Weak Systems of Set Theory related to HOL

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Contents

This is a later (1998) expanded and corrected version of a survey article appearing in the proceedings of HUG94, Springer lecture notes in Computer Science **859** pp 193–204. It is designed for the interested nonspecialist, by which I mean *nonspecialist Set Theorist*; it was written for a HOL conference and assumes familiarity with HOL. Although it doesn't contain any proofs of novel results, it does contain announcements (of novel unpublished results) and proofs (of frequently underregarded trivialities). For the reader who wishes to take this material further the chief advantage of this essay will be the bibliography, which would be very hard for a naïve reader to assemble from scratch. I would like to thank my friend and colleague Juanito Camilleri for the invitation which led me to write this essay.

I recompiled and copy-edited it in the southern winter of 2023.

1 HOL and TST

In virtue of theorem 1 all the set theories that are related to HOL are related to it *via* a type system called 'TST'. This system is descended from the type system of Russell and Whitehead [22] and is due in its present form to Ramsey[1] [20]. Traditionally the initials spell *T*heory of *S*imple *T*ypes, but *T*yped *S*et *T*heory would be better.

TST is expressed in a language with a type for each non-negative integer, an equality relation at each type, and between each pair of consecutive types n and $n+1$ a relation $\in_n$. The axioms are an axiom of extensionality at each type

$$\forall x_{n+1} \forall y_{n+1} \, (x_{n+1} = y_{n+1} \longleftrightarrow \forall z_n (z_n \in_n x_{n+1} \longleftrightarrow z_n \in_n y_{n+1}))$$

and (at each type) an axiom scheme of comprehension

$$\forall \vec{x} \exists y_{n+1} \forall z_n \, (z_n \in_n y_{n+1} \longleftrightarrow \phi(\vec{x}, z_n))$$

with 'y_{n+1}' not free in 'ϕ'.[2]

TST$_k$ is like TST except that there are only k types, labelled 0,..., $k-1$. The "theory of negative types"[3] (T$\mathbb{Z}$T[4] ([29]) and its language are defined analogously, except that the types are indexed by $\mathbb{Z}$. T$\mathbb{Z}$T is in some ways a more convenient theory to deal with than TST—there is no danger of *falling off the bottom* as it were—and the two theories are equiconsistent by a simple compactness argument. For the purposes of this paper all theories will be assumed to have the axiom of infinity as an axiom.

TSTI is TST with the axiom of infinity for the bottom level. (We have to make explicit that we mean the bottom level, for it is possible to have a Dedekind-finite set whose power set has a countably infinite subset, so we can have models of type theory which are infinite above some level but Dedekind-finite below it.) Similarly we will have *TSTI*$_k$. In general, the result of appending an '*I*' to the name of a theory denotes the result of adding the axiom of infinity to that theory. If x is a set in a model of some minimal sensible set theory (Zermelo set theory will do), $\langle\langle x \rangle\rangle$ is the structure $(x, \mathcal{P}(x), \mathcal{P}^2(x), \ldots)$ thought of as a model of TST.

The syntax of TST is evidently much simpler than that of HOL. TST is what one gets from HOL if one decides to implement ordered pairs and functinos as sets: a lot of the type structure collapses. It doesn't even matter very much *how* one interprets functions and ordered pairs as sets: any implementation of functions and pairs as sets will result in an interpretation of HOL into TST. Accordingly we have the following:

THEOREM 1 HOL *and* TST *can be interpreted in one another.*

Proof:

It is very easy to interpret TST in HOL because we can interpret type 0 of TST as being any type we please and thereafter type $n+1$ is interpreted as type $\alpha \rightarrow$ `:bool`

[1] Hence Sheffer's joke about ramified set theory becoming Ramsified set theory

[2] I have written '$\vec{x}$' to avoid writing '$x_1 \ldots x_n$: writing a list of variables in the latter style would cause confusion between subscripts that pertain to types and subscripts that denote position in a list. We have to have the first, so we should avoid the second if possible.

[3] This is actually a misnomer: strictly he should have called it the theory of *positive and* negative types.

[4] In 2023 I changed this notation from 'TNT' to conform to modern practice.

where α is the HOL type that interprets type n of TST. The other direction is equally elementary but much more complicated. First we must take note that any level of a model of TST can be embedded in all higher levels by means of an iteration of the singleton functioon. For notational convenience we will sometimes write singletons functionally, using the function letter ι so that $\iota(x) = \{x\}$. $\iota^2(x) = \{\{x\}\}$ and $\iota``x = \{\{y\} : y \in x\}$. Then one can embed type $m$ into type $m + n$ by the function $\lambda x.\iota^n``x$. In fact we have to reinterpret the membership relation as well. If we let[5] $RUSC(R)$ be $\{\langle\{x\}, \{y\}\rangle : \langle x, y\rangle \in R\}$ we interpret $\in$ by $RUSC^n(\in)$ composed with a suitable iterate of ι

For the other direction we need to interpret HOL in TST. We embed `:ind` and `:bool` in type 0. We embed `:num` in the smallest type containing an interpretation of the naturals. Even tho' we are assuming the axiom of infinity (and therefore that type 0 is infinite) this does not mean that type 0 is the smallest type in which we can interpret the naturals, since there will not be interpretations for the definite descriptions of the numerals in type 0. Frege-Russell numerals are to be found in types 2 and above.

