, equivalent to formulæ that are stratifiable-mod-n but are not equivalent to any stratifiable formula.

So the picture is: we only have to worry about stratifiability-mod-p for p prime, and the various stratifiabilities-mod-p are the weakest conditions; stratifiability-mod-mn is stronger than stratifiability-mod-n, and all these are weaker than stratifiability *tout court*, which is the conjunction of them all. The various stratifiabilities-mod-p with p prime all seem to be equally weak, and they are all of minimal strength.

It may be worth noting that we cannot strengthen remark 1 by modifying the asssumption on the formula to being merely *equivalent* both to a formula that is stratifiable-mod-n and to a formula that is stratifiable-mod-m, because of the Axiom of Counting. For every n, the Axiom of Counting is equivalent (modulo NF) to a formula that is stratifiable mod n (we will see a proof of this on p 20) so the analogue of remark 1 part (ii) would tell us that it is equivalent to a stratifiable formula. However, it is known that it is not equivalent (modulo NF) to any stratifiable formula. However, the Axiom of Counting is invariant, so it might be possible to strengthen remark 1 by modifying the asssumption on the formula to being merely *equivalent* (mod NF) both to a formula that is stratifiable-mod-n and to a formula that is stratifiable-mod-m, if the conclusion we want to infer is that the formula in question is merely invariant (modulo NF) rather than actually stratifiable.

Explain 'invariant'?

2.1 Wrapping up miscellaneous definitions

Finally we wrap up some definitions and notations. Some of them are standard in an NF context but a clear summary of them can do no harm.

We write '$x \,\Delta\, y$' for the symmetric difference of two sets x and y.

ι is the singleton function: $\iota(x) = \{x\}$. If $\iota \restriction x$ exists we say x is **strongly cantorian**.

We write 'Symm(X) for the full symmetric group on X.

In practice X is always V.

$j : \mathrm{Symm}(V) \to \mathrm{Symm}(V)$ is defined so that $j(\sigma)(x) = \sigma``x$.

Let us use lower-case $\mathfrak{fraktur}$ characters for variables ranging over conjugacy classes.

If there is σ such that $(j\sigma)^{-1} \cdot \tau \cdot \sigma = \pi$ then we say τ and π are **skew-conjugate**. Observe that this relation of skew-conjugacy is in fact an equivalence relation. However the definition is not stratified and (in NF) the graph of the relation is not a set and the equivalence classes are not sets. The skew-conjugacy class of $\mathbb{1}$, the identity relation, is the class of internal automorphisms and it should be easy to show that that need not be a set (tho' i have not done so so far!)

The significance of this relation is that skew-conjugate permutations give rise to isomorphic permutation models, as follows. Suppose τ and π are skew-conjugate; then $x \in \tau(y)$ iff $x \in (j\sigma)^{-1} \cdot \pi \cdot \sigma(y)$ iff $\sigma(x) \in \pi \cdot \sigma(y)$ which is as much as to say that σ is an $\in$-isomorphism between V^τ and V^π.

However skew-conjugacy doesn't seem to be a congruence relation for very much. Certainly not for the group-theoretic operations of product or inverse. As far as i can see σ skew-conjugate to π doesn't imply σ^{-1} skew-conjugate to π^{-1}, tho' i do not have any counterexamples to hand. It might be an idea to find some.

Minding your ps and qs:

A **partition** $\mathbb{P}$ of a set X is a set of pairwise disjoint subsets of X s.t. $\bigcup \mathbb{P} = X$. The members of $\mathbb{P}$ are **pieces** (of $\mathbb{P}$). I shall use the letter '$\mathbb{P}$' to range over partitions; 'Π' will be used for products (in particular for products of (pairwise disjoint) transpositions, as in '$\Pi_{x \in A}(x, V \setminus x)$'). $\mathcal{P}(x)$ is the power set of x. $B(x) = \{y : x \in y\}$, the principal ultrafilter in the powerset algebra $\mathcal{P}(V)$. $ƀ(x) = \{y : y \cap x \neq \emptyset\}$ and is thus dual to $\mathcal{P}$, in the sense that $b(x) = V \setminus (\mathcal{P}(V \setminus x))$—which is why we write it with an upside-down '$\mathcal{P}$'. The fact that $B(x) = ƀ(\{x\})$ also helps.

3 Motivating stratification-mod-n

3.0.1 The $\in$-game

The $\in$-game G_x in [11] is played by two players—`I` and `II`—and is initiated by player `I` picking a member of x; thereafter the players move alternately, each picking an element of the other's previous choice until one of them attempts to pick a member of the empty set and thereby loses. (That is the only way the game can end). This subject matter has a naturally stratifiable-mod-2 flavour: "Player `I` has a Winning strategy in G_x" and "Player `II` has a Winning strategy in G_x" are both stratifiable-mod-2. The first is

$$(\forall y)(ƀ(\mathcal{P}(y)) \subseteq y \to x \in y)$$

which is as much as to say that x belongs to the $\subseteq$-least fixed point for $X \mapsto \text{Ƅ}(\mathcal{P}(X))$. The second is

$$(\forall y)(\mathcal{P}(\text{Ƅ}(y)) \subseteq y \to x \in y).$$

which is as much as to say that x belongs to the $\subseteq$-least fixed point for $X \mapsto \mathcal{P}(\text{Ƅ}(X)))$. (Recall from page 7 that $\text{Ƅ}(x)$ is $\{y : y \cap x \neq \emptyset\}$.)

It's worth asking how much of this stuff we can describe in formulæ that are stratifiable-mod-($2n$). Start with the case $n = 2$ to keep things simple. Suppose X is the $\subseteq$-least fixed point for $X \mapsto \text{Ƅ}(\mathcal{P}(\text{Ƅ}(\mathcal{P}(X))))$. Suppose $x \in X$; then player `I` can pick something $y \in x$ s.t. any $z \in y$ that `II` picks will be in $\text{Ƅ}(\mathcal{P}(X))$. Similarly suppose $x \in \text{Ƅ}(\mathcal{P}(X))$; then player `I` can pick something $y \in x$ s.t. any $z \in y$ that `II` can pick will be in $\text{Ƅ}(\mathcal{P}(\text{Ƅ}(\mathcal{P}(X))))$ which is of course X. Thus, for any $x \in X$, the only way player `II` can prevent the play of G_x from returning them to X is for `II` to lose the game *en route*. But it's easier than that. Any fixed point for $X \mapsto \text{Ƅ}(\mathcal{P}(X))$ is also a fixed point for $X \mapsto \text{Ƅ}(\mathcal{P}(\text{Ƅ}(\mathcal{P}(X))))$. Therefore It will suffice to show that the $\subseteq$-least fixed point for $X \mapsto \text{Ƅ}(\mathcal{P}(\text{Ƅ}(\mathcal{P}(X))))$ is also a fixed point for $X \mapsto \text{Ƅ}(\mathcal{P}(X))$, because it will then be the least. I think all we have to do is show that if there are any fixpoints for $X \mapsto \text{Ƅ}(\mathcal{P}(X))$, then the least fixpoint for $X \mapsto \text{Ƅ}(\mathcal{P}(\text{Ƅ}(\mathcal{P}(X))))$ will be one of them. (We have to be careful how we state this because there are functions without fixpoints whose squares have fixpoints.)

This still needs some work done on it!

3.0.2 The Axiom of Counting

The axiom of counting is unstratified and not equivalent modulo NF to any stratifiable formula but is, for each concrete n, equivalent modulo NF to a formula that is stratifiable-mod-n. It's also invariant. The same goes for $\text{AxCount}_{\leq}$ (with a bit more work) since—for any concrete k—$\text{AxCount}_{\leq}$ can be written as '$(\forall n \in \mathbb{N})(n \leq T^k n)$'.

see also section 5

4 Preservation Results for Stratification-mod-n

We start with a definition from [5].

Definition 3 $H(0, \tau) =: \mathbb{1}_V$; $\quad H(n+1, \tau) =: (j^n \tau) \cdot H(n, \tau)$.

This H notation will only ever be used with concrete naturals in first argument place.[3]

The effect of this notation is that, for any τ and any concrete n, $(\forall xy)(x \in \tau(y) \longleftrightarrow H(n, \tau)(x) \in H(n+1, \tau)(y))$. The intention behind the design of this family of permutations derived from a single τ is to prove that, when ϕ is

[3]so we shouldn't use these purely concrete chaps as arguments; they should be hidden in the syntax? The trouble with this policy is that we don't want footnotesized things like '$LCM(n, m)$'.

stratifiable, ϕ^τ is equivalent to the result of replacing every occurrence of each free variable 'v' with '$H(n_v, \tau)(v)$' where n_v is the concrete natural number associated to the variable 'v' in a fixed stratification of ϕ. In the treatment here, our stratifications are functions from $vbls(\phi)$ to the $\mathbb{Z}$-gon or the n-gon and do not take numbers as values. This can be remedied by composing a stratification with a decoration-by-numbers (satisfying the obvious adjacency condition) of the gon in question.

It might be worth minuting other facts about the family of permutations engendered in this way from a permutation σ. For example $H(n+m, \sigma) = j^m(H(n,\sigma)) \cdot H(m,\sigma)$. We don't think there is a nice formula for $H(n \cdot m, \sigma)$. This is another manifestation of the fact that there is no natural arithmétic structure on the set of type indices.

We have a theorem of Scott that stratifiable formulæ are preserved under the Rieger-Bernays permutation construction. This is an assertion of the form

$$(\forall \pi)(F(\pi) \to (\forall \phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi))) \tag{A}$$

or equivalently

$$(\forall \phi)(\phi \in \Gamma \to (\forall \pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)))$$

Assertions like (A) have converses of the form

$$(\forall \pi)[(\forall \phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)] \tag{B}$$

and of the form

$$(\forall \phi)[(\forall \pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma] \tag{C}$$

In this section we consider the project of proving assertions like these where Γ is the set of formulæ that are stratifiable-mod-n. This will involve us in identifying interesting properties of permutations to serve as the 'F' in the statement of the results

4.1 Instances of (A): $(\forall \pi)(F(\pi) \to (\forall \phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)))$

Proposition 1 *If ϕ is stratifiable-mod-n then it is preserved under all Rieger-Bernays constructions using setlike permutations π s.t. $H(n, \pi) = \mathbb{1}$.*

Proof:

The proof is a straightforward adaptation of the proof given by Henson. Reference?

In Henson's treatment of the stratified case we fix a stratification s for ϕ. [In that treatment stratifications take values in $\mathbb{Z}$, not in the $\mathbb{Z}$-gon.] Then, whenever we look at a subformula '$x \in \sigma(y)$' in ϕ^σ we replace it by '$H(n,\sigma)(x) \in H(n+1,\sigma)(y)$' where n is the type given to the variable 'x' by the stratification s. We then observe that, for every variable, all occurrences of that variable in the rewritten version of ϕ^σ are prefixed by a '$H(n, \sigma)$' where n is the type given to 'x' by the stratification s. Then we appeal to the fact that $H(n,\sigma)$ is a permutation, so we can reletter '$H(n,\sigma)(x)$' as 'x', and this manipulation turns ϕ^σ back into ϕ. The difference here, in this case, is that our subscripts are no longer integers but are integers-mod-n, so that if $i \equiv j \pmod{n}$ we must have $H(i,\sigma) = H(j,\sigma)$. This is equivalent to requiring that $H(n,\sigma)$ be the identity. ■

4.2 Instances of (C): $(\forall\phi)[(\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma]$

There is a theorem, proved by Pétry and Forster ([7], [15], [16]) to the effect that: if a formula is preserved under all Rieger-Bernays constructions using setlike permutations then it is equivalent to a stratifiable formula.

Is there an analogous result to the effect that if a formula is preserved under all Rieger-Bernays constructions using setlike permutations $\sigma = H(n,\sigma)$ then it is equivalent to a formula that is stratifiable-mod-n? Something like that ought to be true, and it's probably worth proving.

4.3 Instances of (B): $(\forall\pi)[(\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)]$

We start with a very easy example:

Remark 2 *If $f : V \to V$ (possibly a proper class) satisfying $\phi \longleftrightarrow \phi^f$ for all stratifiable expressions then f must be a setlike permutation.*

Proof: The axiom of extensionality is stratifiable, and any f that preserves it must be onto. If f preserves an $(n+1)$-stratifiable formula then $H(n,f)$ has to be defined, so f has to be n-setlike. ■

One might expect that if π is a permutation that preserves all formulæ that are stratifiable-mod-n then $H(n,\pi) = \mathbb{1}$. Something with that sort of flavour should be true. The following is a straw in the wind.

Remark 3 *If $H(n,\sigma) = \mathbb{1}$ and $H(k,\sigma) = \mathbb{1}$ then $H(HCF(n,k),\sigma) = \mathbb{1}$.*

Proof: This is because, for every σ, the class of naturals n s.t. $H(n,\sigma) = \mathbb{1}$ is closed under subtraction[4] so we can, as it were, perform Euclid's algorithm. If $H(n,\sigma) = \mathbb{1}$ and $H(k,\sigma) = \mathbb{1}$, with $n > k$ then reflect that $H(n,\sigma)$ is $(j^k H(n-k,\sigma)) \cdot H(k,\sigma)$. So $j^k H(n-k,\sigma) = H(n,\sigma) \cdot H(k,\sigma)^{-1} = \mathbb{1} \cdot \mathbb{1} = \mathbb{1}$. But then $H(n-k,\sigma) = \mathbb{1}$ as well. ■

This doesn't actually say that if σ both preserves formulæ that are stratifiable-mod-n and preserves preserves formulæ that are stratifiable-mod-k) then it preserves formulæ that are stratifiable-mod-$HCF(n,k)$, but it has that flavour.

[4] *prima facie* we cannot expect this thing to be a set, since it is defined by an unstratified expression.

One wants to say that a permutation that preserves *all* closed formulæ must be an $\in$-automorphism, but that doesn't seem to be strictly true. At any rate we don't know how to prove it! Perhaps we can prove it by reasoning about Ehrenfeucht games. What we *do* know how to prove is that, if $V \simeq V^\sigma$, then σ is skew-conjugate to the identity. The only permutation that preserves *all* expressions (i.e., including open formulæ) is $\mathbb{1}$.

And, once we have identified predicates F that appear in theorems of flavour (B), one wants to find a structure for the set of all permutations on V such that, for each F, the class of permutations that are F is a sub*structure* not a mere sub*class*.

One thing one might hope to prove is that if ϕ is stratifiable-mod-n and is logically equivalent to a formula that is stratifiable-mod-m then it is logically equivalent to a formula that is stratifiable-mod-nm ...

Thinking aloud about this...

If ϕ is equivalent to something that is stratifiable-mod-n then ϕ is equivalent to $(\forall\sigma)(H(n,\sigma) = \mathbb{1} \to \phi^\sigma)$. Now let$\tau$ be a permutation satisfying $H(m,\tau) = \mathbb{1}$ and consider what happens in V^τ. We have

$(\forall\sigma)(H(n,\sigma) = \mathbb{1} \to \phi^\sigma)^\tau$

This requires thought! "$H(n,\sigma) = \mathbb{1}$" can be thought of as a stratified formula with n occurrences of 'σ', one at each of n distinct adjacent types, namely

$$j^n(\sigma) \cdot j^{n-1}(\sigma) \cdot \cdots \cdot j^{n-2}(\sigma) \cdot \sigma = \mathbb{1}$$

$$j^n(H(k,\tau)(\sigma)) \cdot j^{n-1}(H(k{+}1,\tau)(\sigma)) \cdot j^{n-2}(H(k{+}2,\tau)(\sigma)) \cdots H(k{+}n,\tau)(\sigma) = \mathbb{1}$$

where we have chosen k large enough so that $H(k+1,\tau)(\sigma)$ is conjugated to $H(k,\tau)(\sigma)$ by τ or $j^i(\tau)$

Now let's think about what happens to ϕ^σ in the permutation model. This is a problem well-known to your humble correspondent.

