Groups: Example Sheet 4 of 4

1. If $A \in M_n(\mathbb{C})$ with entries A_{ij}, let $A^\dagger \in M_n(\mathbb{C})$ have entries $\overline{A_{ji}}$. A matrix is called *unitary* if $AA^\dagger = I_n$. Show that the set $U(n)$ of unitary matrices is a subgroup of $GL_n(\mathbb{C})$. Show that
$$SU(n) = \{A \in U(n) \mid \det A = 1\}$$
is a normal subgroup of $U(n)$ and that $U(n)/SU(n) \cong S^1$. Show that Q_8 is isomorphic to a subgroup of $SU(2)$.

2. Suppose that N is a normal subgroup of $O(2)$. Show that if N contains a reflection then $N = O(2)$.

3. Which pairs of elements of $SO(3)$ commute?

4. Write the following permutations as products of disjoint cycles and compute their order and sign.
   (a) $(12)(1234)(12)$;
   (b) $(123)(45)(16789)(15)$.

5. What is the largest possible order of an element in S_5? What about in S_9? Show that every element in S_{10} of order 14 is odd.

6. Show that if H is a subgroup of S_n containing an odd permutation then precisely half of the elements of H are odd.

7. Show that S_n is generated by each of the following set of permutations:
   (a) $\{(j, j+1) \mid 1 \leq j < n\}$;
   (b) $\{(1, k) \mid 1 < k \leq n\}$;
   (c) $\{(12), (123\cdots n)\}$.

   Given $1 \leq k < n$ show that $\{(1, 1+k), (123\cdots n)\}$ generates S_n if and only if k and n are coprime.

8. Show that A_5 has no subgroups of index 2, 3 or 4.

9. Let N be a normal subgroup of a finite group G of prime index p.
   (i) Show that if H is a subgroup of G then $H \cap N$ is a normal subgroup of H of index 1 or p.
   (ii) Suppose the conjugacy class of x in G is a subset of N. Show that either the conjugacy class of x in G coincides with its conjugacy class in N or is a disjoint union of p conjugacy classes in N of equal sizes.

10. Show that $G = SL_2(\mathbb{R})$ acts on $\mathbb{C}_\infty$ by Möbius transformations. Compute the orbits and stabilisers of the points 0, i and $-i$. Let
$$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \right\}.$$
Show that $H \leq G$ and compute $\mathrm{Orb}_H(i)$. Deduce that every element g of G can be written $g = hk$ with $h \in H$ and $k \in SO(2)$. How many ways can this be done?

11. Prove that S_n has a subgroup isomorphic to Q_8 if and only if $n \geq 8$. Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to Q_8?

12. Let K be a normal subgroup of order 2 in a group G. Show that K is a subgroup of the centre $Z(G)$ of G. Show that if n is odd then $O(n) \cong SO(n) \times C_2$. Why doesn't a similar argument work if n is even?

13. * Let G be a finite non-trivial subgroup of $SO(3)$. Let X be the set of points on the unit sphere in $\mathbb{R}^3$ fixed by some non-trivial element of G. Show that G acts on X and that there are either 2 or 3 orbits. What can you say about the G that can arise in each case?