For the recursion we assume that we have interpreted two types α and β into two types m and n repectively where $m < n$. We compose this interpretation of α into m with an iterated singleton map to get them both interpreted into type n. Then by means of Wiener-Kuratowski pairs one interprets $\alpha \times \beta$ into[6] type $n + 2$ and therefore $\alpha \to \beta$ into $n + 3$ (or $n + 1$ using Quine pairs). ■

The equiconsistency of HOL and TST is one of two results central to the programme of this paper. Theorem 1 is the bridge between type theories and set theories and provides the context in which all the rest of the paper operates.

Retaining the type-structure of HOL enables one to reason in a way that is independent of decisions about how to implement natural numbers and ordered pairs. Type theory captures what can be done in *all* implementations; Set Theory captures what can be done if one is allowed to choose one's implementation.

2 Predicativity, truth definitions and consistency proofs

What do we mean by the 'strength' of a system? Because of the second incompleteness theorem no recursively axiomatisable system can prove its own consistency[7]: if $T \vdash Con(S)$ then $S \nvdash Con(T)$ so $T \vdash Con(S)$ is clearly a relation we will want to look at. Another relation of interest is "S can be interpreted in T". This relation in contrast is not asymmetrical.

Much of the early work on interpreting theories in other theories dates from the time when the canonical work on set theory was Russell-Whitehead and is therefore informed by a type-theoretical intuition that is, once again, the flavour of the hour.

If we can interpret T_1 into T_2 in such a way that we can prove in T_2 that every theorem of T_1 is true then clearly $T_2 \vdash con(T_1)$. What sort of conditions on T_2 are sufficient for this to be possible? For a start, we must be able to define in T_2 a truth

[5] This notation is Rosser's

[6] Or, by means of Quine ordered pairs, into type n

[7] We are not going to study systems that are not recursively axiomatisable!

predicate for the expressions in the range of the interpretation. Now a truth predicate is an inductively defined set of ordered pairs, and we can illustrate the complications which arise here by reference to the simplest case, namely the natural numbers $\mathbb{N}$. If we are working in a theory that has a concept of cardinal number and we know what 0 is and what Succ is , we can say x is a natural number iff

$$(\forall Y)([0 \in Y \wedge (\forall y \in Y)(\mathsf{Succ}(y) \in Y)] \to x \in Y)$$

This molecular formula capturing a property of cardinal number contains a quantifier over *sets* of cardinals (not merely cardinals!). This is an example of an **impredicative**[8] definition. This particular example is not *terribly* impredicative (it involves quantifying only over sets of cardinals not sets of sets of cardinals) but its impredicativity is a recurring and central theme. This is because the declaration of $\mathbb{N}$ as a recursive datatype is merely one of many, and so *all* recursive datatype declarations involve impredicative set-existence axioms like this. In particular—as we shall see in the next section—the satisfaction relation on which truth-definitions and consistency proofs rely is an inductively defined set (recursive datatype) in exactly the same was as $\mathbb{N}$.

2.1 Truth definitions

We will assume that our variables, rather than being x, y, z etc, are all x's with numerical subscripts. This clearly makes no difference to us, *qua* language users, since it is a trivial relettering, but it does make life a lot easier for us *qua* students of the language. The subscripts are quite important. We call them indices. The purpose of this change in notation is to make visible to the naked eye the fact that we can enumerate the variables: it is much clearer that this is the case if they are written as "x_1, x_2 ..." than if they are written as "x, y ..." In fact I think we will also have to assume that no variable is bound more than once in any formula, and that there are no occurrences of any variable outside the scope of any quantifier that binds some other occurrence of that variable. Thus we will outlaw

$$((\forall x)F(x)) \vee ((\forall x)G(x))$$

and

$$F(x) \vee (\forall x)(Gx)$$

even though they are perfectly good wffs. It will make life easier later.

Naturally you expect that a notion of interpretation will crop up if we are trying to define what it is for a sentence to be true in a structure. There are actually two gadgets we need here which the reader should keep distinct in her mind. A *finite assignment function* is a function that assigns elements of M to finitely many indices. Computer scientists will recognise this immediately as the logician's version of their concept of *state*. They will also recognise that the partial assignment functions form a chain-complete CPO. I have (see above) carefully arranged that all our variables are

[8] 'Impredicative' is a word coined by Russell to describe definitions of properties of widgets that make reference to objects of higher type: sets of widgets, sets of sets of widgets and so on. The word is unevocative to modern ears, and to understand why he coined it we need to know that—at the time—Russell thought that such definitions were not legitimate, did not define predicates and so were—**im**predicative.

orthographically of the form x_i for some index i, so we can think of our assignment function f as being defined either on *variables* or on *indices*, since they are identical up to 1-1 correspondence. It is probably better practice to think of the assignment functions as assigning elements of M to the *indices* and write "$f(i) = \ldots$", since any notation that involved the actual *variables* would invite confusion with the much more familiar "$f(x_i) = \ldots$" where f would have to be a function defined on the things the variables range over.

