Asenjo's System LP has no connective obeying *Modus Ponens*

Thomas Forster

April 9, 2012

This brief note is the result of a talk given by Jc Beall in Auckland in 2011, in which he asked the question answered in the title. I am indebted to him for showing me this amusing puzzle and for supplying historical background.

A much expanded version of this note (co-authored with Jc Beall and Jeremy Seligman) which includes some speculative material on the significance of the result for the dialethist project, and a very nice strengthening by Seligman of the main result here (quoted at the end of the paper) has been accepted by NDJFL.

We consider the system LP of [1]. It has connectives $\neg$, $\wedge$ and $\vee$. There are two truth-values, 0 and 1 (false and true). A *valuation* for LP is a function v from the primitive propositions to $\mathcal{P}(\{0,1\}) \setminus \{\emptyset\}$. That is to say, a value of a valuation must be $\{0\}$ or $\{1\}$ or $\{0,1\}$. A *deterministic* valuation is one that doesn't ever take the value $\{0,1\}$.

This semantics is equivalent to the usual semantics that takes LP-valuations to have *three* truth-vales: 0, 1/2 and 1. In our scheme 0 corresponds to our $\{0\}$; 1/2 corresponds to $\{0,1\}$ and 1 to $\{1\}$.

`Eval` is a function that takes a valuation and an LP-formula and returns a truth-value. We define it by the obvious recursion:

$$
\begin{aligned}
\texttt{eval}(v,p) &= v(p) \text{ when } p \text{ is a propositional letter;}\\
\texttt{eval}(v,\neg\phi) &= \{1-n : n \in v(\phi)\};\\
\texttt{eval}(v,\psi\wedge\phi) &= \{\min\{n,m\} : n \in v(\phi) \text{ and } m \in v(\psi)\};\\
\texttt{eval}(v,\psi\vee\phi) &= \{\max\{n,m\} : n \in v(\phi) \text{ and } m \in v(\psi)\}.
\end{aligned}
$$

We say that a valuation *satisfies* a formula—"sat(v,ψ)"—iff $1 \in \texttt{eval}(v,\psi)$. (This matches the definition of satisfaction in [1].)

If a valuation v sends a propositional letter p to $\{0,1\}$ we say v **equivocates on** p.

We will need the following elementary observation:

REMARK 1 *For any valuation v and any formula ϕ, we must have at least one (and possible both) of sat(v,ϕ) and sat$(v,\neg\phi)$.*

Proof: The recursive definition of `eval` enables us to prove by structural induction that $\texttt{eval}(v, \phi)$ is never the empty set, so if $\texttt{eval}(v, \phi)$ doesn't contain 1 [that is to say, we do not have $\text{sat}(v, \phi)$] then it must contain 0. But, if $0 \in \texttt{eval}(v, \phi)$, the recursion tells us that $1 \in \texttt{eval}(v, \neg\phi)$—which is to say $\text{sat}(v, \neg\phi)$. ■

Conditionals obeying Detachment

We say a formula ϕ (in the language $\mathcal{L}(p, q)$ of formulæ containing only the propositional letters p and q) **obeys detachment** iff every valuation that satisfies both p and ϕ also satisfies q. We can write this as $\{p, \phi\} \models q$, using the notation $\gamma \models \phi$ to mean that every valuation satisfying everything in Γ will also satisfy ϕ.

It is well-known that the obvious candidate for a ϕ obeying detachment—to wit $\neg p \vee q$—fails to obey detachment because of the valuation v defined by $v(p) = \{0, 1\}$ and $v(q) = \{0\}$. This annoying valuation will be the key to showing the result below. We need one more notion. A formula ϕ might obey detachment *trivially*, for example if it is $p \wedge q$. "Trivially"? Every valuation that satisfies this ϕ satisfies q—even if it does not satisfy p. These cases are of no interest to us: we will say that they are *trivial*.