$(x \in \sigma(y))^\tau$ becomes

$H(k-1,\tau)(x) \in H(k+1,\tau)(\sigma)(H(k,\tau)(y))$ becomes

$x \in (\tau_k)^{-1}$“$\tau_{k+2}(\sigma)(\tau_{k+1}(y))$

Need to continue rewriting.

Now we reletter '$\tau_k(\sigma)$' as 'σ' throughout.

$j^n(\tau_k(\sigma)) \cdot j^{n-1}(\tau_{k+1}(\sigma)) \cdot \cdot j^{n-2}(\tau_{k+2}(\sigma)) \cdots \tau_{k+n}(\sigma) = \mathbb{1}$ becomes

$$j^n(\sigma) \cdot j^{n-1}(\sigma^\tau) \cdot \cdot j^{n-2}(\sigma^{\tau_2} \cdots \sigma^{\tau_n}) = \mathbb{1}$$

which is

$$j^n(\sigma) \cdot j^{n-1}\tau j^{n-1} \cdot \sigma \cdot j^{n-1}\tau^{-1}) \cdot \cdot j^{n-2}(\sigma^{\tau_2}) \cdots \sigma^{\tau_n} = \mathbb{1}$$

and we can do some cancellation...

Definitely work to be done in section 4; not sure what was going on there(!)

5 Stratifiable mod n for every n

Given a theory T, there is a natural class consisting of those formulæ that, for each n, are equivalent modulo T to a formula that is stratifiable-mod-n. It is larger than the class of stratifiable formulæ and even (tho' this is less obvious and of course depends on T) contains formulæ that are not T-invariant. Whether or not there are formulæ that are T-invarinat but are not in our class i do not know at this stage.

We will consider the following formulæ: stcan(x), WF(x), $\bigcup x \subseteq x$

As we shall see, the axioms TCl and TCo of transitive closure and transitive containment are both of this class.

Definition 4
*Let us say x is n-**hemitransitive** iff $(\forall y)(y \in^{n+1} x \to y \in x)$.*

Thus ordinary transitivity is 1-hemitransitivity. It is easy to see that n-hemitransitivity is stratifiable-mod-n.

It is also Horn, so there is a notion of the n-hemitransitive closure of a set. Observe that if x is n-hemitransitive then $x \cup \bigcup x \cup \ldots \bigcup^{n-1} x$ is transitive. So "There is an infinite transitive set" is, for each n, equivalent to "There is an infinite n-hemitransitive set" ... which of course is stratifiable-mod-n. So it ought to be invariant. Suppose then that we are working in a model of NF + $\mathrm{AxCount}_{\leq}$ that contains V_ω. Holmes' clever permutation will kill off V_ω but it would leave behind an infinite transitive wellfounded set. This doesn't seem frightfully plausible.

Now we can see that the fact that if x is n-hemitransitive then $\bigcup^{<n} x$ is transitive means that the axiom of transitive containment belongs to our special class of formulæ.

Let TCl_n say that every set has an n-hemitransitive closure. Consideration of $\bigcup^{<n} X$ shows that this implies TCl. For the other direction we will need unstratified separation—annoyingly. TCo_n implies TCo; for the other direction we do not need any unstratified separation, since any transitive set is n-hemitransitive for any n.

Consider the sets—call them X_n for the nonce—where X_n is the least fixpoint for $x \mapsto (\mathcal{P}_{\aleph_0})^n(x)$. X_n is the collection of sets of (finite) rank a multiple of n. The assertion that X_n exists is stratifiable-mod-n, and V_ω is of course $\bigcup^{\leq n} X_n$. Thus the assertion that V_ω exists is, for each n, equivalent (over NF) to a formula that is stratifiable-mod-n. However, beco's of Holmes' clever permutation, it is not invariant!

This shows that, for $T = \mathrm{NF} + \mathrm{AxCount}_{\leq}$ at least, there are formulæ that, for each n, are T-equivalent to something that is stratifiable-mod-n but are not T-invariant. This doesn't prove the same for NF, but the damage is done.

Can we do the same for NF? Clearly one wants to put Holmes' clever permutation to use. Let A_n be the assertion that there is an infinite n-hemitransitive subset of X_n. Is A_n equivalent to the assertion that there is an infinite transitive subset of V_ω? I can only see the implication one way.

Probably the most important unstratified set-theoretic property is wellfoundedness; it cannot be captured by any stratfiable formula but can it be captured by a formula that is stratifiable-mod-n? The following elementary observation took me by surprise.

PROPOSITION 2 *"x is wellfounded" is, for every n, equivalent to a formula that is stratifiable-mod-n.*

Proof:

In fact there is a parametrised family of such formulæ. The typical formula is $(\forall y)(\mathcal{P}^n(y) \subseteq y \to x \in y)$, or $\mathrm{WF}_n(x)$ for short. Notice that for $n = 1$ this gives the natural inductive definition of the class of wellfounded sets as the least fixpoint for the power set function. $\mathrm{WF}_n(x)$ is stratifiable-mod-n all right, but is it equivalent to $WF(x)$? One direction is easy: $\mathcal{P}^n(y) \subseteq y$ is a weaker condition than $\mathcal{P}(y) \subseteq y$ so if you belong to everything satisfying the weaker condition you certainly belong to everything satisfying the stronger condition. So $\mathrm{WF}_n(x)$ implies $\mathrm{WF}(x)$. What about the other direction?

Let us say that a set y s.t. $\mathcal{P}^n(y) \subseteq y$ is n-**fat**[5]. Observe that if y is n-fat so is $\mathcal{P}(y)$. Suppose now that every member of x belongs to every n-fat set. Then x is included in every n-fat set, and so is a member of the power set of any n-fat set. So it is a member of every n-fat set. Thus we can prove by $\in$-induction that every wellfounded set is WF_n. ■

"$(\forall y)(\mathcal{P}^n(y) \subseteq y \to x \in y)$" seems to make sense only in NF-like contexts, where separation fails and sets can be supersets of their own power sets. However if we contrapose and replace y by $V \setminus y$ we obtain

$$(\forall y)(x \in y \to (\exists z \in y)(z \notin \mathcal{P}^n(V \setminus y))).$$

which make sense in a context with full separation. This development is analogous to the way in which one obtains the concept of *regular set* from the natural inductive (least-fixpoint) definition of wellfounded set as $(\forall y)(\mathcal{P}(y) \subseteq y \to x \in y)$.

While we are about it (tho' perhaps this observation could be better placed elsewhere) this shows that altho' stratified parameter-free $\in$-induction seems to be quite weak (it is open whether or not it proves anything more than the nonexistence of a universal set) it is nevertheless the case that, for each n, parameter-free $\in$-induction for formulæ that are stratifiable-mod-n implies full $\in$-induction.

6 Cylindrical Types

Stratifiable formulæ of the language of set theory are those from which one can obtain wffs of TST by decorating the variables with indices indicating what

[5]This terminology is generalised from that in [1].

levels they belong to. There is an analogous move to be made with formulæ that are stratifiable-mod-n: one can obtain from them formulæ that are wffs of a typed theory of sets whose levels are indexed by n-gons. The more properties we succeed in capturing with formulæ that are stratifiable-mod-n the greater the expressive power of these typed theories of sets will be.

We should note that—in contrast to stratification *tout court*—stratification-mod-n is not a useful notion from the point of view of comprehension principles in a one-sorted language, since there are paradoxical objects that are the extension of formulæ that are stratifiable-mod-n; one thinks of the n-fold Russell class $\{x : x \not\in^n x\}$—being the extension of the formula '$x \not\in^n x$' (which is stratifiable-mod-n) which is a paradoxical object even in mere first-order logic. This is discussed in section 4 of [4] and also below). Also, as we have just shown (proposition 2), wellfoundedness is capturable by a formula that is stratifiable-mod-n for any n (and is therefore expressible in $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$). Of course there are no known paradoxical objects defined by stratifiable set abstracts.

So *that*'s a dead end, but there is an obvious link from formulæ that are stratifiable-mod-n to the theory T$\mathbb{Z}$T+ Amb^n. The usual Specker equiconsistency analysis leads one thence to type theories whose levels are indexed by the n-gon. One could perhaps call these theories "type theory mod n", and that is what i shall do here; the proper name will be "$\mathrm{TC}_n\mathrm{T}$" ("theory of n cylindrical types").

Let's be formal about it.

Definition 5 *The language $\mathcal{L}(TC_nT)$, where n is a concrete natural number, has two binary relation symbols: '$=$' and '$\in$'. Its variables each have a type index as an integral part, and those type indices are precisely the elements of the n-gon.*

The axioms of $\mathrm{TC}_n\mathrm{T}$ are extensionality at each type, as with T$\mathbb{Z}$T, but there is a subtlety with the set comprehension axioms. One cannot allow $(\exists x)(\forall y)(y \in x \longleftrightarrow y \not\in^n y)$ to be an axiom (for obvious reasons) even tho' this formula is a wff of $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$ and has the syntactic form of a comprehension axiom, and '$y \not\in^n y$' is a wff of the language. One allows set comprehension only for the old T$\mathbb{Z}$T axioms. To be formal about it, a wff that looks like a comprehension axiom is adopted as an axiom only if it is possible to rejig the type indices in it so that the resulting formula is an axiom of T$\mathbb{Z}$T.

Thus the axioms of $\mathrm{TC}_n\mathrm{T}$ are "closed under rotation", or *ambiguous* in traditional parlance.

It may be worth noting that $\mathrm{TC}_n\mathrm{T}$ can expressed as a theory in the usual one sorted first-order language $\mathcal{L}(\in, =)$ of set theory. However, since we will not be making any use of this fact, we feel under no obligation to provide a proof.

The various analogues of Russell's paradox prevent us from adopting as our comprehension scheme for $\mathrm{TC}_n\mathrm{T}$ the obvious scheme of all expressions of the form $(\forall \vec{x})(\exists y)(\forall z)(z \in y \longleftrightarrow \phi(\vec{x}, z))$ that belong to $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$. Of course the mere fact that the existence of $\{x : x \not\in^n x\}$ is not a comprehension axiom does not *ipso facto* mean that the sets $\{x^i : x^i \not\in^n x^i\}$ cannot exist at any of the

n levels, though it will be shown below that first-order logic by itself suffices to show that they cannot exist at all levels at once. TC$_n$T has comprehension axioms and can prove that they cannot exist at even one level. This fact is probably worth minuting.

Remark 4 *$TC_nT \vdash R^i_n = \{x^i : x^i \not\in^n x^i\}$ does not exist for any $i \leq n$.*

Proof:
Reasoning in TC$_n$T we pick on any level i and consider the possibility of the existence of $R^i_n = \{x^i : x^i \not\in^n x^i\}$. Consider $\iota^{n-1}(R^i_n)$; is it a member of R^i_n or not? If it is, then it belongs to an $\in$-loop of circumference n, so it is barred from membership of R^i_n. So it isn't a member of R^i_n. So there are $x_1 \ldots x_{n-1}$ with $\iota^{n-1}(R_n) \in x_1 \in x_2 \in \ldots \iota^{n-1}(R_n)$ an $\in$-loop of circumference n. But then $x_1 = R_n$ (peel off the brackets) showing that $R_n \in R_n$ after all. ■

Notice that we have not used very much comprehension. All we have used is the assumption that every set has a singleton. In fact all we need is that for every x there is a nonempty subset of $\mathcal{P}(x)$, the point being that every set of loop-free sets is itself loop-free.

Readers might like to note the curiosity that inside first-order logic pure and simple (without using any set theory at all) we can show that $\{x^i : x^i \not\in^n x^i\}$ must fail to exist at least one level i. This, too, is probably worth minuting.

Remark 5 *It is a theorem of First Order Logic that that no model of TC_nT can contain $R^i_n = \{x^i : x^i \not\in^n x^i\}$ for all $i \leq n$.*

Proof:
Suppose $\{x : x \not\in^n x\}$ exists at every level. Let us write 'R^i' for its manifestation at level i. Let i be an arbitrary concrete natural $\leq n$. Suppose $R^i \not\in R^{i+1}$. Then R^i belongs to an $\in$-loop of circumference n, and there must be x^{i-1} in R^i in this loop. But $x^{i-1} \in R^i$ implies that x^{i-1} cannot belong to any such loop. Thus we conclude $R^i \in R^{i+1}$. But i was arbitrary. So there is an $\in$-loop of circumference n consisting entirely of the R^i and this clearly cannot happen. ■

It doesn't seem to be possible to spice up this proof to show (in first-order logic) that *none* of the R^i exist. The nonexistence of $\{x : x \not\in^n x\}$ is a theorem of first-order logic that is stratifiable-mod-n, but i know of no globally stratifiable-mod-n cut-free proof. This fact (if it is a fact) is almost certainly related to the fact (if it is a fact) that we cannot prove that $\{x : x \not\in^n x\}$ exists at no level (tho' we can show that it doesn't exist at all). If we had a proof of the nonexistence of $\{x : x \not\in^n x\}$ in FOL that was globally stratifiable-mod-n then we could run it at any level and show that $\{x : x \not\in^n x\}$ exists at no level. The following reflection suggests that there is no such proof. Consider the two-lobed model with precisely one inhabitant in each lobe: a `yin` set that is a member of the `yang` set (but not the other way round. The `yang` set (but not the `yin` set) is a double-Russell class.

This does rather suggest that first-order logic holds no globally stratifiable-mod-2 nonexistence proof for the double Russell class... *and that this is true even if we allow cut.*

6.1 Rieger-Bernays Permutation methods for $\mathrm{TC}_n\mathrm{T}$

Rieger-Bernays methods generalise smoothly to $\mathrm{TC}_n\mathrm{T}$. R-B methods in NF enable one to obtain from any model of NF a new model which satisfies the same stratifiable sentences but tweaks the truth-values of some formulæ that are not stratifiable. In the $\mathrm{TC}_n\mathrm{T}$ context we have the same notion of stratifiable, but the role of non-stratifiable formulæ is played by formulæ that are stratifiable-mod-n. Formulæ that are frankly unstratified *don't enter into it* as the man in the parrot shop would say. The R-B method which we develop below for $\mathrm{TC}_n\mathrm{T}$ will enable us to obtain from a model $\mathfrak{M}$ of $\mathrm{TC}_n\mathrm{T}$ a model that satisfies the same stratifiable sentences as $\mathfrak{M}$ but satisfies *different* sentences that are merely stratifiable-mod-n.

It goes as follows. Let $\mathfrak{M}$ be a model of $\mathrm{TC}_n\mathrm{T}$. To each of the n levels of $\mathfrak{M}$ associate an internal permutation τ of that level. Thus we have a *suite* of permutations. Then we declare a new membership relation between levels i and $i+1$ by $x_i \in_{new} x_{i+1}$ iff $x_i \in \tau(x_{i+1})$. The relettering now proceeds as in the proof of Henson's lemma. For this we naturally need all the permutations in the suite to be setlike, just as in the original R-B setting. Realistically we can take them to be sets of the model.

Observe the two special cases: NF and TST. NF is the special case $\mathrm{TC}_1\mathrm{T}$. There is only permutation, and we are in the standard situation with R-B models for NF. TST is the case $\mathrm{TC}_\infty\mathrm{T}$ and we have infinitely many permutations, one for each level. In this case nothing happens, because there are no wellformed formulæ that this process could possibly change the truth-value of. No wonder nobody noticed it before!

