Linear Algebra: Example Sheet 3 of 4

1. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

The second and third matrices commute; find a basis with respect to which they are both diagonal.

2. By considering the rank of a suitable matrix, find the eigenvalues of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. Hence write down the determinant of A.

3. Let α be an endomorphism of the finite dimensional vector space V over $\mathbb{F}$, with characteristic polynomial $\chi_\alpha(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0$. Show that $\det(\alpha) = (-1)^n c_0$ and $\operatorname{tr}(\alpha) = -c_{n-1}$.

4. Let V be a vector space, let $\pi_1, \pi_2, \ldots, \pi_k$ be endomorphisms of V such that $\mathrm{id}_V = \pi_1 + \cdots + \pi_k$ and $\pi_i \pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \cdots \oplus U_k$, where $U_j = \operatorname{Im}(\pi_j)$.
Let α be an endomorphism on the vector space V, satisfying the equation $\alpha^3 = \alpha$. Prove directly that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

5. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\operatorname{Ker}(\alpha - \lambda\iota)$ and $\operatorname{Ker}(\alpha^2 - \lambda^2\iota)$ necessarily the same?

6. (Another proof of the Diagonalisability Theorem.) Let V be a vector space of finite dimension. Show that if α_1 and α_2 are endomorphisms of V, then the nullity $n(\alpha_1\alpha_2)$ satisfies $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$. Deduce that if α is an endomorphism of V such that $p(\alpha) = 0$ for some polynomial $p(t)$ which is a product of distinct linear factors, then α is diagonalisable.

7. Let A be a square complex matrix of finite order — that is, $A^m = I$ for some $m > 0$. Show that A can be diagonalised.

8. Show that none of the following matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

similar to any of them? If so, which?

9. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

10. (a) Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.
(b) Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal and characteristic polynomials? If A is not diagonalisable, how many possible JNFs are there for A?

11. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Show that there is a vector y such that $(y, \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y))$ is a basis for V.

Show that if β is an endomorphism of V which commutes with α, then $\beta = p(\alpha)$ for some polynomial p. *[Hint: consider $\beta(y)$.]* What is the form of the matrix for β with respect to the above basis?

12. Let α be an endomorphism of the finite-dimensional vector space V, and assume that α is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of α^{-1} in terms of those of α.

13. Prove that that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan normal form a Jordan block $J_m(\lambda^{-1})$. For an arbitrary invertible square matrix A, describe the Jordan normal form of A^{-1} in terms of that of A.

    Prove that any square complex matrix is similar to its transpose.

14. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

15. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_n & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \cdots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}.$$

    Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

16. Let V denote the space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$ and let α be the differentiation endomorphism $f \mapsto f'$.
    (i) Show that every real number λ is an eigenvalue of α. Show also that $\ker(\alpha - \lambda\iota)$ has dimension 1.
    (ii) Show that $\alpha - \lambda\iota$ is surjective for every real number λ.