

## Representation Theory — Examples Sheet 3

*On this sheet all groups are finite and all representations are complex representations*

- Find all the characters of  $S_5$  obtained by inducing irreducible representations of  $S_4$ . Use these to reconstruct the character table of  $S_5$ . Then repeat, replacing  $S_4$  by the subgroup  $\langle (12345), (2354) \rangle$  of  $S_5$  of order 20.
- Recall the character table of  $D_{10}$  from sheet 2. Explain how to view  $D_{10}$  as a subgroup of  $A_5$  and then use induction from  $D_{10}$  to  $A_5$  to reconstruct the character table of  $A_5$ .
- Let  $H$  be a subgroup of a group  $G$ . Show that for every irreducible representation  $(\rho, V)$  of  $G$  there is an irreducible representation  $(\rho', W)$  of  $H$  such that  $\rho$  is an irreducible component of  $\text{Ind}_H^G W$ .  
Deduce that if  $A$  is an abelian subgroup of  $G$  then every irreducible representation of  $G$  has dimension at most  $|G/A|$ .
- Obtain the character table of the dihedral group  $D_{2m}$  by using induction from the cyclic group  $C_m$ ; you will want to split into two cases according as  $m$  is odd or even.
- Prove that if  $H$  is a subgroup of a group  $G$ , and  $K$  is a subgroup of  $H$ , and  $W$  is a representation of  $K$  then  $\text{Ind}_K^G W \cong \text{Ind}_H^G \text{Ind}_K^H W$ .
- Calculate  $S^2V$  and  $\Lambda^2V$  for the two-dimensional irreducible representations of  $D_8$  and of  $Q_8$ . Which has the trivial representation as a subrepresentation in each case?
- Let  $\rho: G \rightarrow GL(V)$  be a representation of  $G$  of dimension  $d$ .
  - Compute  $\dim S^n V$  and  $\dim \Lambda^n V$  for all  $n$ .
  - Let  $g \in G$  and  $\lambda_1, \dots, \lambda_d$  be the eigenvalues of  $\rho(g)$ . What are the eigenvalues of  $g$  on  $S^n V$  and  $\Lambda^n V$ ?
  - Let  $f(t) = \det(tI - \rho(g))$  be the characteristic polynomial of  $\rho(g)$ . What is the relationship between the coefficients of  $f$  and  $\chi_{\Lambda^n V}$ ?
  - What is the relationship between  $\chi_{S^n V}(g)$  and  $f$ ? (Hint: start with case  $d = 1$ ).
- Let  $G = S_n$  act naturally on the set  $X = \{1, \dots, n\}$ . For each non-negative integer  $r$ , let  $X_r$  be the set of all  $r$ -element subsets of  $X$  equipped with the natural action of  $G$ , and  $\pi_r$  be the character of the corresponding permutation representation. If  $0 \leq l \leq k \leq n/2$ , show that

$$\langle \pi_k, \pi_l \rangle_G = l + 1.$$

Deduce that  $\pi_r - \pi_{r-1}$  is a character of an irreducible representation for each  $1 \leq r \leq n/2$ . What happens for  $r > n/2$ ?

- Suppose  $\rho: G \rightarrow GL(V)$  is an irreducible representation of  $G$  with character  $\chi$ . By considering  $V \otimes V$ ,  $S^2V$  and  $\Lambda^2V$  show that

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \begin{cases} 0 & \text{if } \chi \text{ is not real-valued} \\ \pm 1 & \text{if } \chi \text{ is real valued.} \end{cases}$$

Deduce that if  $|G|$  is odd then  $G$  has only one real-valued irreducible character.

- Suppose that  $G$  is a Frobenius group with Frobenius kernel  $K$ . Show that if  $\chi$  is a non-trivial irreducible representation of  $K$  then  $\text{Ind}_K^G \chi$  is also irreducible. Hence, explain how to construct the character table of  $G$  given the character tables of  $K$  and  $G/K$ .
- Suppose that  $V$  is a faithful representation of a group  $G$  such that  $\chi_V$  takes  $r$  distinct values. Show that each irreducible representation of  $G$  is a summand of  $V^{\otimes n}$  for some  $n < r$ .
- Suppose  $G$  is a finite group of odd order and with  $k$  conjugacy classes. Show that  $|G| \equiv k \pmod{16}$ .

Comments and corrections to S.J.Wadsley@dpmms.cam.ac.uk.