Next we define what it is for a partial assignment function to satisfy a sentence p, (written "f *sat* p"). We define *sat* first of all on atomic sentences. First a word on use and mention. Notice that in

$$f \; sat \; x_i = x_j$$

we have a relation between a function and an expression, not a relation between f and x_i and x_j. This is usually made clear by putting quotation marks of some kind round the expressions to make it clear that we are mentioning them not using them. Now precisely what *kind* of quotation mark is a good question. Our first clause will in fact be something like

$$f \; sat \; `x_i = x_j\text{'} \text{ iff}_{\text{df}} \; f(i) = f(j)$$

But how like? Notice that, as it stands, it contains a name of the expression which follows the next colon:

$$x_i = x_j$$

Once we have put quotation marks round this, the i and j have ceased to behave like variables (they were variables taking indices as values) because quotation is a referentially opaque context. But we still want them to be variables, because we want the content of this clause to read, in English, something like: "for any variables i and j, we will say that f *sat* the expression whose first and fourth letters are 'x', whose third and fifth are i and j respectively (whatever i and j are in this case) and whose middle letter is '=', iff $f(i) = f(j)$". Notice (and this is absolutely crucial) that in the piece of quoted English text 'x' and '=' appear with single quotes round them and 'i' and 'j' do not. Now to achieve this, ordinary single quotes will not do. Quine invented a new notational device in [18], which he modestly calls "corners" and which are nowadays known more usually as "Quine quotes" (or "quasi-quotes") which operate as follows: The expression after the next colon:

$$\ulcorner x_i = x_j \urcorner$$

being an occurrence of '$x_i = x_j$' enclosed in quine quotes is an expression which does not, as it stands, name anything. However, i and j are variables taking integers as values, so that whenever we put constants (numerals) in place of i and j it turns into an expression which will name the result of deleting the quasi-quotes. This could also be put by calling it a variable name.

SLOGAN:

> *Putting quine-quotes round a compound of names of wffs gives you a name of the compound of the wffs named.*

A good way to think of quine quotes is not as a funny kind of quotation mark, for quotation is referentially opaque and quine quotation referentially transparent, but rather as a kind of diacritic, not unlike the LATEXcommands I am using to write this paper. Within a body of text enclosed by a pair of quine quotes, the symbols '$\wedge$' '$\vee$' etc. do not have their normal function of composing *expressions* but instead compose *names of expressions*. This also means that Greek letters within the scope of quine quotes are being used to range over expressions (not sets, or integers). Otherwise, if we think of them as a kind of funny quotation mark, it is a bit disconcerting to find that, as Quine points out, $\ulcorner\mu\urcorner$ is just μ. The reader is advised to read pages 33-37 of Quine [18] where this gadget is introduced. Let α and β be variables taking expressions as values. We say

f *sat* α iff$_{\text{df}}$
$f\ R\ \alpha$ for every R satisfying (i) - (vii)

(i) $f\,R\ulcorner\, x_i = x_j\urcorner$ iff$_{\text{df}}$ $f(i) = f(j)$

(ii) $f\,R\ulcorner\, x_i \in x_j\urcorner$iff$_{\text{df}}$$f(i) \in f(j)$

(iii) if $f\,R\,\alpha$ and $f\,R\beta$ then $f\,R\ulcorner\alpha\wedge\beta\urcorner$

(iv) if $f\,R\,\alpha$ or $f\,R\beta$ then $f\,R\ulcorner\alpha\vee\beta\urcorner$

(v) if for no g extending f does $g\,R\ulcorner\alpha\urcorner$ hold then $f\,R\ulcorner\neg\alpha\urcorner$

(vi) if there is some g extending f such that $g\,R\ulcorner F(x_i)\urcorner$ then $f\,R\ulcorner(\exists x_i)(F(x_i))\urcorner$

(vii) if for every g extending f with $i \in dom(g)$, $g\,R\ulcorner F(x_i)\urcorner$ then $f\,R\ulcorner(\forall x_i)(F(x_i))\urcorner$

Then we say that ϕ is true in $\mathfrak{M}$ iff the empty partial assignment function *sat* ϕ.

3 Consistency Proofs

Once we have a formal notion of truth-in-a-structure $\mathfrak{M}$ and a formal notion of theorem-of-T we have the possibility of formally proving (or refuting) assertions like "All theorems of T are true in $\mathfrak{M}$". The obvious way to prove such assertions is by structural induction on the recursive datatype of theorems of T. There are in fact very many proofs of this kind. Now consider two typed set theories T_1 and T_2 both with extensionality and both with an axiom scheme of comprehension at each type. T_1's axiom scheme is

$$\forall\vec{x}\exists y\forall z(z \in y \longleftrightarrow \phi(z, \vec{x}))$$

where no bound variables may appear in ϕ that are of higher type than 'y', and the axiom scheme for T_2 lacks this restriction.

One would expect that T_2 should be able to prove the consistency of T_1 , and this is in fact the case.

