Γ-INVARIANT IDEALS IN IWASAWA ALGEBRAS

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Abstract. Let kG be the completed group algebra of a uniform pro-p group G with coefficients in a field k of characteristic p. We study right ideals I in kG that are invariant under the action of another uniform pro-p group Γ. We prove that if I is non-zero then an irreducible component of the characteristic support of kG/I must be contained in a certain finite union of rational linear subspaces of $\operatorname{Spec} \operatorname{gr} kG$. The minimal codimension of these subspaces gives a lower bound on the homological height of I in terms of the action of a certain Lie algebra on G/G^p. If we take Γ to be G acting on itself by conjugation, then Γ-invariant right ideals of kG are precisely the two-sided ideals of kG, and we obtain a non-trivial lower bound on the homological height of a possible non-zero two-sided ideal. For example, when G is open in $\mathrm{SL}_n(\mathbb{Z}_p)$ this lower bound equals $2n-2$. This gives a significant improvement of the results of Ardakov, Wei and Zhang [2] on reflexive ideals in Iwasawa algebras.

1. Introduction

1.1. **Prime ideals in Iwasawa algebras.** In recent years, several attempts have been made to understand the structure of prime ideals in non-commutative Iwasawa algebras. These are the completed group algebras Ω_G of compact p-adic analytic groups G with coefficients in the finite field $\mathbb{F}_p$; we refer the reader to the survey paper [1] for definitions and more details about these algebras.

Perhaps the first result in this direction was obtained by Venjakob in [15] with the classification of all prime ideals in Ω_G where G is a non-abelian soluble pro-p group of rank two. After this, the second author showed in [16] that if G is a Heisenberg pro-p group with centre Z then every non-zero prime ideal in Ω_G must contain the kernel of the map $\Omega_G \to \Omega_{G/Z}$. More significantly in [2] and [3] Wei, Zhang and the first author showed that there are no reflexive ideals in Ω_G if G is a uniform pro-p group of Chevalley type. The methods of this paper are very much in the spirit of the latter works.

1.2. **Γ-invariant right ideals.** Our basic set-up is that we have a uniform pro-p group G and a group Γ acting on G by group automorphisms. This action will induce an action of Γ on Ω_G and we are interested in studying the Γ-invariant right ideals in Ω_G. When $\Gamma = G$ and the action is the natural conjugation action then of course the Γ-invariant right ideals of Ω_G are just ordinary two-sided ideals.

As in [2] and [3] the basic strategy is to show that under certain conditions, a Γ-invariant right ideal is controlled by the subring Ω_{G_2}, where G_2 is the second term of the lower p-series of G. Since G_2 is a characteristic subgroup, the action of Γ on G restricts to an action on G_2 and the intersection of our ideal with Ω_{G_2} is still Γ-invariant. We then aim to show inductively that any ideal satisfying our

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conditions is controlled by Ω_{G_n} for every n. In this way we deduce that our original ideal is 0.

In order to prove a control theorem of this type we associate to each right ideal I in Ω_G a "failure of control module" $F_I := I/(I \cap \Omega_{G_2})\Omega_G$. Then I is controlled by Ω_{G_2} precisely if $F_I = 0$.

1.3. **Microlocalisation.** Whenever we have a complete filtered ring A whose associated graded ring B is commutative noetherian, for each multiplicatively closed subset T of B consisting of homogeneous elements we can define the microlocalisation of A at T, which is a certain completion of an Ore localisation of A. Heuristically, this gives rise to a 'sheaf' $\mathcal{O}_A$ of complete filtered non-commutative rings on $\mathrm{Spec}_{\mathrm{gr}}(B)$ — the set of homogeneous prime ideals in B — viewed as a subspace of $\mathrm{Spec}(B)$ with its usual topology.

Similarly, for each finitely generated A-module M we can define a 'sheaf' $\mathcal{M}$ of $\mathcal{O}_A$-modules by microlocalisation in an analogous way. As usual we say such a sheaf is supported on a subset C if each stalk $\mathcal{M}_P = 0$ for each P not in C. We call the minimal such set $\mathrm{Ch}(M)$, the *characteristic support* of M; see §2.4.