If we want to preserve formulæ that are stratifiable-mod-k then we require certain equations to hold between the permutations τ that we use. Consider $\mathrm{TC}_3\mathrm{T}$, the simplest case that is complicated enuff to partake of the general flavour. We have a suite π, τ, σ of permutations. In the permutation model $x_1 \in x_2$ becomes $x_1 \in \tau(x_2)$; $x_2 \in x_3$ becomes $x_2 \in \pi(x_3)$ and $x_3 \in x_1$ becomes $x_3 \in \sigma(x_1)$. To reletter we have to rewrite $x_2 \in \pi(x_3)$ as $\tau(x_2) \in (j\tau) \cdot \pi(x_3)$ and then rewrite $x_3 \in \sigma(x_1)$ as $(j\tau) \cdot \pi(x_3) \in (j^2\tau) \cdot j\pi \cdot \sigma(x_1)$. If we want to be able to eliminate π, σ and τ from formulæ that are stratifiable-mod-3 (but not stratifiable) then we will need $(j^2\tau) \cdot j\pi \cdot \sigma = \mathbb{1}$. Call this **The Equation For n.**

OK, so we have a model of $\mathrm{TC}_n\mathrm{T}$, and we decorate it with permutations of each level. This R-B construction preserves all stratifiable expressions. What about expressions that are stratifiable-mod-$(n \cdot k)$? Then we have The Equation For $n \cdot k$. This is an equation $w = \mathbb{1}$ where w is a product of $n \cdot k$ things, with k occurrences of each permutation. Persisting for the moment with the $n = 3$ example, we find that if we want our suite of permutations to preserve formulæ that are stratifiable-mod-6, then we need τ, σ and π to satisfy

$$(j^5\tau) \cdot (j^4\pi) \cdot (j^3\sigma) \cdot (j^2\tau) \cdot j\pi \cdot \sigma = \mathbb{1}.$$

This looks messy, but i think it is correct.

There is also the small matter of proving an analogue of the PHF theorem for $\mathrm{TC}_n\mathrm{T}$. We can obtain a model of $\mathrm{T\mathbb{Z}T}$ from a model of $\mathrm{TC}_n\mathrm{T}$ in the same was as we obtain one from a model of NF. The analogue will then say that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy the same stratifiable sentences iff the two models of $\mathrm{T\mathbb{Z}T}$ obtained from them have stratimorphic ultrapowers.

But what about stratifiable-mod-n? What condition on two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathrm{TC}_n\mathrm{T}$ corresponds to them satisfying the same formulæ that are stratifiable-mod-$n \cdot m$? I'm guessing it will be the following. Any model of $\mathrm{TC}_n\mathrm{T}$ can be turned into a model of $\mathrm{TC}_{m\cdot n}\mathrm{T}$ in an obvious way. So: let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two models of $\mathrm{TC}_n\mathrm{T}$. Obtain two models of $\mathrm{TC}_{m\cdot n}\mathrm{T}$. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy the same formulæ that are stratifiable-mod-$n \cdot m$ iff these two models have isomorphic ultrapowers. I should be able to prove this but i am old and tired and i have multiple infarct dementia.

We should probably try to find something to say about expressions that, for every n, are logically equivalent to a formula that is stratifiable-mod-n. It's not true that all such sentences are stratified, because the axiom of counting is a counterexample. An interesting example of a sentence of this kind is "every wellfounded set is finite". It's not known if this allegation is invariant.

However, at least some such expressions are not invariant.

6.2 Possible Equiconsistency of $\mathrm{TC}_n\mathrm{T}$ and NF

Two fundamental questions:

(i) Is $\mathrm{TC}_n\mathrm{T}$ equiconsistent with NF?

(ii) Are there models of $\mathrm{TC}_n\mathrm{T}$ that are not ambiguous? Equivalently, are there models of $\mathrm{T\mathbb{Z}T}+\mathrm{Amb}^n$ that are not models of Amb?

I'm guessing that the answer to both is 'yes', but i have no idea how to prove it. Part of the trouble is that i can't think of a stratified formula for which Amb might fail while Amb^2 holds.

Is $\mathrm{TC}_n\mathrm{T}$ equiconsistent with NF? One direction is easy. We can obtain a model of $\mathrm{TC}_n\mathrm{T}$ from a model of NF by making n copies of the model of NF and decorating them appropriately. The obvious way is as follows. Let level n of $\mathfrak{M}$ be $V \times \{n\}$ and let us declare that $\mathfrak{M} \models x_n \in y_{n+1}$ iff let $x_n = \langle x, n\rangle$ in let $y_{n+1} = \langle y, n+1\rangle$ in $x \in y$. $\mathfrak{M}$ is clearly an ambiguous model.

However one would not expect *every* model of $\mathrm{TC}_n\mathrm{T}$ to be ambiguous, because that would mean that Amb^n implies Amb, and that surely cannot be true. It would be nice to obtain a model of $\mathrm{TC}_2\mathrm{T}$ that violates ambiguity. A simple observation is that no model of $\mathrm{TC}_2\mathrm{T}$ can contain a Boffa atom in one lobe and a Boffa antiatom in the other. This means that if we can find a model $\mathfrak{M}$ that has a Boffa atom and a Boffa antiatom in one lobe then $\mathfrak{M}$ must violate ambiguity, because an ambiguous model with a Boffa atom plus antiatom in one lobe must also contain a Boffa atom plus antiatom in the other, and this is impossible, as we have just observed.

We can do this by a simple tweak of the obvious construction. Let τ be some permutation that adds a Boffa atom and a Boffa antiatom, such as $(\emptyset, B(\emptyset))(V, \overline{BV})$. Then we set both `yin` and `yang` to be V and we set $x_{\texttt{yin}} \in y_{\texttt{yang}}$ iff $x \in y$ and $x_{\texttt{yang}} \in y_{\texttt{yin}}$ iff $x \in \tau(y)$. Let us call this model $\mathfrak{M}$.

$\mathfrak{M}$ is clearly extensional. If we pinch ourselves to keep in mind that the comprehension axioms of $\mathrm{TC_2T}$ are the fully stratifiable instances of comprehension and not the (larger) class of comprehension axioms that are stratifiable-mod-2 (Beware the double Russell class) then we can see that all the instances of comprehension for $\mathfrak{M}$ follow smoothly from comprehension in the model of NF in which we are working. (See also p 22.)

[One potentially useful piece of clarification.... What happens if we use τ on *both* lobes, so that we set $x_{\texttt{yin}} \in y_{\texttt{yang}}$ iff $x \in \tau(y)$ and $x_{\texttt{yang}} \in y_{\texttt{yin}}$ iff $x \in \tau(y)$? Clearly we do not get Boffa atoms plus antiatoms in *both* lobes—beco's we can't—but it might help to show what becomes of V and $\emptyset$ in each lobe. They probably become something annoying that is *almost* an atom or an antiatom.]

However this construction does not resolve the question. Hitherto all discussions of ambiguity were in the context of TST. The scheme was: take any stratifiable formula $\phi \in \mathcal{L}(\in, =)$, decorate it with type indices on the variables, and assert biconditionals between the results. The point is that *all* formulæ of $\mathcal{L}(TST)$ arise from formulæ in $\mathcal{L}(\in, =)$ by this process of decoration. **However** $\mathcal{L}(\mathrm{TC}_n\mathrm{T})$ has extra formulæ that can be decorated in this way, namely the formulæ that are stratifiable-mod-n. Ambiguity fails in the model $\mathfrak{M}$ that we have just considered, but the failure we have exhibited concerns not formulæ that arose from *stratifiable* formulæ of $\mathcal{L}(\in, =)$, but a formula that arose from a formula in $\mathcal{L}(\in, =)$ that was *stratifiable-mod-n*. I claim that $\mathfrak{M}$ satifies ambiguity for all formulæ that arose from *stratifiable* formulæ in $\mathcal{L}(\in, =)$. Let ϕ be any closed stratifiable formula of $\mathcal{L}(\in, =)$. Fix a stratification of it. This stratification awards every variable a decoration that is either an even natural or an odd natural. We can now turn ϕ into a formula of $\mathcal{L}(\mathfrak{M})$ in two ways: make every variable with an *even* decoration into a variable of type `yin` and every variable with an *odd* decoration into a variable of type `yang` or *vice versa*. But then in both these cases any variable v that ever appears in a context "$\ldots \in \tau(v)$" only ever appears in such contexts, and so can be relettered.

If we think of the task of finding a model of $\mathrm{TC_2T}$ that is not ambiguous as the task of finding a model of TZT that satisfies Amb^2 but does not satisfy Amb then it perhaps becomes clearer. This second task clearly remains undone.

It's an old result (it was in Forster's Ph.D. thesis, with a much improved proof by Crabbé [2] subsequently) that $\mathrm{T\mathbb{Z}T}+\mathrm{Amb}^n$ refutes AC, and by essentially the same mechanism as does $\mathrm{T\mathbb{Z}T}+$ Amb. The best guess is that all the theories $\mathrm{TC}_n\mathrm{T}$ are equiconsistent with NF.

I noted above, in definition 5, that we have to make sure that our comprehension axioms are only those formulæ which become axioms of $\mathrm{T\mathbb{Z}T}$, lest we get Russell-style paradoxes. It might be worth thinking a bit about how one might cautiously relax this restriction to admit some more comprehension axioms. There is an analogue of *strongly cantorian* and altho' one obviously cannot allow the class of analogue-stcan sets to be a set (for the usual reasons concerning the Burali-Forti paradox) there doesn't seem to be any objection to the collection of *finite* analogue-stcan sets being a set.

Since "x is wellfounded" can now be captured by a formula that is stratifiable-mod-n and separation for wellfounded sets is safe for many expressions we should sort that out. In this next section we consider the property "$\iota^n \restriction x$ exists" which is stratifiable-mod-n.

7 Modulo-n analogues of *strongly cantorian*

7.1 Analogues in NF

In this section we work in NF.

"$\iota^n \restriction x$ exists" is an analogue of x *is strongly cantorian*. Lots of things to be said about it. Is this generalisation of strong cantorian-ness a good notion of small set? In the categorial sense, that is?

I noticed years ago the fact that altho' the existence of $\iota \restriction x$ clearly implies the existence of $\iota^n \restriction x$, the converse does not seem to hold. If $\iota^2 \restriction x$ exists then certainly $x \sqcup \iota``x$ is cantorian but that (and its analogues for $n > 2$) seem to be as far as one can go. It would appear that, in principle, there might be sets x s.t. $\iota^n \restriction x$ exists for some n but which are nevertheless not strongly cantorian.

The property "$\iota^n \restriction x$ exists" is inherited by subsets in the same way that strong-cantorianness is, so it is an *analogue* of 'strong cantorian' rather than a mere *weakening* of it—unlike 'cantorian' which (being a mere weakening) is not inherited by subsets in the same way.

The possible existence of such sets is worth noting in the present context, since for them one can prove an analogue of *subversion of stratification* for formulæ that are stratifiable-mod-n.

Subversion of stratification says that, if M is a strongly cantorian set, and ϕ an arbitrary formula, then $\{x \in M : \phi^M(x)\}$ exists. (ϕ^M is the result of restricting all quantifiers in ϕ to M.) The analogue here would say that, if $\iota^n \restriction M$ exists and ϕ is stratifiable-mod-n, then $\{x \in M : \phi^M(x)\}$ exists. Of course this will hold in $\mathrm{TC}_n\mathrm{T}$ ...which may be the correct setting for this observation: $\mathrm{TC}_n\mathrm{T}$ has subversion of stratification for x s.t. $\iota^n \restriction x$ exists, in the sense that the following holds.

Remark 6 *If $\iota^n \restriction M$ exists, and ϕ is stratifiable-mod-n then $\{y \in M : \phi^M(y)\}$ exists.*

Proof: ■

Should really write out a proof.

Just as subversion for strongly cantorian sets gives us interpretations into (extensions of) NF of fully unstratified set theories, subversion for sets x for which $\iota^n \restriction x$ exists will give us interpretations into (extensions of) NF of set theories satisfying syntactic contraints correspondingly less onerous than full stratification.

Subversion of stratification enables us to cutely finitise the restriction of the scheme of Δ_0 separation to formulæ that are stratifiable-mod-n. We know how to finitely axiomatise stratifiable Δ_0 separation (see the second edition of the monograph [6]), and we can get full Δ_0 separation from that axiomatisation simply by adding the existence of $\iota \restriction x$ for all x. The obvious thing to do is augment the kit of rudimentary functions by adding a new rudimentary function which gives $\iota^n \restriction x$, and then rely on subversion.

Does this open up a vein of novel, more delicate, relative consistency proofs? Possibly, but not if we are adopting an axiom of infinity: the assumption that there is an (infinite) x s.t. $\iota^n \restriction x$ exists is as strong as the assumption that there is an infinite strongly cantorian set. This triviality is worth minuting because we will make use of it elsewhere (see p. 6).

Remark 7
(i) If x is a wellorderable set s.t. $\iota^n \restriction x$ exists then x is strongly cantorian.
(ii) If there is an infinite x and a concrete n such that $\iota^n \restriction x$ exists then the axiom of counting holds.

Proof:
(i) If x is a wellorderable set s.t. $\iota^n \restriction x$ exists then the order type of any worder of x is certainly going to be less than all of Ω, $\Omega_1 \ldots$[6], so we can assume without loss of generality that x is an initial segment X of the ordinals. This means that $\iota^n \restriction X$ exists, and that in turn means that $T^n \restriction X$ exists, and that in turn means that we can prove by induction on the ordinals that $T^n \restriction X$ is the identity. So, for every $\alpha \in X$, $T^n\alpha = \alpha$. For every ordinal α (and so in particular for every $\alpha \in X$) we have $\alpha = T\alpha \vee \alpha < T\alpha \vee \alpha > T\alpha$. The second disjunct implies (apply T to both sides) $T\alpha <^2 \alpha$ giving $\alpha < T\alpha < \ldots T^n\alpha$ contradicting $T^n\alpha = \alpha$; the third disjunct is refuted similarly. So $T \restriction X$ exists beco's it is the identity, so $\iota \restriction X$ exists as well.
(ii) This property "$\iota^n \restriction x$ exists" is preserved by power set as well as by subset, so if there is even one infinite set which has it then $\mathbb{N}$ will have it as well. (Just as: $\mathbb{N}$ is strongly cantorian if there is even one infinite strongly cantorian set). But $\mathbb{N}$ is wellordered, so we can apply part (i). ■

The other direction (inferring "$\iota^n \restriction \mathbb{N}$ exists" for arbitrary concrete n from the axiom of counting) is easy. Thus, for every (concrete) n, the axiom of counting is equivalent modulo NF to a formula that is stratifiable-mod-n. If ϕ is, for each n, equivalent (modulo NF) to something that is stratifiable-mod-n must it be (NF)-invariant? No, because the property of being a wellfounded set is, for each n, equivalent (modulo NF) to something that is stratifiable-mod-n but is not invariant. But what about closed formulæ? No, that doesn't work either, as we shall see below.

Let Mac_n be Mac with separation restricted to formulæ that are Δ_0 and stratifiable-mod-n. Analogues of the result in [10] to the effect that Mac +

[6] Ω is the order type of the set of ordinals; $\Omega_1 = T\Omega$, and so on.

TCl can be interpreted into KF can be obtained, saying that Mac_n + TCl can be interpreted into KF, but these results are weaker than the result in [10]. However these refined constructions could turn out to be useful should there turn out to be theories of the form $\mathrm{Mac}_n \cup \{A\}$ (where A is some formula not a theorem of Mac). However no such examples leap to mind. Not to the authors' mind anyway: ∃NO might have sounded like a starter but is is inconsistent with the existence of $\iota^n \restriction x$ for all x. (This last follows from remark 7 part (i).)