There are at least three ways of thinking of valuations in LP. The first (which, it seems, is the most usual) is to think of valuations as functions from (the set of) primitive propositions to the set $\{0, 1/2, 1\}$ of three truth-values. The second, which we have set out above, considers that there are only *two* truth-values but views valuations as things whose values are *sets* of truth-values. A third way—which is probably the most helpful in this context—regards valuations as *nondeterministic* functions from (the set of) primitive propositions to the set $\{0, 1\}$ of two truth-values. We say an assignment of a truth-value to a propositional letter is *compatible* with a valuation v if that truth-value is one of the values that v can take at that propositional letter. It is helpful because of the parallel with the study of nondeterministic finite state automata (NFAs). Just as we describe the state of an NFA $\mathfrak{M}$ started in state σ and given a string w as a *set* of states of the NFA (to wit, the set of those states that $\mathfrak{M}$ might be in after having been started in σ and having been fed successively the characters in w) so we can consider the truth-value of an LP-formula ϕ under a valuation v as a *set* of truth-values, namely the set of those truth-values that ϕ can obtain under all assignments of truth-values to propositional letters in ϕ that are compatible with v. More fully: if v is a valuation and ϕ a formula, what is the truth-value of ϕ according to v? Answer: it is the set of all those truth-values that can be obtained in the following way. Assign to each occurrence of a propositional letter in ϕ a truth-value compatible with v, then compute the resulting truth-value of ϕ (in the usual way). Observe that, under a nondeterministic valuation, distinct occurrences of any particular variable may receive differing truth values.

The second way of thinking about valuations in LP can be seen as arising from the third way by the same ruse that is used in the standard demonstration that to every nondeterministic finite automaton there corresponds a deterministic finite automaton with the same recognition behaviour.

Theorem 2 *No formula nontrivially obeys detachment.*

Proof:

Let p and q be two propositional letters, and let ϕ be a formula in the language $\mathcal{L}(p, q)$ and consider the valuation v defined by $v(p) = \{0, 1\}$ and $v(q) = \{0\}$. What is $\texttt{eval}(v, \phi)$?

This is where we need the third way (see above) of thinking of valuations for LP as *nondeterministic (single-valued)* functions from letters to $\{0, 1\}$. Computing `eval` of a valuation-and-a-formula is rather like computing a trajectory of a nondeterministic finite automaton. The set $\texttt{eval}(v, \phi)$ of possible truth-values of ϕ under the (nondeterministic) valuation v is simply the set of possible values ϕ might take given the possible values given to p and to q by v. Consider the various occurrences of 'p' inside ϕ. Since $v(p) = \{0, 1\}$ we can think of them as each being given truth value 0 or 1 independently. We want $\texttt{eval}(v, \phi)$ to contain 1. (This will ensure that v is a valuation that is witness to the fact that ϕ does not nontrivially obey detachment). It will do so as long as there is *even one* allocation of 0 and 1 to the various occurences of p inside ϕ that makes ϕ come out true (given that $v(q) = \{0\}$). So suppose there is no such allocation. Now this is simply to say that no valuation that satisfies $\neg q$ can satisfy ϕ. So every valuation v' that satisfies ϕ must fail to satisfy $\neg q$. But—by remark 1—every valuation must either satisfy q or satisfy $\neg q$, so v' must satisfy q. So any valuation that satisfies ϕ also satisfies q—so ϕ obeys detachment only trivially.

■

Here is an alternative proof for those who do not like NFAs.

Let v be the bad valuation, the one that sends q to false and equivocates on p. One of two things must happen. Either v satisfies ϕ (in which case ϕ does not obey detachment) or it doesn't. If v doesn't satisfy ϕ then there is no way of assigning 0s and 1s to occurrences of p in ϕ to make ϕ come out true. So in particular the two (deterministic) valuations v_1 (which makes all occurrences of p true and all occurrences of q false) and v_2 (which makes all occurrences of p false and all occurrences of q false) do not satisfy ϕ either. But these three valuations we have just considered exhaust all valuations that do not satisfy q (on the assumption that ϕ obeys detachment) so the only valuations left are valuations that do satisfy q. So every valuation that satisfies ϕ satisfies q. So: according to the first horn ϕ does not satisfy detachment; according to the second horn ϕ satisfies detachment trivially. ■

On being shown the analysis above, Jeremy Seligman immediately refined it to obtain the following result, included here with his permission.

Let us write '$\mathcal{PROP}(\psi)$' for the set of propositional letters occurring in ψ.

Theorem 3 *("Strong Interpolation for LP")*

If $\Gamma \models \phi$ *then* $\{\psi \in \Gamma : \mathcal{PROP}(\psi) \cap \mathcal{PROP}(\phi) \neq \emptyset\} \models \phi$.

References

[1] Asenjo, F. G., "A calculus of antinomies Notre Dame Journal of Formal Logic, vol. 16 (1966), pp. 103105. 1