This is illustrated beautifully by the results in McNaughton [15] and the general treatment in Wang [28]. Another standard result with the same flavour was proved by Shoenfield [24] and by Novak [17] . The set theory GB is obtained from ZF by adding a scheme of class existence (so that the class of all x such that ϕ exists as long as ϕ contains no bound class variables) and substituting for the replacement scheme an axiom that says that the image of a set in a class is a set. GB is consistent if ZF is. In fact Shoenfield shows that there is a primitive recursive function that will accept the Gödel number of a proof of a theorem-about-sets (in GB) and return the Gödel number of a ZF proof of the same theorem. However, if we allow bound class variables to appear in the class existence scheme we obtain a new theory, nowadays commonly called *Morse-Kelly*, which is stronger than ZF. Quine [19] is good on this point. See also Wang [27]. One consequence of this is that, for any sensible system of type theory, one is liable to find that one can prove the consistency of any proper initial segment of it in some larger initial segment. Let us go into a little detail on this, and take the example of the construction in TST of a truth-definition for the theory of the bottom three types of a model of TST. By means of iterating the singleton relation (as in section 1) we can represent the first three types all as sets of the same level (probably level 5), and the satisfaction predicate will be a set of ordered pairs a few levels higher. Thus we have the theorem

THEOREM 2 *For all k and all sufficiently large n,* $\mathrm{TST}_{k+n} \vdash Con(\mathrm{TST}_k)$

This gives rise to interesting complications when we introduce polymorphism, which is the subject of the next section.

3.1 More quantifiers

Before we leave the subject of truth-definitions altogether we should mention Levy [13]. In this beautiful monograph Levy makes *inter alia* the point that if we are trying to define a satisfaction relation on a set of formulæ that is not itself a recursive datatype, then the satisfaction relation we are trying to define is not itself an inductively defined set of ordered pairs, and so can be defined without appeal to a comprehension principle using quantification over objects of higher type. What he shows is the following. Let Σ_n be the class of formulæ with no more than n unrestricted quantifiers. Then a truth definition for Σ_n formulæ belongs to Σ_{n+1}. This has the important consequence that if we increase the number of (unrestricted) quantifiers we allow to appear in instances of the comprehension scheme of our theory, we increase its consistency strength. See Levy [13].

3.2 Polymorphism

Polymorphism is more general than we tend to think. It is really the phenomenon of theories and languages with endomorphisms and automorphisms. There are several forms polymorphism can take. People who can read German should read Specker [25], which is the best introduction to this topic.[9] He starts with the example of duality between points and lines in projective geometry. There is an automorphism (in fact an involution) of the language of projective geometry that swaps quantifiers over points with quantifiers over lines, and swaps "x and y intersect at z" with "z goes through x

[9] Those who can't could consult Scott's review of it in Mathematical Reviews. [23]

and y". Let us write this automorphism as Specker does, with an asterisk. Clearly if ϕ is an axiom of projective geometry, so is ϕ^*. Indeed * extends to a endomorphism defined on the recursive datatype of proofs, and this enables us to prove by induction on that datatype that ϕ^* is a theorem of projective geometry iff ϕ is. The important point is that this is not the same as saying that $\phi \longleftrightarrow \phi^*$ is a theorem. Specker says that this scheme of biconditionals is actually the same as adopting Pappus's theorem as an axiom).[10]. Another example of an automorphism of a language is the automorphism of the language of first-order predicate calculus obtained by replacing every atomic formula by its negation. Like the projective geometry example this automorphism is an involution.

In general a theory T can have an automorphism σ (so that $T \vdash \phi$ iff $T \vdash \sigma(\phi)$ without this implying $T \vdash \sigma(\phi) \longleftrightarrow \phi$. Specker gives some examples which won't be covered here but the involution mentioned in the last paragraph ("negate all atomics and negatomics") is an easy example which is to hand. These are all examples of Specker's first kind of typical ambiguity: and automorphism σ of the language giving rise to a theorem scheme to the effect that $\vdash \phi$ iff $\vdash \sigma(\phi)$.

Elegant though these examples are, they is a little remote from our concerns here. Closer to HOL is the theory T$\mathbb{Z}$T, which is defined above. The language in which it is expressed has an automorphism too, like the language of projective geometry. In fact it has an infinite group of them, all generated by one which we will notate with an asterisk and which arises as follows. Simply raise every type index attached to a variable in a formula ϕ by one to obtain a new formula ϕ^*. For example, asterisk of

$$(\forall x_2 y_2)(x_2 = y_2 \longleftrightarrow (\forall z_1)(z_1 \in x_2 \longleftrightarrow z_1 \in y_2))$$

is

$$(\forall x_3 y_3)(x_3 = y_3 \longleftrightarrow (\forall z_2)(z_2 \in x_3 \longleftrightarrow z_2 \in y_3))$$

(The reason for working with T$\mathbb{Z}$T rather than TST at this point is to ensure that * is not an endomorphism but an automorphism, as with projective geometry). As with projective geometry we notice that ϕ is a theorem of T$\mathbb{Z}$T iff ϕ^* is. As before we prove this by induction of the recursive datatype of proofs. (indeed * gives rise to an automorphism of this datatype, though this automorphism is of infinite order and is not an involution as it was with projective geometry). This form of polymorphism, which is the kind we find in HOL and in the type theory of Russell and Whitehead was called by them "Typical Ambiguity"[11]: since the axioms (and therefore the theorems) are the same at each type, there is no need to put in the type indices.