The upshot of this is that ∃NO is incompatible with Mac_n, the point being that $\iota^n \restriction$the representative set of wellorderings would exist and that the quotient would be strongly cantorian.

Lemma 1 *For all concrete n and k, $(\forall x)(\iota^n \restriction x$ exists) implies $(\forall x)(\iota^{n\cdot k} \restriction x$ exists).*

Proof: We know that RUSC(R) always exists, so $\mathrm{RUSC}^k(R)$ exists for all R and all concrete k, so $\mathrm{RUSC}^k(\iota^n \restriction x)$ exists and so $\iota^n \restriction x$ composed with $\mathrm{RUSC}^n(\iota^n \restriction x)$ exists, and that is $\iota^{n\cdot 2} \restriction x$. And so on for all the other multiples of n. ■

7.2 Ambiguity in $\mathrm{TC}_n\mathrm{T}$

Take a simple example: $\mathrm{TC}_2\mathrm{T}$. Since every formula that is stratifiable-mod-4 is also stratifiable-mod-2 we can assert in the language of $\mathrm{TC}_2\mathrm{T}$ that the `yin` collection that would be the quartic Russell class $\{x : x \not\in^4 x\}$ exists. Ambiguity for formulæ that are stratifiable-mod-4 would then say that the corresponding `yang` set exists. See the discussion on page **??**.

It might be an idea to write out a proof that the quartic Russell classes $\{x : x \not\in^4 x\}$ cannot both exist.

If we are right about all ambiguity^n schemes being of equal consistency strength then it should be easy to prove the consistency of $\mathrm{TC}_n\mathrm{T}$ + Ambiguity for formulæ that are stratifiable-mod-$(m \cdot n)$ relative to $\mathrm{TC}_n\mathrm{T}$. Yeah right.

7.3 CO models for $\mathrm{TC}_n\mathrm{T}$

It is simplicity itself to cook up a CO model of (the version of) $\mathrm{TC}_n\mathrm{T}$ that corresponds to AST. (For definition of AST and more on CO models in general see [8].) Let $\langle \mathbb{N}, E\rangle$ be the standard Oswald model. Define a new relation E' on $\mathbb{N}$ by

$$2n\, E'\, (2m+1) \text{ iff } n\, E\, m$$

and

$$(2n+1)\, E'\, 2m \text{ iff } n\, E\, m.$$

That way even numbers are `yin` and odd numbers are `yang`. I think the double Russell class will turn out to contain precisely the wellfounded sets. . . but this will need to be checked. It's clear how to do the same for $\mathrm{TC}_n\mathrm{T}$ for $n > 2$. You partition $\mathbb{N}$ into the n residue classes mod n and you say that i is a member of j in the new sense if $i + 1 \equiv j$ mod n and $(i \texttt{ DIV } n)\ E\ (j \texttt{ DIV } n)$.

Of course there is nothing special about E. We can do this for any Oswald model at all.

What we might be able to do is get a model of the AST version of $\mathrm{TC}_2\mathrm{T}$ with a Boffa antiatom in one lobe but not in the other. It might be an instructive exercise to write this out in some detail.

We'll have two copies of $\mathbb{N}$: `yin` naturals and `yang` naturals. And we'll put a Boffa antiatom into level `yang` but not into level `yin`. In n is a `yin` natural and m a `yang` natural then we ordain than m is a member of n in the new sense iff $m\,E\,n$, where E is the membership relation of the Oswald model. Membership of `yang` naturals echoes the construction of CO models containing moieties. You look at `yang` naturals mod 4: that is to say, peel off the two least significant bits of a `yang` natural m and use them as a flag, which of course is $0, 1, 2$ or 3.

> If the flag is 0 then we say n belongs to m in the new sense iff the nth bit of the truncation is 1;
>
> If the flag is 1 then we say n belongs to m in the new sense iff the nth bit of the truncation is 1;
>
> If the flag is 2 then we say n belongs to m in the new sense iff (the nth bit of the truncation is 1 iff n belongs to the complement of the Boffa antiatom);
>
> If the flag is 3 then we say n belongs to m in the new sense iff (the nth bit of the truncation is 1 iff n belongs to the Boffa antiatom).

But questions of whether or not any given `yin` n belongs to the `yang` Boffa antiatom are answered by examining whether the Boffa antiatom is a member of n. And membership of `yang` sets in `yin` sets is unproblematic.

7.4 Generalise a Result of Specker?

Specker shows that in the situation where our language admits an automorphism $*$ of order 2, a conjunction of finitely many assertions of the form $\phi \longleftrightarrow \phi^*$ is another expression of that form. See Chad Brown's discussion of this question. Can we do anything similar here? Does it matter?

8 Applications to Duality

The special case of stratification-mod-n which will concern us here is $n = 2$. The context throughout this section is NF.

Definition 6 *The dual $\widehat{\phi}$ of a formula ϕ is the formula obtained from ϕ by replacing all occurrences of '$\in$' in ϕ by '$\notin$'.*

It has been known for some time that $\phi \longleftrightarrow \widehat{\phi}$ is a theorem of NF whenever ϕ is a closed stratifiable formula. Permutation models can be found in which $\phi \longleftrightarrow \widehat{\phi}$ *fails* for some unstratifiable ϕ, but it remains an open question whether or not there are models in which $\phi \longleftrightarrow \widehat{\phi}$ *holds* for all ϕ. The natural conjecture is that there should be such models.

We do not prove the full conjecture here but we can prove the relative consistency of the scheme $\phi \longleftrightarrow \widehat{\phi}$ at least for all ϕ that are stratifiable-mod-2. This will be theorem 6 below, and it is the principal aim of this section to prove it.

However, in preparation for theorem 6 we need to do a lot of bush-clearing in regard to NF's theory of permutations (and specifically involutions) of V, and this necessitates a few subsections of prolegomena.

First we reflect that the duality scheme might in principle be witnessed by the existence of an antimorphism.

Definition 7

An **antimorphism** *is a permutation τ of V satisfying* $(\forall x, y)(x \in y \longleftrightarrow \tau(x) \notin \tau(y))$.

An antimorphism that is an involution is a **polarity**.

Clearly if there is an antimorphism then duality follows.

8.1 Transversals, Partitions, Conjugacy and the Axiom of Choice for sets of pairs

Definition 8 *A* **transversal** *for a disjoint family is a set that meets every member of the family on a singleton.*

Bowler [1] has found an injection i from the set of pairs into the set of singletons: send $\{x, y\}$ to $\{(x \times y) \cup (y \times x)\}$. This enables us to infer (2) from (1):

1. Every set of disjoint pairs has a choice function;
2. Every set of pairs has a choice function.

Let P be a set of pairs. We desire a choice function for it, but we know only (1)—not (2). The set

$$\{p \times i(p) : p \in P\}$$

is a family of disjoint pairs and therefore, by (1), has a choice function, f. We can recover a choice function f^* for P by $f^*(p) =: \texttt{fst}(f(p \times i(p)))$. ■

We will also need the equivalence of (3) and (4):

3. Every partition of V into pairs has a choice function;
4. Every set of disjoint pairs has a choice function.

If we are given a set of pairs we can make disjoint copies of it by the trick we used earlier. In fact—by using an i whose range is a moiety[7] of singletons—we can ensure that the sumset $\bigcup P$ of the disjoint family P of pairs we construct by this method has a complement that is the same size as V. The complement $V \setminus \bigcup P$ therefore has a partition $\mathbb{P}'$ into pairs. Then $P \cup \mathbb{P}'$ is a partition of V into pairs. Any selection set for this partition will give us a choice function for the partition we started with.

Two more propositions:

5. Whenever we partition V into pairs we get the same number of pairs;

6. Whenever we partition V into pairs the two partitions are conjugate.

It turns out that (6) is equivalent to AC_2. (I mention 5 only as a foil, lest a reader think i'm talking about 5 when i am in fact talking about 6).

• $AC_2 \to 6$

Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two partitions of V into pairs. By AC_2 we have a selection set S for $\mathbb{P}_1$ and $\mathbb{P}_1$ is obviously a bijection between S and $V \setminus S$. So $|S| = |V|$ and $|\mathbb{P}_1| = T|V|$. Naturally we argue for $\mathbb{P}_2$ in the same way. So there is a bijection π between $\mathbb{P}_1$ and $\mathbb{P}_2$. For each $p \in \mathbb{P}_1$ there are precisely two bijections between p and $\pi(p)$ and we use AC_2 to pick one. The union of all such chosen bijections is a permutation conjugating $\mathbb{P}_1$ and $\mathbb{P}_2$.

• $6 \to AC_2$.

Assume 6. If $\mathbb{P}$ is a partition of V into pairs then by 6 it will be conjugate to the partition $\{\{x, V \setminus x\} : x \in V\}$. That is to say, there is a permutation π of V such that, for all $p \in P$, π“p is a pair $\{x, V \setminus x\}$. But clearly the partition $\{\{x, V \setminus x\} : x \in V\}$ has a choice function f (“pick the element that contains $\emptyset$”) so the choice function for $\mathbb{P}$ that we want is $p \mapsto \pi^{-1}(f(\pi\text{“}p))$. ■

So we have established:

Remark 8 *The following are equivalent:*
Every set of pairs has a choice function (AC_2);
Every set of disjoint pairs has a choice function;
Any two partitions of V into pairs are conjugate;
Every partition of V into pairs has a choice function.

If we partition V into pairs how many do we get? No more than $T|V|$ (by this result of Bowler's in [1]) but can we get fewer? We must try to connect this with the question of whether or not $|V|$ is decomposable.

Definition 9 *An involution with no fixed points and no transversal set is* **bad**.

The thinking behind this perjorative notation is that an involution without fixed points is a partition of V into pairs and will have a transversal as long

[7]Moieties are supposed to be the same size as V

as AC_2 holds. If there are bad involutions then AC_2 fails. We will make much use of the fact that an involution without fixed points can be thought of as a partition of V into pairs.

I assume the reader can work out for themselves that every polarity is a bad involution.

Lemma 2 *Any two involutions-without-fixed-points whose corresponding partitions-of-V-into-pairs have transversals are conjugate.*

Proof:

First we establish that if $\mathcal{T}$ is a transversal for a partition $\mathbb{P}$ of V into pairs then its cardinality is $|V|$. Clearly $|\mathbb{P}| = T|\mathcal{T}|$, since we can send each piece of $\mathbb{P}$ to the unique singleton $\subset \mathcal{T}$ that meets it. Observe that there is a bijection between $\iota``V$ and $\mathbb{P} \times \{0, 1\}$, as follows. For each x there is a unique $p_x \in \mathbb{P}$ with $x \in p_x$. If $x \in \mathbb{P}$ we send $\{x\}$ to $\langle p_x, 0\rangle$; if $x \notin \mathbb{P}$ we send $\{x\}$ to $\langle p_x, 1\rangle$.

Finally if π_1 and π_2 are two involutions-without-fixed-points equipped with transversals $\mathcal{T}_1$ and $\mathcal{T}_2$, then not only do we have $|\mathcal{T}_1| = |\mathcal{T}_2| = |V|$ but π_1 and π_2 are conjugate, as follows. $\mathcal{T}_1$ and $\mathcal{T}_2$ are in bijection, by a map π^*, say. Any such π^* can be extended to a permutation π of the universe by adding all the ordered pairs $\langle \pi_1(x), \pi_2(\pi^*(x))\rangle$ for $x \in \mathcal{T}_1$.

■

Some minor points:

(i) The proof of lemma 2 as given above tells us nothing about permutations that conjugate π_1 and π_2 beyond the fact that they exist. However the construction is effective and can be mined for more information. In lemmas 3 and 8 we consider a particular case in which we need more information and we go into more detail.

(ii) Notice that in lemma 2 the assumption on the two involutions is that the corresponding partitions have transversals. It is not the weaker assumption that the corresponding partitions are the same size. *Might* it be possible to prove in NF that any two partitions of V into pairs are the same size...? After all—as mentioned above—Nathan Bowler [1] has shown us a proof in NF that there are as many pairs as singletons.

Sadly no, not unless NF $\vdash AC_2$.

Remark 9 *If whenever σ and τ are two involutions-without-fixpoints whose two partitions of V into pairs are of the same cardinality then σ and τ are conjugate, then AC_2 follows.*

Proof:

Let π be a set of pairs without a choice function. Without loss of generality the pairs are disjoint. Take the disjoint union of $\bigcup \pi$ with V. The result is the same size as V and can be canonically split into pairs using c (on the copy of V) and π (on the copy of $\bigcup \pi$). Copy this over into a partition of V into pairs. We have $T|V|$-many pairs, which is the same as the number of pairs in the partition corresponding to c. So—*if any two partitions of V into the same* *number of pairs are conjugate*—then this π must have a choice function. But π was arbitrary. ■

We now need Nathan Bowler's fruitful idea of a *universal involution*. That in turn relies on a notion of permutation morphism due to Bowler:

Definition 10
For permutations σ and τ of sets X and Y, a **map of permutations** *from σ to τ is a function $\pi : X \to Y$ such that $\pi \cdot \sigma = \tau \cdot \pi$.*

If π is injective, we call it an **embedding of permutations**.

An involution is **universal** *if every involution embeds into it.*

We will write "$\sigma \leq_B \tau$." to say that there is an embedding of permutations from σ to τ.

In all the cases of interest to us we have $X = Y$ but we shouldn't forget that Bowler's definition is more general.

Think of a permutation as a digraph wherein every vertex has indegree one and outdegree one, and loops at vertices are allowed. An embedding-of-permutations must send n-cycles onto n-cycles, and when you look at it like that the Cantor-Bernstein theorem becomes much more obvious.

For the moment we need definition 10 only for involutions, and we will speak of *involution-embeddings* or *embeddings of involutions*. In due course we will prove (lemma 4) that there are universal involutions, and give examples. We do not address the question of whether there are permutations that are universal for other classes of permutations, interesting tho' that question is. It looks quite hard.

[There is a case for a digression on this topic. There probably is a universal permutation, and it has $T|V|$ n cycles for every n (and $T|V|$ infinite cycles too). No coincidence that this is the type of $j\sigma$ for σ any permutation with an infinite cycle. Take a wee bit of time to think how much AC one needs to prove that this cycle type has only one conjugacy class. It's presumably the principle i have elsewhere called GC.

This is probably also a theatre within which one can use the theorem of Bowler-Forster [1] that if $|X| = |X|^2$ then the symmetric group on X has no normal subgroups of small index. So every permutation is a product of *universal* involutions, or of involutions from any congruence class, co's each of these congruency classes (presumably!) generate the whole group.]

We will need the following analogue of Cantor-Bernstein for embeddings-of-permutations.

Lemma 3 *(Bowler)*
If σ is a permutation of X and τ a permutation of Y with $\sigma \leq \tau \leq \sigma$ then σ and τ are conjugate.

Proof: (Bowler, edited by tf)

Suppose $\sigma \leq \tau$ in virtue of $\rho : X \hookrightarrow Y$ and $\tau \leq \sigma$ in virtue of $\pi : Y \hookrightarrow X$. Consider the map $\mathcal{P}(X) \hookrightarrow \mathcal{P}(X)$ defined by $S \mapsto X \setminus \rho``(Y \setminus \pi``S)$. By Tarski-Knaster this map has a least fixed point, which we will call P. Then the map $X \hookrightarrow Y$ given by $\pi \restriction P \cup \rho^{-1} \restriction X \setminus P$ conjugates σ to τ. ■

Notice that the map that conjugates σ and τ has a stratifiable definition in terms of them, so if they are definable it is too, and so is its least fixed point. It won't matter that there is a least fixed point, but it will matter that there is a fixed point that is definable in terms of ρ and π, and the lfp is one such.

