# Representation Theory — Examples Sheet 4

*On this sheet all representations are complex representations unless stated otherwise*

1. Suppose $G$ is a finite group of odd order and with $k$ conjugacy classes. Show that $|G| \equiv k \mod 16$.

2. Let $G = SU(2)$ and $V_n$ be the vector space of complex homogeneous polynomials of degree $n$ in the variables $x$ and $y$.

   (a) Describe how to view $V_n$ as an irreducible representation of $SU(2)$. What is its character?

   (b) Show that $V_n$ is isomorphic to its dual $V_n^*$.

   (c) Decompose the representations $V_4 \otimes V_3$, $V_3 \otimes V_3$, $S^2V_3$ and $\Lambda^2 V_3$ into irreducibles.

   (d) How do $V_1^{\otimes n}$, $S^nV_1$, $S^2V_n$ and $\Lambda^2 V_n$ decompose into irreducibles for $n \geq 1$. What about $S^3V_2$?

3. Let $SU(2)$ act on the space $M_3(\mathbb{C})$ of $3 \times 3$ complex matrices by

$$A\colon X \mapsto A_1 X A_1^{-1},$$

   where $A_1$ is the $3\times3$ block diagonal matrix with block diagonal entries $A, 1$. Show that this defines a representation of $SU(2)$ and decompose it into irreducibles.

4. Let $\chi_n$ be the character of the irreducible representation of $SU(2)$ of dimension $n+1$. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} K(z)\overline{\chi_n}\chi_m \, \mathrm{d}\theta = \delta_{nm},$$

   where $z = e^{i\theta}$ and $K(z) = -\frac{1}{2}(z - z^{-1})^2$.

5. Let $G$ be a compact group. Show that there is a continuous group homomorphism from $G$ to the orthogonal group $O(n)$ if and only if $G$ has an $n$-dimensional representation over $\mathbb{R}$.

   By considering the action of $SU(2)$ by conjugation on the $2 \times 2$ complex matrices $A$ such that $A = -\overline{A}^T$ and $\operatorname{tr} A = 0$, construct a continuous group homomorphism $SU(2) \to SO(3)$. Deduce that $SU(2)/\{\pm I\} \cong SO(3)$ as topological groups.

6. Write down a Haar measure on $SU(2)$ and prove that it is translation invariant and normalised correctly.

7. The *Heisenberg group* is the group $G$ of order $p^3$ of upper unitriangular matrices over the field with $p$ elements. Show that $G$ has $p$ conjugacy classes of size 1 and $p^2 - 1$ conjugacy classes of size $p$. Find $p^2$ characters of $G$ of degree 1.

   Find an abelian subgroup $H$ of $G$ of order $p^2$. By induction of characters from $H$ to $G$ show that $G$ has $p-1$ irreducible characters of degree $p$. Write down the character table of $G$.

8. Let $\mathbb{F}$ be the field with $2^n$ elements for some $n \geq 1$. Construct the character table of $GL_2(\mathbb{F})$. Deduce that $PGL_2(\mathbb{F})$ is simple for $n \geq 2$. What can you say about $PGL_2(\mathbb{F})$ when $n = 1$?

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