One reaction to the fact that the theorems are the same at each type is to omit the type indices on the grounds that no information is provided by putting them in. Another is to quantify universally over the type indices. We can do this conservatively only if the type indices are variables of the language. For Russell and Whitehead [22] and for Church [2] they were not. Nor are they for any of the systems we consider here.

[10] Specker even shows how to express the conjunction of finitely many expressions of the form $\phi \longleftrightarrow \phi^*$ as another expression of that form. This depends on * being an involution and doesn't apply in the cases below.

[11] I know of no good reason for this term to have been replaced by 'polymorphism': people who study TST continue to use the old word. I assume this is another example of a neologism arising because people are unfamiliar with the literature. The original reference is Russell and Whitehead vol 1 end of ch 2.

In some ways the situation is a bit like that with regard to first-order predicate calculus. In that language we cannot quantify over predicates, and so expressions like

$$(\forall F)((\forall x)F(x) \to (\exists y)(F(y)))$$

simply do not make sense. However they can be *given* a sense: the expression $(\forall x)(F(x) \vee \neg F(x))$ is a valid wff of first order logic. That means that $(\forall x)(F(x) \vee \neg F(x))$ for all choices of F . One could, if one felt like it, hang a '$(\forall F)$' in front of the formula to express this fact.

We can distinguish between polymorphisms of theories and polymorphisms of models. What we have considered so far is polymorphism of theories. We can also consider the following. Consider a model $\mathfrak{M}$ of T$\mathbb{Z}$T. Ask yourself: if we alight on a type, can we tell which type we have alighted on? This is equivalent to the question: is there a sentence true at a unique type? For suppose there is a sentence ϕ which is true only at type n, say. Then ϕ^* is true only at type $n-1$, ϕ^{**} is true only at type $n-2$ and so on. The lack of any sentence true at a unique type is a kind of polymorphism: i tried calling it "ergopdic ambiguity" but it never caught on. This is because it is too weak to be interesting. It is not a first-order property of $\mathfrak{M}$, and standard model-theoretic techniques will allow us to construct models that satisfy this without adding any axioms to T$\mathbb{Z}$T. We will be able to do this even if T proves $\phi \longleftrightarrow \neg\phi^*$!

Returning again to typical ambiguity of theories, a much stronger kind of polymorphism/typical ambiguity is the assertion that all types look the same. If Γ is a class of formulæ in $\mathcal{L}_{TNT}$, the language of T$\mathbb{Z}$T, we call the scheme $\phi \longleftrightarrow \phi^*$, for $\phi \in \Gamma$, "Γ-ambiguity, $Amb(\Gamma)$. Ambiguity for all formulæ is just *Amb*. The full scheme *Amb* is strong, and is not known to be consistent. We can prove $Amb(\Gamma)$ consistent for various natural classes Γ. I will mention only three of these, and only one of those three will be pursued here.

Finally there is a much stronger kind of ambiguity, which is considered by Specker [1958]. A theory T is ambiguous in this strong sense if every model of T has an automorphism that respects *. Projective geometry plus Pappus's theorem has this property: every model of this theory has an automorphism that exchanges points and lines.) I know of no consistent version of T$\mathbb{Z}$T has this feature and it is not of any concern to us here.

In Set theory/Type Theory the historically earliest example found by anyone of an instance of the second kind of typical ambiguity is the class of all formulæ that mention only two types (Grishin [7]). The background to this is that all one can say in a typed set theory with two types can be said in the first-order theory of infinite atomic boolean algebras, and this is known to be a complete theory.

For us the most important example of a Γ for which $Amb(\Gamma)$ is known to be consistent relative to T$\mathbb{Z}$T is the class of all formulæ of the form $(\forall x_1 \ldots x_n)\phi$ where ϕ is built up from atomics by the usual quantifiers and connectives and restricted quantifiers in the style of Levy [13] and quantifiers $(\forall x \in \mathcal{P}(y)$ and $(\exists x \in \mathcal{P}(y)$ (where $\mathcal{P}(y)$ is the power set of y). This is in Kaye-Forster [5]. In the terminology of that paper, Γ is $\Sigma_1^{\mathcal{P}}$. This result leads us directly to the material of the next section. This section concludes with three tangential minor topics which can be safely skipped. In the next section we will see how schemes of typical ambiguity give rise to consistency results for untyped set theories.

3.3 Extensionality

Interestingly, in view of the way in which extensionality is proof-theoretically problematic, one can show that if the axiom of extensionality is weakened to allow lots of empty sets (or urelemente) but retained for nonempty sets (so that distinct nonempty sets have distinct members) to obtain a system which we call TSTU, then the axiom scheme $\phi \longleftrightarrow \phi^*$ can be added without any extra consistency strength being gained. This is in Jensen's revolutionary paper [9].

