Groups: Example Sheet 2 of 4

1. Show that if a group G contains an element of order six, and an element of order ten, then G has order at least 30.

2. Show that the set $\{1, 3, 5, 7\}$ forms a group under multiplication modulo 8. Is it isomorphic to $C_2 \times C_2$ or C_4.

3. How many subgroups does the quaternion group Q_8 have? What about the dihedral group D_8?

4. Let H be a subgroup of a group G. Show that there is a (natural) bijection between the set of left cosets of H in G and the set of right cosets of H in G.

5. What is the order of the Möbius map $f(z) = iz$? What are its fixed points? If h is another Möbius map what can you say about the order and the fixed points of hfh^{-1}? Construct a Möbius map of order 4 that fixes 1 and -1.

6. Show that $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$; $(t, (x, y)) \mapsto (e^t x, e^{-t} y)$ defines an action of $(\mathbb{R}, +)$ on $\mathbb{R}^2$. What are the orbits and stabilisers of this action? There is a differential equation that is satisfied by each of the orbits. What is it?

7. Suppose that G acts on X and that $y = g \cdot x$ for some $x, y \in X$ and $g \in G$. Show that $\mathrm{Stab}_G(y) = g\mathrm{Stab}_G(x)g^{-1}$.

8. Suppose that Q is a quadrilateral in $\mathbb{R}^2$. Show that its group of symmetries $G(Q)$ has order at most 8. For which n is there a $G(Q)$ of order n? *Which groups can arise as a $G(Q)$ (up to isomorphism)?

9. Let G be a finite group and let X be the set of all its subgroups. Show that $(g, H) \mapsto gHg^{-1}$ defines an action of G on X. Show that for $H \in X$, $|\mathrm{Orb}_G(H)| \leq |G/H|$. Deduce that if $H \neq G$ then G is not the union of all conjugates of H.

10. Show that D_{2n} has one conjugacy class of reflections if n is odd and two conjugacy classes of reflections if n is even.

11. Let G be the group of all symmetries of a cube. Show that G acts on the 4 lines joining diagonally opposite pairs of vertices. Show that if l is one of these lines then $\mathrm{Stab}_G(l) \cong D_6 \times C_2$.

12. Show that every group of order 10 is cyclic or dihedral. Suppose that p is any odd prime. *Can you extend your proof to groups of order $2p$?

13. Let G be a finite abelian group acting faithfully on a set X. Show that if the action is transitive then $|G| = |X|$.

14. Let p be a prime. By considering the conjugation action show that every group of order p^2 is abelian. Deduce that there are precisely two groups of order p^2 up to isomorphism.