In fact—for the moment—we will need lemma 3 only for involutions.

Corollary 1 *Any two universal involutions of V are conjugate.*

■

We observe without proof that if π is an embedding of permutations from σ to τ then $j(\pi)$ is an embedding of permutations from $j(\sigma)$ to j(τ).

Lemma 3 is telling us that the intersection of the quasiorder $\leq_B$ and its converse $\geq_B$ is the equivalence relation of conjugacy. This makes it sort-of OK to abuse notation by additionally using '$\leq_B$' to denote the partial ordering induced on the quotient. The quotient is a directed poset because of disjoint unions of copies of V. Is it an upper semilattice? It certainly supports a $+$ operation, but whether or not $[\sigma] + [\tau]$ is the sup of $[\sigma]$ and $[\tau]$ is another matter!

The following elementary facts will loom large.

Remark 10
(i) Conjugacy is a congruence relation for j;
(ii) j is $\leq_B$-order-preserving.

Proof:

(i) is obvious (and skew-conjugacy, too, is a congruence relation for j, tho' that is not as important here).

For (ii) Observe that if π is an embedding of permutations from σ to τ then $j(\pi)$ is an embedding of permutations from $j(\sigma)$ to j(τ). ■

We will need this in the proof of the second part of lemma 4.

We begin by giving some examples of universal involutions of V.

Lemma 4 *(Bowler unpublished)*
For all i, $j^i(c)$ is a universal involution.

Proof:

First we prove that $j(c)$ is universal.

There are bijections $V \longleftrightarrow \{x : \emptyset \not\in x\}$; in what follows fix θ to be one of them—it won't matter which.

For any involution σ of any set X we define an embedding of involutions π from σ to $j(c)$ by

$x \mapsto j(\theta)(x) \cup j(c \cdot \theta)(\sigma(x))$.

The function π is injective, with left inverse $y \mapsto j(\theta^{-1})(\{z \in y : \emptyset \notin z\})$.

To see that π is a map of involutions from σ to $j(c)$ we calculate as follows:

(1)	$(j(c) \cdot \pi)(x) = j(c)(\pi(x))$	Expand $\pi(x)$ to get
(2)	$= j(c)[j(\theta)(x) \cup j(c \cdot \theta)(\sigma(x))]$	distribute jc over $\cup$ to get;
(3)	$= j(c)(j(\theta)(x)) \cup jc \cdot j(c \cdot \theta)(\sigma(x))$	
(4)	$= j(c \cdot \theta)(x) \cup j(c \cdot c \cdot \theta)(\sigma(x))$	
(5)	$= j(c \cdot \theta)(x) \cup j(\theta)(\sigma(x))$	reorder the set unions to get
(6)	$= (j\theta)(\sigma(x)) \cup j(c \cdot \theta)(x)$	which gives
(7)	$= (j\theta)(\sigma(x)) \cup j(c \cdot \theta)(\sigma(\sigma(x)))$	beco's σ is an involution;
(8)	$= \pi(\sigma(x))$	

The fact that (6) $\rightarrow$ (7) relies on σ being an involution promises to complicate the endeavour to find universal permutations of other orders.

The reordering between (5) and (6) suggests that for permutations that are universal for permutations of order n the corresponding π constructor will take values that are unions of n terms.

For the main result we argue as follows.

Clearly any involution into which a universal involution can be embedded is also universal, and any involution conjugate to a universal involution is again universal.

Since $j(c)$ is universal, there is an embedding of c into $j(c)$. This lifts to embeddings of $j^i(c)$ into $j^{i+1}(c)$, and composing these embeddings we get embeddings of $j(c)$ into $j^i(c)$ for any $i \geq 1$. Thus $j^i(c)$ is universal for any $i \geq 1$.

■

(It might be an idea to properly write out a proof that j lifts in this way. We should also check that $j(c)$ and $j^2(c)$ are conjugate and do it *by hand* as it were, so that we can tell whether or not they are conjugated by anything definable.)

So $j(c)$ is a universal involution. In the medium term we are going to be interested in the possibility of permutations that are universal for other classes of permutations. For example (the simplest next case) is there a universal permutation of order 3? It looks as if we'd have to have a particular candidate in mind, and there is nothing obvious in the way $j(c)$ was an obvious candidate for a universal involution, $j(c)$ is rather special beco's it arises from the logic rather than the set theory. There is nothing of three-character that arises from the logic! I don't think the prospects are good. There was no obvious reason why there should be universal involution, tho' there was an obvious thing to try. Similarly there doesn't seem to be any obvious reason why there should be a universal permutation of order p, and nor is there an obvious suspect to try.

Every permutation is a product of involutions; is every permutation a products of universal involutions?

For the moment we record the following.

Remark 11 *There is a universal involutions-without-fixpoints.*

Proof:

Of course (as we have seen) if AC_2 holds then there is a *unique* conjugacy class of involutions-without-fixpoints: *all* involutions-without-fixpoints would be universal. However we are not going to assume AC_2. Recall that $j(c)$ is a universal involution. In particular, if π is an involution without fixpoints there is a permutation-embedding from π into $j(c)$, and any such embedding must send π into that part of $j(c)$ that consists of pairs not singletons. Let X be the union of all the pairs in $j(c)$ ($j(c)$ tho'rt of as a set of pairs and singletons). It's easy to check that there are $|V|$-many sets that are not closed under complementation (*) so X is the same size as V and *that* part of $j(c)$ can be copied over to V to give us a permutation τ of V that has no fixed points. And the construction of τ ensures that π above embeds into it. And τ is definable!

For (*) reflect that $\{x : V \in x \wedge \emptyset \notin x\}$ is a subset of the collection of sets not closed under complementation, so it will suffice to show that it is of size $|V|$. But it's a moiety of a moiety, in the sense that $B(V)$ is a moiety and provably the same size as V, and its members fall into one two pieces depending on whether or not they contain $\emptyset$, and these two pieces are of course the same size as each other and the same size as V. ■

This merits some reflection. This τ gives us a definable partition of V into pairs which is a kind of ϵ-object for bad pairs: if there any counterexamples to AC_2 then τ is one of them. I don't think this is going to help us prove AC_2, but it is quite striking. I think the set $\{\{x, c``x\} : x \neq c``x\}$ is another such. But there's no significance to that: the set of all pairs is another such. Duh.

[In the medium term we are going to be interested in finding automorphisms thare are not involutions, that have other cycle types. The cycle types we have to consider are actually quite special. Every automorphism is a fixed point for j, and that tells us quite a lot about the cycle type. For any n, the number of things belonging to n-cycles is either $|V|$ or 0. If there is an n-cycle and $m|n$ then there is an m-cycle. For these purposes every natural number divides the order of an infinite cycle. This give us $\omega+1$ cycle types we have to worry about, one for each cantorian natural and one for the presence of infinite cycles. We are interested in "universal" permutations of these cycle types and not in any other. It would be nice to show that each of these flavours has a "universal" (top) type. There are some details to be nailed down about the cantorian nature of all these cycles but that is for later.

Is there anything analogous one can say about the cycle types of antimorphisms?]

9 Working towards Antimorphisms

We start with the observation that no antimorphism can have any odd cycles. One might think this is obvious but it isn't. Things are complicated by the fact that a cycle need not be cantorian!

REMARK 12 *No antimorphism can have an odd cycle.*

Proof:

What one *wants* to say is this: suppose τ is an antimorphism and x belongs to a $(2n+1)-$cycle. One then has

$$x \in x \longleftrightarrow \tau^{2k+1}(x) \notin \tau^{2k+1}(x),$$

so we cannot have $\tau^{2k+1}(x) = x$. Unfortunately the biconditional one obtains is not that, but instead is

$$x \in x \longleftrightarrow \tau^{2k+1}(x) \notin \tau^{2Tk+1}(x).$$

However one can do the following. Suppose x belongs to an o-cycle ('o' for '$o$dd'). We seek a natural number k such that k and Tk are both divisible by o. We then have $x \in x \longleftrightarrow \tau^k(x) \notin \tau^{Tk}(x)$. Now, since k and Tk are both divisible by o, we have $\tau^k(x) = \tau^{Tk}(x) = x$ whence $x \in x \longleftrightarrow x \notin x$ and we have the contradiction we desired. Q: But what is k? A: LCM$(o, T^{-1}o)$.

■

Two points:

(i) Notice that we have not assumed that τ is a set; so this holds for external antimorphisms as well.

(ii) The proof we have given was complicated by the need to allow for noncantorian cycles. Might it be possible to prove that any cycle of an antimorphism is in fact cantorian? I suspect it is, but it might be quite fiddly. Sounds as if we ought to be able to prove that any cycle of an $\in$-automorphism must be cantorian.... We do at least know that the order of any $\in$-automorphism is cantorian and every set of $\in$-automorphisms is stcan. .

Recalling that one of the aims of this investigation is to understand antimorphisms we remind ourselves that τ is an antimorphism iff $\tau = j(\tau) \cdot c$. This fact gives us an interest in permutations of the form $j\tau \cdot c$ and, in particular, in how the cycle type of τ controls the cycle type of $j\tau \cdot c$. It's quite easy to see how the cycle type of τ controls the cycle type of $j\tau$. We remark (without proof for the moment):

if τ has an n-cycle, $j\tau$ has a Tn-cycle;
if τ has infinite cycles, $j\tau$ has cycles of all sizes;
if τ has cycles of arbitrarily large finite sizes, then $j\tau$ has infinite cycles;
if all τ-cycles have lengths in $I \subset \mathbb{N}$ with I finite then, for n large enough, $j^n\tau$ has cycles of all sizes that divide $LCM(I)$. (With a few 'T's scattered around)

How many fixed points does $j\tau$ have? Clearly j of a transposition (a, b) has $|V|$-many fixed points (every subset of $V \setminus \{a, b\}$ is fixed). If τ is a bad involution, how many fixed points hath $j\tau$?

... $j\tau \cdot c$ is a lot harder. Lemma 5 is a taster.

Lemma 5 *AC_2 implies that, for all permutations τ, $j\tau \cdot c$ has fixed points iff τ has no odd cycles.*

Proof:

R $\rightarrow$ L

Suppose X is a fixed point for $j\tau \cdot c$. Then, for each τ-cycle C, we must have $\tau``(X \cap C) = C \setminus X$ and that means that $|C|$ must be even (or infinite). This direction does not need AC_2.

L $\rightarrow$ R

This direction needs AC_2. Suppose τ has no odd cycles. Each τ-cycle splits into precisely two τ^2 cycles. Use AC_2 to pick, for each τ-cycle, one of the two τ^2-cycles into which it splits. The union of the set of chosen τ^2-cycles is a fixed point for $j\tau \cdot c$.

■

The converse is true too. Suppose τ is a permutation with no odd cycles, and assume the consequent. Then $j\tau \cdot c$ has a fixed point. τ itself of course has no fixed point. The fixed point for $j\tau \cdot c$ is a transversal for τ!

Another fairly easy observation in the thread of cycle-type-of-τ-controlling-cycle-type-of-$j\tau \cdot c$ is that...

Remark 13

(1) If τ is of order $2n$ then $j\tau \cdot c$ is of order $T2n$;
(2) If τ is of order $2n+1$ then $j\tau \cdot c$ is of order $T4n+2$;
(3) If τ has a $\mathbb{Z}$-cycle then so does $j\tau \cdot c$.

Proof:

It's obvious that if τ is of order n then $j\tau$ is of order Tn, but composing with c embroils us in slightly more work.

(1) Suppose τ is of order $2n$. c commutes with $j\tau$, so in $(j\tau \cdot c)^{T2n}$ we can rearrange to make all the cs adjacent and all the $j\tau$ adjacent so they all cancel.

(2) What if the order of τ is odd? A similar calculation shows that if τ is of order $2n+1$ then $(j\tau \cdot c)^{T2n+1}$ becomes, with rearrangement-followed-by-cancellation, $(j\tau)^{T2n+1} \cdot c = j(\tau^{2n+1}) \cdot c = \mathbb{1} \cdot c = c$. This is not the identity! However, its square is.

(3) Let x belong to a τ $\mathbb{Z}$-cycle and consider $\{x\}$.

■

Of some interest will be the sequence of permutations: $\mathbb{1}$, c, $jc \cdot c$, $j^2c \cdot jc \cdot c$ ..., where c is the complementation permutation. The superscripts are all small (they are all concrete numerals, in fact), so—rather than persist with the more general but slightly unwieldy $H(c, i)$ notation of [4] introduced above—we will revert to the simpler (original) notation of Henson, in which these permutations are written 'c_i', thus: $c_1 := c$; $c_{i+1} := j(c_i) \cdot c$.

Suppose AC_2 fails, so that there are ("bad") involutions with neither fixed points nor transversals. If τ is a bad involution then, by remark 15, $j\tau \cdot c$ has no fixed points. (It might have a transversal and not be bad...). And, if there are bad involutions, then any involution that is maximal among involutions without fixed points will be bad.

What we might be able to do is this: If AC_2 fails then there is a bad involution, so any involution that is universal for involutions-without-fixpoints (uiwf) is bad. So all we need to show is that if τ is uiwf then $j\tau \cdot c$ is uiwf. The operation $\tau \mapsto j\tau \cdot c$ respects conjugacy. (I don't think it is $\leq_B$-monotone). Certainly if $\pi^{-1}\tau\pi = \sigma$ then $(j\pi)^{-1}(j\tau) \cdot c \cdot (j\pi) = (j\sigma) \cdot c$. That doesn't sound obviously impossible. If τ has no fixed points and no transversals then $j\tau \cdot c$ has no fixed points (any fixed point would be a transversal for τ). It is true that there doesn't seem to be anything to prevent $c \cdot j\tau$ having a transversal, so there is work to do. The point of all this, of course, is that if there is a uiwf τ s.t. $c \cdot j\tau$ is also uiwf then we get a permutation model containing an antimorphism. It a classic fixed-point-for-a-tro-obtained-by-permutations–situation.

Recall that we use lower-case $\mathfrak{fraktur}$ characters for variables ranging over conjugacy classes.

Consider the poset $\langle \mathfrak{P}, \leq_B \rangle$ of conjugacy classes of involutions-without-fixpoints. It is closed under j (which is order-preserving) and $\sigma \mapsto j\sigma \cdot c$ (which isn't). It has a top element which is the conjugacy class of universal involutions; call it $\mathfrak{c}_1$. There are also (i) the conjugacy class (call it $\mathfrak{c}_3$) of universal involutions-without-fixpoints, and (iii) the conjugacy class of the involutions that have a transversal, call it $\mathfrak{c}_2$. Evidently $\mathfrak{c}_2 \leq_B \mathfrak{c}_3$. AC_2 is simply the assertion that $\mathfrak{c}_3 = \mathfrak{c}_2$. And if they are the same then there is only one conjugacy class of involutions-without-transversals. And that's an iff. What happens if AC_2 fails? Then there is more than one conjugacy class. Can we prove that $\mathfrak{c}_2$ is always the bottom element? If there are involutions-without-fixpoints that have fewer than $T|V|$ pairs (and there might be, for all i know) then the answer would be: no!