3.4 Automorphisms of type algebras

The idea of polymorphism or typical ambiguity for a type theory is of course tied up with the idea of an automorphism of what one might call the type algebra of the theory under consideration. The most straightforward case is T$\mathbb{Z}$T. Its types are indexed by $\mathbb{Z}$ not by $\mathbb{N}$, so that the type algebra is the monad $\mathbb{Z}$. Asserting the biconditional $\phi \longleftrightarrow \phi^*$ for all ϕ has the same effect as asserting the biconditional $\phi \longleftrightarrow \phi^n$ (where ϕ^n is the result of applying n asterisks to ϕ) for all ϕ and all n. This second, more inclusive scheme is probably what one would naturally think of as an axiom scheme of polymorphism but it follows from the weaker version because the automorphism group of the type algebra $\mathbb{Z}$ is cyclic.[12] The type algebras of even quite simple elaborations of TST (consider for example the Church-style type theory with only one type constructor (namely function types) and where every type is a function type) are much more complicated. However it is known that the automorphism group of the type algebra of this last theory is a finitely generated simple group.[13] If we wish to express ambiguity schemes for theories like Church's simple type theory we will need to express them in the second form.

Note that in order to have here a concept of "higher type" which behaves like our concept of higher type in TST we will need a natural definable partial order of the type algebra.

3.5 T$\mathbb{Z}$T: The theory of (positive and) negative types

Theorem 2 tells us that TST proves the consistency of all its proper initial segments. This means that T$\mathbb{Z}$T proves the consistency of TST$_k$, for each k. Now we were able to infer the consistency of T$\mathbb{Z}$T from the consistency of TST by a simple compactness argument (any proof of an inconsistency in T$\mathbb{Z}$T can be reproduced inside TST$_k$ for some k) so we know that T$\mathbb{Z}$T $\nvdash$ *Con*(TST). If there were such a consistency proof, we could reproduce the compactness argument inside T$\mathbb{Z}$Tand T$\mathbb{Z}$T would prove its own consistency. Therefore T$\mathbb{Z}$T $\vdash (\forall k)Con(TST_k)$. Therefore T$\mathbb{Z}$T is ω-incomplete. Although any consistent recursively axiomatisable system extending arithmetic is incomplete, the ω-incompleteness is not always this apparent. It is even an open question

[12] Both the type algebra and its automorphism group are naturally called Z!

[13] R. C. Thompson's group—Thank you John Conway!

whether or not T$\mathbb{Z}$T $+ Amb \vdash (\forall k)Con(TST_k)$. T$\mathbb{Z}$T is a bit odd in other ways. Although (as we have seen) its consistency follows by a compactness argument, it has no standard model. See Forster [4]. Although TST and T$\mathbb{Z}$T are equiconsistent, and TST can be interpreted in T$\mathbb{Z}$T (obviously!) it seems fairly clear that there is no interpretation of T$\mathbb{Z}$T in TSTI. (Certainly none that commute with *!) This shows that a result of Harvey Friedman [6]) to the effect that two equiconsistent finitely axiomatisable theories can be interpreted in one another cannot be strengthened by dropping the italicised condition. We can detour here briefly to have a sketch of a proof that T$\mathbb{Z}$T + typical ambiguity for all formulæ implies the axiom of infinity. We can implement cardinal arithmetic at each type, and we can make various assertions about the cardinal number of the universe at each type. One thing we can prove is that $|V_{n+1}| = 2^{|V_n|}$ at each type n. Also, for any number k, we can consider the sequence $\langle k, log_2 k, log_2(log_2 k), \ldots \rangle$ and ask how many of this sequence are whole numbers. Let us do this to $|V_n|$. It turns out that we need the negation of the axiom of infinity to ensure that the length of this sequence is well-defined, preferably in the form "$|V_n|$ is a natural number". Clearly, since $|V_{n+1}| = 2^{|V_n|}$ we know that the length of the sequence we obtain starting at $|V_n|$ is one less than the length of the sequence starting at $|V_{n+1}|$. In particular their lengths are of different parities (remainder mod 2). Let ϕ be the assertion that the sequence obtained in this way is of odd length. So as long as we can express ϕ within the theory, it looks as if we should be able to prove $\phi \longleftrightarrow \neg\phi^*$. It turns out that this is indeed the case, and so complete ambiguity proves the axiom of infinity.

4 Untyped Set Theories: KF, Mac, Z and ZF

4.1 The Kaye-Specker lemma

Recently Richard Kaye has proved a very useful theorem which enables us to infer the consistency of one sorted theories from typed theories with ambiguity schemes to which they are related. This is a strengthening of a theorem in Specker [26].

LEMMA 1 Kaye. [10] 1991
Suppose that $M = \langle M_0, M_1, M_2 \ldots \rangle$ is a structure for the language of TST *and that Σ is the class of formulae of the form "$\exists\vec{x}\Phi(\vec{x},\vec{y})$" for Φ in some class Δ which contains all atomic formulae and is closed under conjunction and substitution of variables and contains $\psi^+(\vec{y})$ whenever it contains $\psi(\vec{x})$.*

Suppose further that $M \models Amb(\Sigma)$. Then there is a structure for the signature $\langle \in, = \rangle$ that satisfies any σ of the form $\forall\vec{y}\Phi(\vec{y})$, where the result of adding suitable type indices to Φ is true in M and the $\mathcal{L}_{TST}$ formula corresponding to Φ is in Σ.

