Henkin on ω-consistency and ω-completeness JSL 1954 and 1957: A commentary by

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This is my attempt to work through [2] and [3]. I started this a while ago, as a result of finding allusions to these two articles in Quine: *Set theory and its Logic* when i was a student. However more recently it has seemed a good idea to examine these two works because (i) i think they predate and perhaps foreshadow Omitting Types (which is something i would like to understand properly)[1] and (ii) when i started looking at Yablo's paradox i found myself being reminded of them.

1 Γ-Consistency

We start with [2], on ω-consistency.

The obvious generalisation of ω-consistency is to the situation of a theory T with a set Γ of constants, and we want to say that:

Definition 1
T is Γ-consistent unless—for some ϕ—$T \vdash \phi(a)$ for every constant a in Γ but nevertheless also proves $(\exists x)(\neg\phi(x))$.

If we write this in the form $\bigvee_{a \in \Gamma} (T \nvdash \phi(a)) \vee T \nvdash (\exists x)(\neg\phi(x))$ it becomes clear that this property is infinitary Horn. I'm not sure how useful this will be. In any case there are several more immediate complications to consider.

First Γ might not be the set of all constants of T. I must confess that it seems to me particularly perverse to jam this particular door open. After all T might prove $\phi(a)$ for every a in Γ and yet prove $\neg\phi(b)$ for some constant b not in Γ. It seems to me that rather than allow Γ to be a proper subset of the constants of T, one should always assume that Γ contains all constants, and if one wants some extra constants, dress them us as one-place predicates with $\exists!$

[1] Omitting Types seems to be in [1] which is 1961.

axioms *à la Quine* if desired.[2] I have taken the liberty of rewriting Henkin's examples with this in mind.

Secondly we should probably consider vectors of variables and polyadic predicate letters.

Finally (and this is what Henkin actually tackles first) is the idea that Γ-consistency is a syntactic notion, and he wants to see how it ties up with the corresponding semantic (model-theoretic) notion. To this end he introduces an idea of T being Γ-**satisfied** by a model, with a view to proving results with the flavour: if T is Γ-consistent then there is a model that Γ-satisfies it. 'Γ' always points to a set of constants.

Definition 2

A theory T is Γ-**consistent** *iff there is no formula $\phi(x)$ (with only one free variable) such that $T \vdash \phi(a)$ for every $a \in \Gamma$ but nevertheless $T \vdash (\forall x)\phi(x)$.*

$\mathfrak{M}$ Γ-**satisfies** *T iff it is a model of T wherein '=' is interpreted by equality and everything in the domain of $\mathfrak{M}$ is the denotation of a constant in Γ.*

Henkin shows that if T is Γ-satisfied by some model $\mathfrak{M}$ then T is Γ-consistent. Also if Γ is finite then the converse holds. These are routine applications of the completeness theorem.

One might fear that if T is a theory without pairing and unpairing then it might be well-behaved for one-placed predicates, so that

> (i) there is no ψ such that $T \vdash (\exists x)\psi(x)$ and $T \nvdash \psi(s)$ for all constants s, but nevertheless ill-behaved for two-place predicates, so that
>
> (ii) there is a formula ϕ such that $T \vdash (\exists x)(\exists y)\phi(x, y)$ but, for all constants s and t, $T \nvdash \phi(s, t)$.

Such a theory would be pathological in that, even tho' it is Γ-consistent, it still wouldn't be Γ-satisfied by any model.

Anyone with such fears would be right. There are indeed such theories, and here is Henkin's example of one. (This is his theory F_0 on p. 185) It has constants a_n for each $n \in \mathbb{N}$, equality, and two unary predicates F and G. Axioms:

1. $a_i \neq a_j$ when $i \neq j$;
2. $(\forall x)(\neg(F(x) \wedge G(x)))$;
3. $(\exists! x)(F(x))$;
4. $(\exists! x)(G(x))$;

[2] Henkin is aware of this. (See bottom of p 189). Indeed he makes the point that one can even use a *single* additional binary predicate letter. I can only assume he prefers a language without predicate letters.

5. $\neg F(a_i) \vee \neg G(a_j)$ for $i \neq j$.

Then $T \vdash (\exists x)(\exists y)(F(x) \wedge G(y))$ but $T \vdash \neg(F(a_i) \wedge G(a_j))$ for all i and j.

First we prove that T admits quantifier-elimination and then we use elimination of quantifiers to show that T is Γ-consistent, that is to say, that whenever $T \vdash \phi(a_i)$ for every i then $T \vdash (\forall x)(\phi(x))$.

Henkin's example is of course a "binary" counterexample. We need to be able to say it is Γ-consistent but not Γ^2-consistent. Let's define this new notion. We obviously want:

DEFINITION 3

- *T is Γ^k-consistent unless—for some k-ary ϕ—$T \vdash \phi(\vec{a})$ for every k-tuple $\vec{a}$ of constants from Γ but nevertheless also proves $(\exists x_1 \ldots x_k)(\neg\phi(x_1 \ldots x_k))$.*
- *T is Γ^∞-consistent if it is Γ^k-consistent for all k.*

By complicating the construction we can come up with theories T_n such that, for some ϕ, $T_n \vdash (\exists x_1 \ldots x_n)(\phi(x_1 \ldots x_n))$ but $T \vdash \neg\phi(\vec{t})$ for each n-tuple $\vec{t}$ of constants. Instead of two unary predicates F and G we have $n+1$, namely $F_1 \ldots F_{n+1}$ and analogous axioms to say that the extensions of the F_i are disjoint singletons. Axiom scheme 5 becomes the scheme

$$\bigvee_{0<i\leq n+1} \neg F_i(\vec{t})$$

... for each tuple $\vec{t}$ of $n+1$ distinct constants.