If AC_2 fails then the congruence class $\mathfrak{c}_1$ of universal involutions consists of bad involutions, and there is the $\leq_B$-minimal class $\mathfrak{c}_2$ which consists of good permutations. In fact it's the equivalence class of c_1—in fact the equivalence class of all cs with odd subscripts. And it's a consequence of corollary 3 that the conjugacy class $\mathfrak{c}_1$ of universal involutions contains all the cs with *even* subscripts. $\pi \mapsto j\pi \cdot c$ swaps you back and forth between these two conjugacy classes. (This is how we know that $\pi \mapsto j\pi \cdot c$ is not $\leq_B$-order-preserving). Thus, among the conjugacy classes of involutions we find the conjugacy class of the c_{2n}

(which is maximum) and the conjugacy class of the c_{2n+1} (which is minimal). $\mathfrak{c}_2$ is $\leq_B$-mini*mal* but perhaps not minim*um*, since—for all we know—if AC_2 fails there may be involutions without fixpoints whose corresponding partitions are *smaller* than $\iota``V$. Indeed, to the best of my knowledge, no-one has ever proved that V is not the union of a wellordered(!) family of finite sets. So we should not expect to be able easily to exclude the possibility of partitioning V into fewer than $T|V|$ pairs.

$\langle \mathfrak{P}, \leq_B \rangle$ admits a $+$ operation, arising from disjoint union. Is it the join in the sense of the poset? Well, it will be if $|V|$ is indecomposible. But is it? What happens if $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$?

Consider the class **BINV** of those involutions that are universal or lack fixed points. **BINV** is closed under $\sigma \mapsto j\sigma \cdot c$, which makes it the correct place to search for fixpoints for $\sigma \mapsto j\sigma \cdot c$. We need a name for this function whose fixed points are antimorphisms. Is **BINV** the correct thing to examine? Or its conjugacy classes? Or perhaps its conjugacy classes in J_1?? The point being that if $j\sigma \cdot c$ and $j\tau \cdot c$ are J_1-conjugate then σ and τ are J_0-conjugate.

In fact this setting seems to be one in which various old festering problems appear and can perhaps be partially processed. Among the bad involutions are there any which have fewer than $T|V|$ pairs? Call such a permutation *small bad*. If there are any, is the collection of conjugacy classes of small bad permutations closed under $+$? This is related to the question of whether or not $|V|$ is decomposible. Clearly if σ and τ are both bad, so is $\sigma \sqcup \tau$. But if $|V|$ is indecomposible, then if $|\sigma \sqcup \tau| = T|V|$, one of σ and τ must also be of size $T|V|$. So might there be a universal small bad permutation?

Let us write 'J_0' for the symmetric group on V, and J_1 for $j``J_0$ (and so on). Thus the triviality is that c is in the centraliser $C_{J_0}(J_1)$ of J_1 in J_0. There is slightly more one can say about this that may be worth recording here.

Remark 14

$$C_{J_0}(J_1) \quad \subseteq \quad \{\sigma : (\forall x)(\sigma(x) = x \ \vee \ \sigma(x) = V \setminus x)\} \quad \subseteq \quad C_{J_0}(\{c, \mathbb{1}_V\})^8.$$

Proof: First inclusion:

Suppose $\sigma \in C_{J_0}(J_1)$. Let τ be any permutation whatever. Then

$$\tau``\sigma(x) = \sigma(x)$$

iff (commutativity)

$$\sigma(\tau``x) = \sigma(x)$$

[8] Actually one can spice this up quite a lot, by reflecting that the centraliser function is antimonotonic, so one can whack '$C()$' in front of each of these and then reverse all the arrows. I was sure i had written this out somewhere but i can't find it.

iff (because σ is a permutation)

$$\tau``x = x$$

So τ fixes $\sigma(x)$ setwise iff it fixes x setwise. But τ was arbitrary. It follows easily[9] that $\sigma(x)$ must be x or $V \setminus x$.

Second inclusion:

Assume $(\forall x)(\pi(x) = x \vee \pi(x) = c(x))$. We will show that $\pi \in C_{J_0}(\{c, \mathbb{1}_V\})$.

If $\pi(x) = c(x)$ then $\pi \cdot c(x) = x$ so $c \cdot \pi(x) = \pi \cdot c(x) = x$.

If $\pi(x) = x$ then $\pi \cdot c(x) = c(x)$ so $c \cdot \pi(x) = c(x) = \pi \cdot c(x)$.

■

Both these inclusions are proper: $\prod\limits_{x \in \iota``V} (x, V \setminus x)$ is a counterexample to the converse of the first inclusion. The second inclusion cannot be reversed because $J_1 \subseteq C_{J_0}(\{c, \mathbb{1}_V\})$.

Note that $(\exists \sigma)(y = \sigma``x)$ is an equivalence relation. Let us write it $\sim_1$, and let us write the equivalence class of x under $\sim_1$ (the orbit of x under J_1) as $[x]$. What we have shown is that, for each $\pi \in C_{J_0}(J_1)$ and for each x, π must either fix all members of $[x]$ or send them all to their complements. That is, we can code members of $C_{J_0}(J_1)$ by the equivalence classes whose members they fix. If we now identify $[x]$ and $[V \setminus x]$ by $\approx$ we see that $C_{J_0}(J_1)$ is precisely the additive part of the boolean ring on $(V/\sim_1)/\approx$.

I think i can prove that $C_{J_0}(\{\sigma : (\forall x)(\sigma(x) = x \vee \sigma(x) = V \setminus x)\}) = \{c, \mathbb{1}_V\}$. The L-to-R inclusion is obvious.

Suppose $a \neq b \neq (V \setminus a)$ and suppose σ is in the centraliser and sends a to b. Then it doesn't commute with the transposition $(a, V \setminus a)$.

Much of what we say below about c goes for any member of $C_{J_0}(J_1)$.

Lemma 6
(i) All the c_i are involutions;
(ii) All the c_i commute with each other.

Proof:

(i) We prove this by induction on i. Suppose c_i is an involution. $c_{i+1} = jc_i \cdot c$. So $(c_{i+1})^2 = (jc_i \cdot c)^2 = jc_i \cdot c \cdot jc_i \cdot c$. Now by the key triviality we can rearrange to $jc_i \cdot jc_i \cdot c \cdot c = \mathbb{1}$.

In fact this even shows that all products of the c_i are involutions.

(ii) We prove by induction on i that, for all j, c_i commutes with c_j.

[9] I suspect the proof that i am eliding is not constructively correct.

Case $i = 0$. $c_0 = c$ and c commutes with $j(\pi)$ for all π. But every c_j is $j(\pi) \cdot c$ for some π, and (compose with c on the right) $j(\pi) \cdot c \cdot c = j(\pi)$ and if we compose with c on the left we get $c \cdot j(\pi) \cdot c$ which, too, is $j(\pi)$ because c commutes with $j(\pi)$.

Now for the induction.

$$c_{i+1} \cdot c_j = j(c_i) \cdot c \ \cdot \ j(c_{j-1}) \cdot c$$

and the RHS simplifies to

$$j(c_i) \cdot j(c_{j-1})$$

which is

$$j(c_i \cdot c_{j-1})$$

which by induction hypothesis is

$$j(c_{j-1} \cdot c_i)$$

which is

$$j(c_{j-1}) \cdot j(c_i).$$

We now sprinkle a couple of cs judiciously—by the triviality we know can insert them anywhere—obtaining

$$j(c_{j-1}) \cdot c \ \cdot \ j(c_i) \cdot c$$

which is of course

$$c_j \cdot c_{i+1}.$$

■

Remark 15
Let σ and τ be involutions of V.

(1) Let τ be an involution without fixpoints. Then $\mathcal{T}$ is a transversal for τ iff $\mathcal{T}$ is a fixpoint for $j\tau \cdot c$;

(2) $\mathcal{T}$ is a fixpoint for σ iff $B(\mathcal{T})$ is a transversal for $j\sigma \cdot c$.

Proof:
(1) Think of τ as a partition of V into pairs. Then, if $\mathcal{T}$ is a transversal, $V \setminus \mathcal{T}$ (which is also a transversal) is precisely $\tau``\mathcal{T}$.

(2) A piece of [the partition] $j\sigma \cdot c$ is a pair $\{x, V \setminus \sigma``x\}$—which of course might be a singleton. If $\sigma(T) = T$ then, for all $x$, precisely one of $x$ and $V \setminus \sigma``x$ will contain T. That is to say, $\{x, V \setminus \sigma``x\} \cap B(\mathcal{T})$ is a singleton, so $B(\mathcal{T})$ is a transversal.

For the other direction ... if $B(\mathcal{T})$ is a transversal for $j\sigma \cdot c$ then, for all x, precisely one of x and $V \setminus \sigma``x$ contains T, which is to say that $\mathcal{T} \in x \longleftrightarrow \sigma(\mathcal{T}) \in x$. In particular let x be $\{\mathcal{T}\}$; then $\mathcal{T} \in \{\mathcal{T}\} \longleftrightarrow \sigma(\mathcal{T}) \in \{\mathcal{T}\}$, so $\sigma(\mathcal{T}) = \mathcal{T}$.

■

I thought this corollary followed but it doesn't.

Error Alert! This is not a corrollary

Corollary 2
τ is bad iff $j\tau \cdot c$ is bad.

Proof:

L $\to$ R: $j\tau \cdot c$ bad implies τ bad:

Suppose $j\tau \cdot c$ is bad. Then it has no transversals. In particular for no $\mathcal{T}$ is $B(\mathcal{T})$ a transversal, so for no $\mathcal{T}$ is $\mathcal{T}$ a fixpoint for τ.

Suppose $j\tau \cdot c$ is bad. Then it has no fixpoints. So τ has no transversals.

R $\to$ L: Now for τ bad implies $j\tau \cdot c$ bad.

τ has no fixpoint so $j\tau \cdot c$ has no transversal[10].

τ has no transversal. Suppose, *per impossibile*, that $j\tau \cdot c$ has a fixpoint, x. Then $x = V \setminus \tau``x$ which says that x is a transversal for τ.

■

The gap could be plugged if there were a way of constructing a fixpoint for an involution τ from a transversal for $j\tau \cdot c$.

The following corollary seems quite striking, but it hasn't borne any fruit just yet.

Corollary 3
(i) For any ultrafilter $\mathcal{U}$ on V, $B^n(\mathcal{U})$ is a transversal for c_{2n+1};
(ii) All the c_{2n+1} are conjugate;
(iii) For all $n \geq 1$, c_n is conjugate to c_{n+2}.

Proof:

(i) We do an induction on n. For the case $n = 0$ any ultrafilter is a transversal for c.

Suppose for the induction that $B^{n-1}(\mathcal{U})$ is a transversal for c_{2n-1}.

Consider

$$c_{2n+1}(A) \in B^n(\mathcal{U}).$$

By definition of B this is the same as

$$B^{n-1}(\mathcal{U}) \in c_{2n+1}(A)$$

Now $c_{2n+1}(A) = V \setminus (c_{2n}``A)$, so we can rewrite the displayed formula as

$$c_{2n}(B^{n-1}(\mathcal{U})) \notin A.$$

By induction hypothesis $B^{n-1}(\mathcal{U})$ is a transversal for c_{2n-1}, which is to say that $B^{n-1}(\mathcal{U})$ is a fixed point for c_{2n}. So rewrite '$c_{2n}(B^{n-1}(\mathcal{U}))$' as '$B^{n-1}(\mathcal{U})$'; this turns our formula-in-hand into

$$B^{n-1}(\mathcal{U}) \notin A$$

which (by definition of B) becomes

$$A \notin B^n(\mathcal{U}).$$

[10]No! It might have transversals that aren't B of any fixpoint for τ

So we have proved

$$c_{2n+1}(A) \in B^n(\mathcal{U}) \longleftrightarrow A \notin B^n(\mathcal{U})$$

...which is to say that $B^n(\mathcal{U})$ is a transversal for c_{2n+1}.

(ii) now follows by lemma 2.

(iii) By induction on n.

The case $n = 1$ we know from (ii).

For the induction step suppose π conjugates c_n to c_{n+2}, which is to say

$$\pi \cdot c_n \cdot \pi^{-1} = c_{n+2}$$

Lift by j:

$$j\pi \cdot j(c_n) \cdot (j\pi)^{-1} = j(c_{n+2})$$

compose both sides with c on the right:

$$j\pi \cdot j(c_n) \cdot (j\pi)^{-1} \cdot c = j(c_{n+2}) \cdot c$$

But c commutes with $(j\pi)^{-1}$ so we can rearrange the LHS, and $j(c_{n+2}) \cdot c = c_{n+3}$ on the RHS giving

$$j\pi \cdot \underline{j(c_n) \cdot c} \cdot (j\pi)^{-1} = c_{n+3}$$

Now $j(c_n) \cdot c$ (underlined) $= c_{n+1}$ giving

$$j\pi \cdot c_{n+1} \cdot (j\pi)^{-1} = c_{n+3}$$

as desired.

■

Also worth minuting is the fact that

Remark 16 *Conjugacy is a congruence relation for the operation $\pi \mapsto j\pi \cdot c$.*

Proof:

Suppose σ and τ are conjugate; so, for some π,

$\pi \cdot \sigma \cdot \pi^{-1} = \tau$;	Then whack it with j:
$j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) = j(\tau)$;	compose with c:
$j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) \cdot c = j(\tau) \cdot c$;	but c commutes with j of anything, giving:
$j(\pi) \cdot j(\sigma) \cdot c \cdot j(\pi^{-1}) = j(\tau) \cdot c$	

which says that $j(\sigma) \cdot c$ and $j(\tau) \cdot c$ are conjugate.

■

Notice that in this construction $j(\sigma) \cdot c$ and $j(\tau) \cdot c$ end up being conjugated by j of something, which is (presumably, demonstrably?) a stronger condition than simply being conjugate. There seems to be no obvious reason why the induced function $[\sigma] \mapsto [j\sigma \cdot c]$ on conjugacy classes should be injective. I think we prove it isn't

Lemma 7 *(Bowler) c_2 is conjugate to $j(c)$ and so is also universal.*

Proof: duplication. . . ??

Given a set of the form $x \bigtriangleup B(\emptyset)$ we can recover x since it is $(x \bigtriangleup B(\emptyset)) \bigtriangleup B(\emptyset)$. So $x \mapsto x \bigtriangleup B(\emptyset)$ is injective. But the same thought reassures us that it is surjective too, so it is genuinely a permutation of V and, actually, an involution. In fact we can write it $\prod_{x \in V}(x, x \bigtriangleup B(\emptyset))$ as a product of disjoint transpositions . . . or π for short. To see that π conjugates c_2 to $j(c)$, we calculate as follows:

$$\begin{aligned}
(j(c) \cdot \pi)(x) &= j(c)(x \bigtriangleup B(\emptyset)) \\
&= j(c)(x) \bigtriangleup j(c)(B(\emptyset)) \\
&= j(c)(x) \bigtriangleup (V \setminus B(\emptyset)) \\
&= j(c)(x) \bigtriangleup (V \bigtriangleup B(\emptyset)) \\
&= (j(c)(x) \bigtriangleup V) \bigtriangleup B(\emptyset) \\
&= (V \setminus j(c)(x)) \bigtriangleup B(\emptyset) \\
&= (c \cdot j(c))(x) \bigtriangleup B(\emptyset) \\
&= c_2(x) \bigtriangleup B(\emptyset) \\
&= (\pi \cdot c_2)(x)
\end{aligned}$$

■

Corollary 4 *Every model of NF has a permutation model with an internal $\in$-automorphism.*

Proof: It follows from corollary 1 that $j(c)$ and $j^2(c)$ are conjugate, making $j(c)$ an example of a permutation which is conjugate to j of itself. It was shown in [6] that any model containing such a permutation π has a permutation model wherein π has become an (internal) $\in$-automorphism. ■

In [6] it is shown that there must be such a π, but that was on the assumption of AC_2, and of course we have here scrupulously eschewed AC_2.