This is the second of the two results central to this paper. It means that whenever we can prove the consistency of $Amb(\Gamma)$ relative to T$\mathbb{Z}$T, we get a consistency result for a one-sorted set theory: it is the bridge between typed set theories and untyped set theories. It can now be applied to the positive results we saw in the previous section to the effect that weak versions of ambiguity were provable or consistent. The first result, Grishin [7], does not concern us greatly here. Applying Kaye's lemma to this gives us the consistency of a funny set theory called NF_3, but this is not of much interest. The second result (from section 3.1), that we can consistently assume full ambiguity if we drop extensionality for empty sets, gives rise to a consistency result for a system called *NF*U. See Jensen [9]. This is much more interesting than NF_3 and is definitely one to watch. *NF*U has an able and active researcher and promoter in the form of Randall Holmes: `holmes@math.idbsu.edu`.

4.2 KF

For present purposes the most important application of Kaye's lemma to an ambiguity scheme is in Forster-Kaye [5]. Starting from a model of $TST + Amb(\Sigma_1^{\mathcal{P}})$ we can obtain a model of the theory that Kaye and I immodestly called KF. First, some terminology: A Δ_0 formula is one that contains no unrestricted quantifiers. Restricted quantifiers are quantifiers in the style "$(\forall x \in y)(\ldots$" and "$(\exists x \in y)(\ldots$". A formula of the language of set theory is *stratified* if it can become a formula of the language of TST by adding type indices consistently to the variables in it.

KFI(= KF+ infinity) is a theory in a one sorted language with two primitives, $\in$ and $=$. It has the following axioms

1. Extensionality: $(\forall xy)(x = y \longleftrightarrow (\forall z)(z \in x \longleftrightarrow z \in y))$
2. Empty set: $(\exists x)(\forall y)(y \notin x)$
3. Pairing: $(\forall xy)(\exists z)(\forall w)(w \in z \longleftrightarrow (w = x \vee w = y))$
4. Union: $(\forall x)(\exists y)(\forall z)(z \in y \longleftrightarrow (\exists w)(z \in w \wedge w \in x))$
5. Powerset: $(\forall x)(\exists y)(\forall z)(z \in y \longleftrightarrow z \subseteq x)$
6. Infinity: There is an infinite set.
7. Stratified Δ_0 separation (Axiom scheme: one instance for each stratified Δ_0 ϕ):

   $(\forall w_1 \ldots w_n)(\forall x)(\exists y)(\forall z)(z \in y \longleftrightarrow (z \in x \wedge \phi(z)))$ (where the '$(\forall w_1 \ldots w_n)$' binds all the remaining free variables in ϕ)

4.3 MacLane Set theory

In fact a refinement extends this to a relative consistency proof of a theory trading under various names in the literature, but which I was brought up by my Doktorvater Adrian Mathias to call 'Mac' after Saunders MacLane, who advocated it as an adequate basis for all of mathematics. Mac is like KF except in not having the restiction to stratified ϕ in axiom scheme 7. The ϕ still have to be Δ_0 though. The equivalence of Mac and TST was first proved by Jensen [9] and clarified in Lake [12]. There are also distinct proofs in Mathias [14] and Kaye-Forster [5] In fact Mathias shows further that all the axioms of Kripke-Platek set theory (see Barwise [1] for an axiomatisation of Kripke-Platek) can be added to Mac without gaining extra consistency strength.

5 Zermelo set theory

It seems that this is as far as one can naturally go without reaching much stronger systems. The next things to consider are the stronger systems formulated by allowing more quantifiers in the separation scheme but these systems have not attracted much attention and do not seem to have any proper names. The next natural step up the ladder is Zermelo set theory which is the union of all those. To be precise Zermelo is like KF except in having no restrictions whatever on the formulæ in axiom scheme 6. Zermelo is of course much stronger than Mac and KF etc. We have just seen that Mac is precisely as strong as TST and HOL, and it is an old result of Kemeny's [11] that we can prove the consistency of TST in Zermelo. Although it is not the first set theory we meet here, Zermelo was historically the first one-sorted axiomatic set theory (if we neglect the naïve set theory of Frege), and there are two natural-looking structures that it can be thought of as axiomatising. We need some definitions to describe them.

$$\beth_0 = \aleph_0; \beth_{n+1} = 2^{\beth_n}$$

H_κ is the set of things hereditarily of size $< \kappa$. Recursively

- The empty set is in H_κ
- If $x \in H_\kappa$ and $|x| < \kappa$ then $x \in H_\kappa$

($|x|$ is the cardinality of x). $ZF \vdash H_\kappa$ is a set.

$$V_0 = \emptyset; V_\alpha = \mathcal{P}(\bigcup_{\beta<\alpha} V_\beta)$$

A set is wellfounded iff it appears in some V_α. The least α such that $x \in V_\alpha$ is the **rank** of x. $V_{\omega+\omega}$ is then the object of interest. Zermelo appears to be the theory of $\bigcup_{n<\omega} H_{\beth_n}$ or $V_{\omega+\omega}$ in the sense that both these structures are natural models for it. However there are facts about small sets of low rank that can only be proved by reasoning about sets of high rank, and Zermelo does not prove the existence of sets of rank $\omega + \omega$ or greater. To reason about sets of rank $\omega + \omega$ or greater we need ZF.