The ϕ we want is

$$\bigwedge_{0<i\leq n+1} F_i(x_i).$$

T_n proves that there is a tuple satisfying this but refutes each instance over the set of constants.

We presumably still use elimination of quantifiers to prove that T_n is $\Gamma^{(n-1)}$-consistent.

I think what will happen is that any theory that is Γ-satisfied by something must be Γ^k-consistent for all k.

The question now is: is T being Γ^∞-consistent sufficient for T to be Γ-satisfied by something? Sadly, the answer is 'no'. (As before, i have doctored Henkin's presentation of his theory by replacing some constants with monadic predicate letters). The theory T_∞ is as follows. There are infinitely many constants $\{a_i : i \in \mathbb{N}\}$ as before, and infinitely many one-place predicates $\{F_i : i \in \mathbb{N}\}$. As before there are axioms to say all the constants have distinct denotations, and that the extensions of the F_i are pairwise disjoint singletons.

The final suite of axioms is not obvious. It's no use taking the union of all the suites we've seen so far. The intersection of all the T_n is in fact Γ^∞

consistent by a horn-clause argument. But that doesn't help. We need an extra one-place predicate, which we will write 'G'. The extra axiom scheme is:

$$G(a_k) \to \neg(F_1(a_{i_1}) \wedge \ldots F_k(a_{i_k}))$$

... for each $k > 0$ and each k-tuple $\langle a_{i_1}, a_{i_2}, \ldots a_{i_k}\rangle$.

Now in a Γ^∞ model of T_∞ there will be a singleton—a_{17}, say—which is the extension of G. This means that the instances of the new suite where $k = 17$ come to life, so we can deduce $\neg(F_1(a_{i_1}) \wedge \ldots F_k(a_{i_{17}}))$ for all 17-tuples of constants. This immediately puts us back in T_{17} as it were, and ...

OK, Leon, we give up: what *is* the fancy consistency condition that implies satisfiability? He replies:

Note that Γ-consistency of T is equivalent to the condition (shades of the ϵ-calculus!) that to every formula $A(x)$ with one free variable one can associate an $a \in \Gamma$ such that $T \not\vdash (\exists x)(A(x) \wedge \neg A(a))$. Γ^∞-consistency of T is is similarly equivalent to the condition that for every n and every n-tuple of formulæ $A_1(x) \ldots A_n(x)$ all with one free variable there is an n-tuple $a_1 \ldots a_n$ all in Γ such that T does not prove the following:

$$\bigvee_{0 \le i \le n} ((\exists x_i)(A(x_i) \wedge \neg A(a_i)))$$

2 Γ-Completeness

In this section we discuss the second paper. We need some definitions.

Definition 4

*A theory T is Γ^n-**complete** iff whenever $\phi(x_1, \ldots x_n)$ (with precisely n free variables) satisfies the condition $T \vdash \phi(\vec{\gamma})$ for all tuples $\vec{\gamma}$ from Γ, then $T \vdash (\forall \vec{x})\phi(\vec{x})$.*

*A theory T is Γ-**saturated** iff whenever ϕ is a sentence of $\mathcal{L}(T)$ which holds in every structure $\mathfrak{M}$ s.t. $\mathfrak{M}$ Γ-satisfies T, then $T \vdash \phi$.*

Henkin remarks that we don't really need the concept of Γ^n-completeness for all n, since these conditions are all equivalent to Γ^1-completeness.

Perhaps i should prove this...

Henkin remarks further that Γ-saturation implies Γ-saturation but not conversely. The failure of the converse is theorem 4 on p. 4. His theory T has a countably infinite set Γ of constants, and a further *un*countable set Λ of constants. And no more constants, and no predicate symbols other than '='. The only axioms are those that say that all members of Λ are distinct.

Clearly every model of T is uncountable and so does not Γ-satisfy T. So, vacuously, *every* sentence in $\mathcal{L}(T)$ is true in every model that Γ-satisfies T—even those senteces not provable in T. This shows that T is not Γ-saturated.

However T is Γ-complete. Suppose $\phi(x)$ is a formula of $\mathcal{L}(T)$ containing only the one free variable 'x', such that $T \vdash \phi(\gamma)$ for each γ in Γ. Let γ' be the first constant in Γ that does not appear in $\phi(\gamma)$. By **UG** we eventually get a T-proof of $(\forall x)\phi(x)$

References

[1] Gzegorczyk, A Mostowski and C Ryll-Nardzewski, "Definability of sets in models of axiomatic theories" Bull. Acad. Polon. Sci Sér. Sci. Math. Astron. Phys **9** 163–167.

[2] Leon Henkin "A Generalisation of the concept of ω-consistency", The Journal of Symbolic Logic **19** No. 3 (Sep., 1954), pp. 183–196

[3] Leon Henkin "A Generalisation of the concept of ω-completeness". The Journal of Symbolic Logic, **22** No. 1 (Mar., 1957), pp. 1–14