Zuhair Abdul Ghafoor Al-Johar has asked me whether the automorphism obtained in this way moves any wellfounded set. Thinking about it for a bit the answer is of course 'no'. For any automorphism σ the set $\{x : \sigma(x) = x\}$ is indeed a set and it extends its own power set, so—by induction—it copntains all wellfounded sets.

For the main result which follows later (corollary 6) we will need involutions σ and τ such that there is a permutation π conjugating σ to $j(\tau) \cdot c$ and τ to $j(\sigma) \cdot c$. The next lemma exhibits such a pair of involutions, taking σ to be c_1 and τ to be c_2.

delete from here (?)

Lemma 8 *There is an involution that conjugates c with c_3 and commutes with c_2.*

Proof:

We begin by choosing a fixed point a of c_2 and setting $b = c_1(a)$. Since a is a fixed point of c_2 we also have $b = c_1(c_2(a)) = j(c)(a)$. For any $s \subseteq \{a, b\}$ we define X_s to be $\{x : x \cap \{a, b\} = s\}$.

$X_\emptyset$ is closed under both $j(c)$ and $j^2(c)$; let $\sigma_\emptyset$ be the restriction of $j(c)$ to $X_\emptyset$ and $\tau_\emptyset$ the restriction of $j^2(c)$. Then there are embeddings of $j(c)$ into $\sigma_\emptyset$ and $j^2(c)$ into $\tau_\emptyset$, so by the results of the last section both $\sigma_\emptyset$ and $\tau_\emptyset$ are universal. Let $\pi_\emptyset$ be an isomorphism from $\sigma_\emptyset$ to $\tau_\emptyset$. Since $j(c) = c_1 \cdot c_2$ and $j^2(c) = c_3 \cdot c_2$ we have the equation $\pi_1 \cdot c_1 \cdot c_2 = c_3 \cdot c_2 \cdot \pi_1$, which we note for future use.

We now define $\pi\colon V \to V$ by

$$x \mapsto \begin{cases} \pi_\emptyset(x) & \text{if } x \cap \{a, b\} = \emptyset \\ x & \text{if } x \cap \{a, b\} = \{b\} \\ c_3(c_1(x)) & \text{if } x \cap \{a, b\} = \{a\} \\ c_3(\pi_\emptyset(c_1(x))) & \text{if } x \cap \{a, b\} = \{a, b\} \end{cases}$$

Then π is a union of bijections from X_s to X_s for each $s \subseteq \{a, b\}$, so it is a bijection.

It remains to check that for any x we have $\pi(c_1(x)) = c_3(\pi(x))$ and $\pi(c_2(x)) = c_2(\pi(x))$. For each equation there are 4 cases, depending on $x \cap \{a, b\}$. We now check these cases for the first equation.

- If $x \cap \{a, b\} = \emptyset$, then $c_1(x) \cap \{a, b\} = \{a, b\}$ and so

$$\pi(c_1(x)) = c_3(\pi_\emptyset(c_1(c_1(x)))) = c_3(\pi_\emptyset(x)) = c_3(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{b\}$ then $c_1(x) \cap \{a, b\} = \{a\}$ and so

$$\pi(c_1(x)) = c_3(c_1(c_1(x))) = c_3(x) = c_3(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a\}$ then $c_1(x) \cap \{a, b\} = \{b\}$ and so

$$\pi(c_1(x)) = c_1(x) = c_3(c_3(c_1(x))) = c_3(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a, b\}$ then $c_1(x) \cap \{a, b\} = \emptyset$ and so

$$\pi(c_1(x)) = \pi_\emptyset(c_1(x)) = c_3(c_3(\pi_\emptyset(c_1(x)))) = c_3(\pi(x))\,.$$

The four cases for the other equation are similar.

- If $x \cap \{a, b\} = \emptyset$ then $c_2(x) \cap \{a, b\} = \{a, b\}$ and so

$$\pi(c_2(x)) = c_3(\pi_\emptyset(c_1(c_2(x)))) = c_3(c_3(c_2(\pi_\emptyset(x)))) = c_2(\pi_\emptyset(x)) = c_2(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{b\}$ then $c_2(x) \cap \{a, b\} = \{b\}$ and so

$$\pi(c_2(x)) = c_2(x) = c_2(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a\}$ then $c_2(x) \cap \{a, b\} = \{a\}$ and so

$$\pi(c_2(x)) = c_3(c_1(c_2(x))) = c_2(c_3(c_1(x))) = c_2(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a, b\}$ then $c_2(x) \cap \{a, b\} = \emptyset$ and so

$$\pi(c_2(x)) = \pi_\emptyset(c_2(x)) = \pi_\emptyset(c_2(c_1(c_1(x)))) = c_2(c_3(\pi_\emptyset(c_1(x)))) = c_2(\pi(x))\,.$$

■

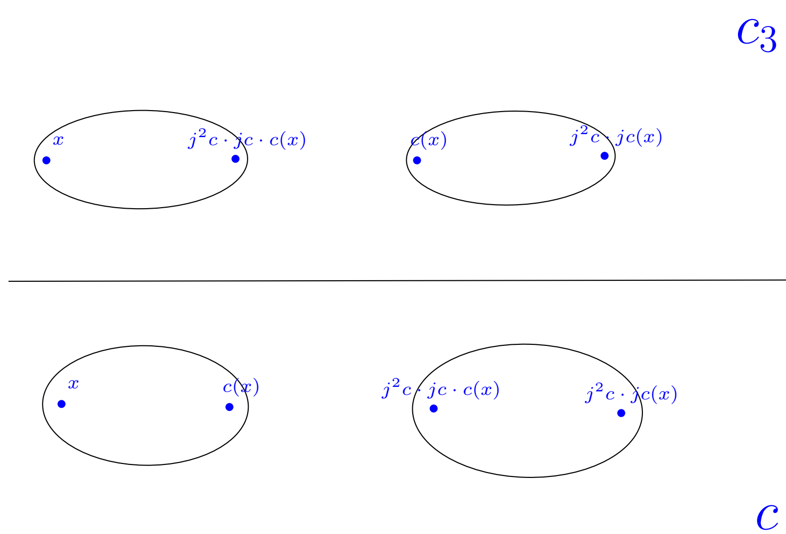
9.1 Finding Permutations that will prove Duality[2]

Lemma 9 *(Bowler)*

There is an involution that conjugates c with c_3 and commutes with $j^2c \cdot jc$.

Proof:

The universe partitions naturally into bundles closed under both c_1 and c_3. Each such bundle contains precisely four sets. We will define a permutation π in such a way that it fixes each bundle setwise. It will turn out that the π we define is the permutation we seek.



In the accompanying picture we have written a typical bundle twice: once below the line where it is divided into two c-cycles and once above the line where it is divided into two c_3-cycles. We have to biject the set of points below the line with the set of points above the line in a way that respects the two partitions into cycles. Evidently this can be done (in eight different ways, as it happens) so we pick one such way for each bundle. By corollary 3 (i) we have transversals for c_3 and c. The transversal for c_3 highlights precisely one element in each pair upstairs, namely that element that contains $B(\emptyset)$. These two highlighted elements cannot—downstairs—belong to different pairs because the downstairs pairs are complements and two complementary sets cannot both contain $B(\emptyset)$.

To illustrate, suppose in the picture that upstairs we highlight x and (therefore) $j^2c \cdot jc(x)$. We tell π to fix these two sets, and that compels it to swap $c(x)$ and $c_3(x)$.

The other possibility is that we highlight $c_3(x)$ and $c(x)$, and then we tell π to fix those two sets and to swap x and $j^2c \cdot jc(x)$.

Either way the net result is that π is

$$\texttt{if } B(\emptyset) \in x \texttt{ then } x \texttt{ else } j^2c \cdot jc(x).$$

Reflect that $B(\emptyset) \in x$ iff $B(\emptyset) \in j^2c \cdot jc(x)$, and $j^2c \cdot c$ is an involution. So, if $B(\emptyset) \in x$, it follows that $\pi(x) = x$ and then $\pi^2(x) = x$; if $B(\emptyset) \notin x$ then $\pi(x) = j^2c \cdot jc(x)$ which does not contain $B(x)$ either. So $\pi^2(x) = \pi(j^2c \cdot jc(x)) = j^2c \cdot jc \cdot j^2c \cdot (x) = x$ and $\pi^2(x) = x$. So π is an involution.

Let us check that π commutes with $j^2c \cdot jc$, that is to say: $j^2c \cdot jc \cdot \pi(x) = \pi \cdot j^2c \cdot jc(x)$ for all x.

There are two cases, depending on whether or not $B(\emptyset) \in x$.

If $B(\emptyset) \in x$ then $\pi(x) = x$ and $j^2c \cdot jc \cdot \pi(x) = j^2c \cdot jc(x)$.

If $B(\emptyset) \in x$ then $B(\emptyset) \in j^2c \cdot jc(x)$ so $j^2c \cdot jc(x)$ is fixed by π.

Either way $j^2c \cdot jc \cdot \pi(x) = \pi \cdot j^2c(x) = \pi \cdot j^2c \cdot jc(x)$

If $B(\emptyset) \notin x$ then $\pi(x) = j^2c \cdot jc(x)$. Then $j^2c \cdot jc \cdot \pi(x) = x$.

If $B(\emptyset) \notin x$ then $B(\emptyset) \notin j^2c \cdot jc(x)$ so $j^2c \cdot jc(x)$ is moved by π, and must be x.

Either way $j^2c \cdot jc \cdot \pi(x) = x = \pi \cdot j^2c \cdot jc(x)$

■

Presumably there is a generalisation that says that there is an involution that conjugates c_i with c_{i+2} and commutes with $j^{i+2}c \cdot j^{i+1}c$. But—presumably—we are not going to need it.

However i think this is completely general. Is it not the case that, in any symmetric group, if σ and τ are conjugate, then they can be conjugated by something that commutes with $\sigma\tau$? Something like that must be true...

Corollary 5

Every model of NF has a permutation model that contains two (internal) permutations σ and τ satisfying $(\forall xy)(x \in y \longleftrightarrow \sigma(x) \notin \tau(y))$ and $(\forall xy)(x \in y \longleftrightarrow \tau(x) \notin \sigma(y))$.

Furthermore any such model satisfies duality for formulæ that are stratifiable-mod-2.

Proof: We use the permutation π from lemma 8, and exploit the two permutations σ and τ that we find in the permutation model V^π.

If a formula ϕ is stratifiable-mod-2 then its variables can be assigned to two types `yin` and `yang` in such a way that in subformulæ like '$x = y$' the two variables receive the same type and in subformulæ like '$x \in y$' the two variables receive different types. Let us associate σ to variables given type `yin` in the assignment and associate τ to variables given type `yang` in the assignment. '$x \in y$' is equivalent to '$\sigma(x) \notin \tau(y)$' and if x is of type `yin` we make this replacement. '$x \in y$' is also equivalent to '$\tau(x) \notin \sigma(y)$' and if x is of type `yang` we make this replacement. We deal with equations analogously. In the rewritten version of ϕ we find that every variable 'x' of type `yin` now appears only as '$\sigma(x)$' and that every variable 'y' of type `yang` now appears only as '$\tau(y)$'. So we can reletter '$\sigma(x)$' as 'x', and '$\tau(y)$' as 'y' and the result is $\widehat{\phi}$. ■

There is a further corollary: no homogeneous formula $\phi(x_1, x_2)$ can define a BFEXT (a well-founded extensional binary relation) on V. Given a definable well-founded extensional binary relation on V we can argue as follows. Let σ be an permutation, assumed to be an $\in$-automorphism. We then prove by wellfounded induction on ϕ that σ is the identity.

Actually we have to be very careful how we state this ...

First we prove that if there is a definable wellfounded extensional relation on the whole of V then there are no nontrivial $\in$-automorphisms.

Suppose σ is an $\in$-automorphism, and that $\phi(x, y)$ defines a wellfounded extensional relation on the whole of V. Fix y and suppose $(\forall x)(\phi(x, y) \to x = \sigma(x))$. Then $(\forall x)(\phi(x, y) \longleftrightarrow \phi(x, \sigma(y))$ whence $y = \sigma(y)$ by extensionality. Then if $\{y : y \neq \sigma(y)\}$ is nonempty it has no ϕ-minimal element, contradicting wellfoundedness of ϕ.

We plan next to exploit corollary 4. The obvious thing to do is to say: suppose ϕ defines a wellfounded extensional binary relation on V; jump into a permutation model containing a nontrivial $\in$-automorphism to prove that it's not a wellfounded extensional binary relation. However for that to work we need the expression "ϕ defines a wellfounded extensional binary relation on V" to be stratified, and for that we need ϕ to be stratified. It doesn't have to be homogeneous, but it does have to be stratified.

to here?

Some questions

Under what operations is the class of universal involutions closed?

Are the universal involutions a normal generating subset of J_0?

Are there maximal permutations? We could start by asking for a maximal permutation of order 3.

Corollary 6 *Every model of NF has a permutation model that satisfies duality for formulæ that are stratifiable-mod-2.*

■

It's worth bearing in mind that σ and τ retain in V^π all the stratified properties they had in their previous life in V, where they were c and c_2. Thus they commute, and $\sigma^2 = \tau^2 = \mathbb{1}$. Observe also that

$$j(\sigma\tau) = j\sigma \cdot j\tau = \tau \cdot c \cdot c \cdot \sigma = \tau\sigma = \sigma\tau,$$

so $\sigma\tau$ is actually an $\in$-automorphism of V^π. It is a nontrivial automorphism beco's σ and τ are not inverse to each other: τ has fixed points and σ does not. By the remark in the proof of part (i) of lemma 6 $\sigma\tau$ is an involution.

This fact is worth recording!

Corollary 7 *Every model of NF has a permutation model containing a nontrivial automorphism of order 2.*

We should be able to express this as a fact inside the base model...

Can we use this technique to obtain models in which duality holds for formulæ that are stratifiable-mod-p for other primes? If we were to rejig the above development to seek a proof for formulæ that are stratifiable-mod-3 then we would be looking for an *antimorphism tuple* (in this case *triple*) σ, τ, π such that there is θ satisfying

$$(j\sigma \cdot c)^\theta = \tau, \ \ (j\tau \cdot c)^\theta = \pi \text{ and } (j\pi \cdot c)^\theta = \sigma.$$

However, as Nathan Bowler has reminded me, the existence of such a triple contradicts AC_2 since τ has an odd cycle iff $j\tau \cdot c$ does not. (That was lemma 5.) And if we are going to ditch AC_2 then we may as well go for outright antimorphisms from day 1.

9.2 Full Duality?

It may be that the set of things fixed by $\sigma\tau$ is a model of NF + full Duality. Something to check!

First we check that $\sigma\tau$ (which is the same as $\tau\sigma$) is an $\in$-automorphism. For all x and y we have $x \in y \longleftrightarrow \sigma(x) \notin \tau(y)$ so $\sigma(x) \notin \tau(y) \longleftrightarrow \tau\sigma(x) \in \sigma\tau(y) = \tau\sigma(y)$ so $\tau\sigma$ is an $\in$-automorphism as desired.

Next we check that if π is an $\in$-automorphism then the set of fixed points is a model of NF. The big gap here is extensionality. We would have to show that every nonempty fixed set has a fixed member.

Finally we check that the set of fixed points of $\sigma\tau$ is additionally a model of duality. Observe that, for all such fixed x we have $x = \sigma(\tau(x))$ whence $\sigma^{-1}(x) = \tau(x)$. But $\sigma^2 = \mathbb{1}$ so $\sigma(x) = \tau(x)$.

Now suppose x and y both fixed. Then $x \in y \longleftrightarrow \sigma(x) \notin \tau(y) = \sigma(y)$. So σ is an antimorphism of the fixed points.