5.1 Zermelo-Fränkel set theory

The difference between Zermelo set theory and Zermelo-Fränkel set theory is the axiom scheme of replacement. This axiom scheme says that the surjective image of a set is a set:

$$(\forall x)(\exists! y)(R(xy)) \rightarrow (\forall w)(\exists z)(\forall y)(y \in z \longleftrightarrow (\exists u \in w)(R(u, y))$$

The motivation behind this is a belief that paradoxes are connected with big collections and so the way to avoid paradox is to ensure that large collections do not turn out to be sets. If this "limitation of size" doctrine is true, then it should certainly be safe to suppose that anything the same size as a set is a set. Actually it turns out that the axiom scheme of replacement has very strong consequences (which rather militates against the limitation of size doctrine). In particular it has strong consequences of the kind mentioned above: in ZF we can prove termination of functions whose termination cannot be proved in Zermelo set theory. It has been known for many years that ZF proves the consistency of Zermelo set theory. Nowadays some quite refined information is coming to light about the precise strengths of different kinds of replacement. A variant of replacement is the axiom scheme of collection:

$$(\forall x \in X)(\exists y)(\psi(x.y)) \to (\exists Y)(\forall x \in X)(\exists y \in Y)(\psi(x, y))$$

It is easy to show that collection and comprehension together imply replacement. To show that replacement implies collection assume replacement and the antecedent of collection, and derive the conclusion. Thus $(\forall x \in X)(\exists y)(\psi(x, y))$. Let $\phi(x, y)$ say that y is the set of all z such that $\psi(x, z)$ and z is of minimal rank with this property. Clearly ϕ is single-valued so we can invoke replacement. The Y we want as witness to the '$(\exists Y)$' in the collection axiom is the sumset of the Y given us by replacement. Notice the use of the axiom of foundation here. We use it to get a set of z which are ψ-related to x. This obstructs the proof of this for stratified formulæ : it is not the case that stratified replacement implies stratified collection. The following counterexample is due to Mathias. Consider the assertion: for every natural number n there is a set of size n consisting of infinite sets all of different sizes. This is provable in Zermelo set theory. However in, say, $Z + V = L$ we can show that there is no set which collects all these together, because the sumset of such a set would be an infinite set of infinite sets of infinitely many different sizes, and we know that Zermelo set theory does not prove the existence of such a set, since in $Z + V = L$ all sets of infinite cardinals are finite. So stratified collection is not provable in Zermelo. However Coret [3] has shown that every stratified instance of the axiom scheme of replacement is provable in Zermelo set theory. Therefore stratified replacement does not imply stratified collection. The motivation for Collection is a lot less obvious than the motivation for Replacement, but it is robust. Recall the concept of Δ_n formula. These contain n restricted quantifiers and as many unrestricted quantifiers as you want. The axiom scheme of collection is precisely what is need to prove a normal form theorem to the effect that all formulæ can be manipulated into a normal form where all the quantifiers are out at the front of the formula (which is standard) and that all the restricted quantifiers come after all the unrestricted quantifiers. The first thing we notice about replacement is that it enables us to prove the existence of sets of size $\beth_\omega$ (the supremum of $\beth_1$, $\beth_2 \ldots$). Consider the following recursively defined function: $f(0) = \mathbb{N}$; $f(n + 1) = \mathcal{P}(f(n))$. We prove easily by induction on the natural numbers that $f(n)$ is defined for all n and is of size $\beth_n$. By replacement, $f``\mathbb{N}$ is a set, and $\bigcup f``\mathbb{N}$ is at least as big as any of the $f(n)$.

Russell and Whitehead remarked in [22] that no construction like this seemed to be possible in their type theory.

6 Current developments and open problems

Holmes [8] considers variants of TST where the types are partially ordered and whenever $\alpha < \beta$ there is a membership relation defined between objects of type α and objects of type β (not just when $\beta = \alpha + 1$!). By Kaye's lemma the full scheme of typical ambiguity is equivalent to the consistency of Quine's NF. It is open whether this theory is consistent. It has already been mentioned that is open whether or not the result of adding full ambiguity to T$\mathbb{Z}$T is a theory that proves $(\forall k)(Con(TST_k))$. It is also open whether or not T$\mathbb{Z}$T has an ω-model; and open whether or not T$\mathbb{Z}$T has a model in which every set is definable. Those interested in realizability interpretations of intuitionistic systems may wish to ponder the following *aperçu* of Holmes. In situations where the innocent kind of typical ambiguity ($\vdash \phi \longleftrightarrow \vdash \phi^*$) holds one always has a simply definable bijection between proofs of ϕ and proofs of ϕ^*. But in realizability interpretations definable maps from proofs to proof are themselves proofs of conditionals so this should give us a means of inferring $\vdash \phi \longleftrightarrow \phi^*$.

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[14] Errata. p 327. Line 11 should read 'and $a \in M$ such that $M \models |\pi\text{‘}a| = |\mathcal{P}(a)|$'. Line 13 the expression following '$M \models$' should be '$|\pi(a)| = |\mathcal{P}(a)|$'. Line 26 '(not just $\pi(a) = \mathcal{P}(a)$)' should read '(not just $|\pi(a)| = |\mathcal{P}(a)|$)'. Line 28 '$\pi(a)$' should read '$|\mathcal{P}(\pi(a))|$'.

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