But this relies on the set of fixed points being extensional. It may be that we can ensure this by a judicious choice of the permutation in lemma 8. We seek a π that conjugates c to $j^2c \cdot jc \cdot c$ and moreover has the extra feature that in V^π the set $\{x : \sigma(x) = \tau(x)\}$ is extensional. Must turn this into a condition on π. . . . We think

$$V^\pi \models (\forall x)(x \neq \emptyset \wedge \sigma\tau(x) = x \to (\exists y \in x)(\sigma\tau(y) = y))$$

is

$$(\forall x)(\pi(x) \neq \emptyset \wedge \sigma\tau(x) = x \to (\exists y \in \pi(x))(\sigma\tau(y) = y))$$

which becomes

$$(\forall x)(x \neq \emptyset \wedge j^2c \cdot jc(x) = x \to (\exists y \in \pi(x))(j^2c \cdot jc(y) = y))$$

where π conjugates c and $j^2c \cdot jc \cdot c$.

Let us write 'F' for $\{x : x = jc \cdot j^2c(x)\}$ to keep things readable. The π we seek has got to inject F into $\{y : y \cap F \neq \emptyset\}$—o/w known (see p. 7) as "$\text{Ƀ}(F)$". Observe that $\text{Ƀ}(x)$ is always a moiety, since it is $V \setminus (\mathcal{P}(V \setminus x))$, and the complement of a power set (of anything other than V) is always the same size as V. This is beco's every set (other than V itself) is included in the complement of a singleton, and the power set of a complement of a singleton is a principal prime ideal and therefore a moiety.

So there's no problem on *that* score.

It's not blindingly obvious to me that it cannot be done.

9.3 Refuting duality

The Lads said:

> **First**: Add a Quine atom by $\tau = (\emptyset, \{\emptyset\})$;
> **Second**: Kill off all Quine atoms by $\tau = \prod_{x \in \iota^2 “V} (x, V \setminus x)$.

Now it should be possible to do it with a single permutation. I think the idea is to swap with their complements-in-the-sense-of-$(\emptyset, \{\emptyset\})$, all those sets that are double singletons in the sense of $V^{(\emptyset, \{\emptyset\})}$. That is to say—writing 'σ' for the transposition $(\emptyset, \{\emptyset\})$ and 'c' for complementation to keep things readable:

$$\tau := \prod_{(x \in \iota^2 “V)^\sigma} (x, \sigma c \sigma(x))$$

is the one-stop permutation we want. (The fact that this definition is legitimate is nontrivial: it's a great help that $\sigma c \sigma$ is an involution. We also need the fact that if x is a double-singleton-in-the-sense-of-σ then its complement-in-the-sense-of-σ cannot be a double-singleton-in-the-sense-of-σ. This ensures that all the transpositions in the big product are disjoint.)

THEOREM 1
Duality fails in V^τ because it contains a Quine antiatom but no Quine atom.

Proof: Clearly the collection $A := \{x : ((\exists z)(x = \{\{z\}\}))^\sigma)\}$ is going to be of interest. Let's process '$(x \in \iota^2 \text{``} V)^\sigma$'.

$$(x \in \iota^2 \text{``} V)^\sigma$$

is

$$(\exists z)(x = \{\{z\}\})^\sigma$$

which is

$$(\exists z)(\sigma\text{``}(\sigma(x)) = \{\{z\}\})$$

which is

$$(\exists z)(\sigma(x) = \sigma\text{``}\{\{z\}\})$$

which is

$$(\exists z)(\sigma(x) = \{\sigma\{z\}\}).$$

Two things to notice

1. Since every Quine atom is fixed by σ every Quine atom belongs to A. Everything that starts life as a Quine atom is moved.
2. Notice too that if $x = \emptyset$ then it belongs to A: $\sigma(\emptyset) = \{\emptyset\} = \{\sigma\{\emptyset\}\}$.

So what is the fate of $\emptyset$ in the new model V^τ? (Let's call it 'a' in order not to confuse ourselves!)

$$(x \in a)^\tau$$

iff

$$x \in \tau(a)$$

Now $\tau(a)$ is the complement-in-the-sense-of-V^σ of a which is $\sigma c \sigma(a) = \sigma c\{\emptyset\} = \sigma(V \setminus \{\emptyset\}) = V \setminus \{\emptyset\}$.

iff

$$x \in (V \setminus \{\emptyset\})$$

iff

$$x \notin \{\emptyset\}$$

iff

$$x \neq \emptyset$$

iff

$$x \neq a$$

So a is a Quine antiatom in the new model V^τ.

Now let's check that there are no Quine atoms in the new model V^τ.

Suppose x is a Quine atom in the sense of V^τ. If x is fixed by τ then it was a Quine atom in the model in which we started. We observed earlier (item 2 p 45) that any object that starts life as a Quine atom is moved by τ. So x is moved. So (x is a Quine atom)$^\tau$ is

$$(\forall y)(y = x \longleftrightarrow y \in \sigma c \sigma x)$$

We need not consider the case where $x = \emptyset$, since we have already dealt with that and seen that x is a Quine antiatom. If $x = \{\emptyset\}$ then the RHS becomes $y \in \sigma c \sigma \{\emptyset\} = V$ which is clearly not equivalent to the LHS; clearly $\{\emptyset\}$ is not a Quine atom in V^τ.

There remain the cases where x is fixed by σ. These give

$$(\forall y)(y = x \longleftrightarrow y \in \sigma(V \setminus x))$$

For x to be a Quine atom in V^τ, $\sigma(V \setminus x)$ will have to be a singleton. This can happen if $x = V$, for then $V \setminus x$ is empty and $\sigma(V \setminus x)$ is $\{\emptyset\}$ so x would have to be both V and $\emptyset$, so the case $x = V$ does not give rise to a Quine atom. The only other way for $\sigma(V \setminus x)$ to be a singleton is for $V \setminus x$ to be a singleton, say $\{z\}$ and for it to be fixed by σ. In that case the condition for (x is a Quine atom)$^\tau$ becomes

$$(\forall y)(y = x \longleftrightarrow y \in V \setminus \{z\}))$$

which is clearly impossible. ■

In contrast, we have not yet found a permutation model that satisfies duality.

10 Work still to do

There remains of course the challenge of proving consistency of duality for all sentences, not merely those that are stratifiable-mod-2. But more to the point are the possibilities of extending to formulæ that are stratifiable-mod-n things known about the rather more restricted class of stratified formulæ—and these we haven't started thinking about. Here are some, in no particular order.

We should show in an NF context that, for each n, the assertion that "there are sets x s.t. $\iota^n \restriction x$ exists" is invariant.

• Is there any interest in versions of Forti-Honsell Antifoundation along the lines "Every set picture that is a n-stratification graph is a picture of a set"?

• If ϕ is, for each n, equivalent (modulo NF) to something that is stratified-mod-n must it be (NF)-invariant?

I briefly thought i had a counterexample, on the grounds that

"$\exists V_\omega$' is, for each n, equivalent to

"The least fixpoint for $x \mapsto (\mathcal{P}_{\aleph_0})^n(x)$ exists"

and that last assertion is stratifiable-mod-n. So it ought to be invariant, but it isn't, beco's of Holmes' clever permutation.

However, the least fixpoint for $x \mapsto (\mathcal{P}_{\aleph_0})^n(x)$ isn't V_ω. It's the set of sets of rank a multiple of n.. Duh.

• Randall has just (4/vi/2016) pointed out to me that $\mathrm{TC}_n\mathrm{T}$ is in some sense the same theory as NFU + $|V| = |\mathcal{P}^n(V)|$. It could be a good idea to spell this out. Evidently any model of NFU + $|V| = |\mathcal{P}^n(V)|$ will give rise to a model of $\mathrm{TC}_n\mathrm{T}$. The other direction looks a lot more complicated.

• In a model of $\mathrm{TC}_k\mathrm{T}$ one can sensibly ask, for any m, whether or not Ambiguity holds for formulæ that are stratifiable-mod-$k \cdot m$.

• André Pétry suggests a generalisation of a result of his-and-mine alluded to earlier ([7], [15], and [16]) to the effect that if two structures are elementarily equivalent for formulæ that are stratifiable-mod-n then they have stratimorphic (as it were) ultrapowers.

• One could investigate whether the construction of [9] could be modified to encompass expressions that are stratifiable-mod-n. That looks messy.

• There are natural settings where one encounters embeddings that are elementary for stratifiable formulæ, and where one might hope to get embeddings that are elementary for some of these larger classes of formulæ. CO models is one setting: the embedding from the ground model into the hereditary low sets is elementary for stratifiable formulæ. (That particular example is probably not a good one, because if the inclusion embedding is elementary for formulæ that are stratifiable-mod-n for even one n then the hereditarily low sets cannot contain any Quine atoms). For another, let $\mathfrak{M}$ be a structure for $\mathcal{L}$. Consider the class of those $m \in M$ s.t. m is fixed by all permutations of M that, for all n, are j^n of something. It's an elementary substructure as long as it's extensional. Now use instead those permutations π of M s.t. $j^m\pi = \mathbb{1}$. Now the class of fixed things is a substructure elementary for expressions that are stratifiable mod m (again, assuming extensionality).

• Str(ZF) is the theory axiomatised by the stratifiable axioms of ZF; by analogy str_n(ZF) will be the theory axiomatised by those axioms of ZF that are stratifiable-mod-n. ZF can be interpreted in str(ZF) + IO. (IO is the axiom "every set is the same size as a set of singletons"). Observe that IO is a theorem of str_n(ZF), since it proves that $\iota^n \restriction x$ exists for all x, so every set is the same size as a set of singletons, so ZF can be interpreted in str_n(ZF). At this stage we cannot see how to prove that str_n(ZF) = ZF. There are parallel questions about the fragments of Mac.

• Stratified parameter-free Δ_0 $\in$-induction seems to prove no more than the nonexistence of a universal set. How about stratifiable-mod-n parameter-free $\in$-induction... what does that do? One might hope that it would prove the nonexistence of $\in$-loops of circumference n but we can't see it offhand. But in any case we should start with the case $n = 2$ in order to not drown *immediately* in the deep end. We noted in section 3 that the collections `I` and `II` as in [11] are both the extensions of expressions that are stratifiable-mod-2. So stratifiable-mod-n parameter-free $\in$-induction will imply $\in$-determinacy. (tho' that induction is not Δ_0...) Needs looking into.

Stratifiable parameter-free $\in$-induction implies the nonexistence of the universal set. (If none of your members are the universal set, you can't be either). It's not known if the converse holds. However the strengthening of the converse one would consider in this context, namely "the non-existence of the universal set implies $\in$-induction for parameter-free formulæ that are stratifiable-mod-n" clearly does not go through: $\in$-induction for parameter-free formulæ that are stratifiable-mod-n" implies $(\forall x)(x \notin^2 x)$, and that clearly doesn't follow from the nonexistence of V.

• Suppose we add to our favourite theory of wellfounded sets a scheme of $\in$-induction for formulæ that are stratifiable-mod-n, for some or all n. Is it the case that any such model is first-order indistinguishable from a wellfounded model? Can we prove anything with that flavour ...? A: by proposition 2 we could prove that every set is wellfounded.

• Every weakly stratifiable theorem of first-order logic has a cut-free weakly stratifiable proof; every stratifiable theorem of first-order logic has a stratifiable proof (Crabbé, [3]); are there analogues for stratification-mod-n? Every theorem of first-order logic that is stratifiable-mod-n has a proof that is stratifiable-mod-n? Crabbé thinks so. Why should it *not* work, after all?

On the other hand we should not expect a stratifiable-mod-n analogue of Crabbé's result that SF is consistent.

• There is an old question about whether the atoms of a model of NFU can be indiscernible. We know that they are indiscernible wrt stratifiable formulæ; now that we've started looking into stratification-mod-n it is natural to wonder whether one might be able to show that the atoms of a model of NFU must be indiscernible wrt expressions that are stratifiable-mod-2. At this stage it's not looking hopeful.

• We should investigate the consistency results relative to T$\mathbb{Z}$T obtained by omitting types, to see how many of them work for TC$_n$T. They make heavy use of Coret's lemma. Coret's lemma tells us how permutations preserve stratifiable formulæ. Any old permutation works. In the NF context we know that if we want to preserve all formulæ then we can't use any-old-permutation but only $\in$-automorphisms. Working in TC$_n$T we want to preserve formulæ that are stratifiable-mod-n, and that means using permutations π s.t. $\pi = j^n(\pi)$, and such permutations are not just lying around. TC$_n$T really does behave more like NF than like T$\mathbb{Z}$T.

• There is the old question of whether or not Amb^n is equiconsistent with NF. Suppose we work in KF, and consider TC$_2$T to keep things simple initially. Suppose we have an x with $|x| = |\mathcal{P}^2 x|$. Is that going to give us a model of NF? Let α be the cardinal of such an x. Can we prove that $\alpha = 2^{T\alpha}$? We suspect not, because that would probably say something about theorems in TC$_2$T. A useful thought is the fact that α is $\beth_n$ of something for all concrete n. So we certainly have $\alpha = \alpha + 1$, $\alpha = \alpha + \alpha$, $\alpha = \alpha \cdot \alpha$. The plan is to use these equations to show that $x \sqcup \mathcal{P}(x)$ gives us a model of NF. So we want $T(2^{\alpha + 2^{T\alpha}}) = \alpha + 2^{T\alpha}$. Now $T(2^{\alpha + 2^{T\alpha}}) = 2^{T\alpha} \cdot 2^{2^{T^2\alpha}}$. $T(2^{\alpha + 2^{T\alpha}}) = 2^{T\alpha} \cdot \alpha$.

So we want $\alpha + 2^{T\alpha} = 2^{T\alpha} \cdot \alpha$, and we hope to get it from the good behaviour of α. We have $\alpha = \alpha^2$ so we get $2^{T\alpha} = 2^{T\alpha^2} = (2^{T\alpha})^{T\alpha}$ which looks hopeful but isn't exactly what we want. The warning sign is that if this worked it would show that $2^{T\alpha}$ absorbs α and that sounds extremely implausible.

But even if $\alpha + 2^{T\alpha} = 2^{T\alpha} \cdot \alpha$ it wouldn't help. We can exploit Bernstein's Lemma to show that we would have $\alpha = 2^{T\alpha}$ or—at the very least—that each $\leq^*$ the other, which is just as bad, as follows.

If we have $\alpha + 2^{T\alpha} = 2^{T\alpha} \cdot \alpha$ then Bernstein's Lemma gives $\alpha \leq 2^{T\alpha} \vee 2^{T\alpha} \leq^* \alpha$ and $\alpha \leq^* 2^{T\alpha} \vee 2^{T\alpha} \leq \alpha$ so a case analysis gives $\alpha = 2^{T\alpha} \vee \alpha \leq^* 2^{T\alpha} \leq^* \alpha$ which gives $2^\alpha = 2^{2^{T\alpha}} = T\alpha$, which is altogether too strong.

One has the impression that KF really does not want to prove that if there is x with $|x| = |\mathcal{P}^n(x)|$ then there is an x with $|x| = |\mathcal{P}(x)|$. The moral of this seems to be that $\mathrm{TC_2T}$ is not as much like NF as it might be.

- Consider "$\Box$(Duality for sentences that are stratifiable-mod-2)"
  Is this consistent? Does it imply AC_2?
- ZF + Foundation and ZF + antifoundation are alike extensions of ZF + Coret's axiom "every set is the same size as a wellfounded set" conservative for stratifiable sentences. (See [12]). Does this hold also for sentences that are stratifiable-mod-n?

Checking this last one should be simple!

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