1.4. **Main results.** Now the associated graded ring of $A = \Omega_G$ with respect to its natural filtration may be naturally identified with the symmetric algebra $B = \mathrm{Sym}(V)$, where $V = G/G_2$. To study F_I we consider it as the global sections of the 'sheaf' $\mathcal{F}_I$; we show in one of our main results, Theorem 4.5, that if I is Γ-invariant then $\mathcal{F}_I$ must be supported on the set

$$\mathcal{X} := \{P \in \mathrm{Spec}_{\mathrm{gr}}(B) : \mathfrak{g}.v \subseteq P \text{ for some } v \in V \backslash 0\}.$$

Here $\mathfrak{g}$ is a certain Lie algebra that acts naturally on V, constructed from the action of Γ on G — see §4.2 for details. Since in this case B is a polynomial algebra and since V is a finite set, $\mathcal{X}$ is contained in the union of finitely many rational (that is, defined over $\mathbb{F}_p$) linear subspaces of the affine space $\mathrm{Spec}(B)$.

This result puts a restriction on the characteristic support of F_I: the codimension of $\mathrm{Ch}(F_I)$, which coincides with the grade of the module F_I, is bounded below by the minimal dimension of a $\mathfrak{g}$-orbit of a non-zero element of V.

Suppose now that in addition I is a prime ideal of A which is not controlled by Ω_{G_2}. Then F_I has the same dimension as A/I by Proposition 2.6 and moreover $\mathrm{Ch}(A/I)$ is geometrically pure by Gabber's purity theorem. Since $\mathrm{Ch}(F_I)$ is always contained in $\mathrm{Ch}(A/I)$ by Proposition 2.5, it follows that an irreducible component of $\mathrm{Ch}(A/I)$ must be contained in one of the aforementioned rational linear subspaces. This gives us a severe restriction on the possible characteristic support of A/I. We can then deduce in Theorem 4.8 that the grade of A/I, also known as the homological height of I, is bounded below by

$$u := \min\{\dim \mathfrak{g} \cdot v : v \in V \backslash 0\}.$$

In fact, the conclusion of Theorem 4.8 holds if we only assume that I is non-zero and Γ-invariant.

1.5. **Consequences for Iwasawa algebras.** The effect of this for the Iwasawa algebras of Chevalley type discussed in [3] is that not only are there no non-zero reflexive prime ideals in Ω_G, (that is, ideals of homological height 1), but in fact non-zero prime ideals of homological height strictly less than u cannot exist: see

Theorem 5.3. We compute the lower bound u in this case in §5.2; here is a table of the values it takes for each root system Φ:

Φ	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\dim G$	n^2+2n	$2n^2+n$	$2n^2+n$	$2n^2-n$	78	133	248	52	14
u	$2n$	$4n-4$	$2n$	$4n-6$	22	34	58	16	6

We include the dimensions of the associated Chevalley groups G in this table, for the convenience of the reader.

1.6. **An outline of the paper.** This paper should be viewed as an appropriate generalisation and strengthening of the method introduced in [2] and [3]. In Section 2, we recall basic facts about the Frobenius pairs framework developed in [2], and discuss the notions of microlocalisation and characteristic support of modules. We also prove some 'geometric' properties of Frobenius pairs in §2.5.

The goal of Section 3 is to establish Theorem 3.5, a control theorem for invariant ideals; this result should be viewed as a generalisation of the control theorem for normal elements, [2, Theorem 3.1]. We introduce a new notion of a 'source of derivations' $\mathcal{S}$ in §3.2 and an appropriate notion of $\mathcal{S}$-closure $J^{\mathcal{S}}$ of a graded ideal J in §3.4; with these in place, the analogue of the derivation hypothesis of [2, §3.5] simply becomes $\mathcal{D}(J^{\mathcal{S}}) \subseteq J$.

In Section 4, we apply Theorem 3.5 to microlocalisations of Iwasawa algebras and prove our main result, Theorem 4.5. The key step in the proof is Proposition 4.3, which shows that the 'derivation hypothesis' holds for the microlocalisation of the Iwasawa algebra at any graded prime ideal P not in $\mathcal{X}$. This step relies heavily on the linear algebra calculations performed in [3, §1]. The geometric properties of Frobenius pairs established in §2 can now be used to prove the lower bound Theorem 4.8, essentially in the way sketched above.

We discuss the two obvious applications of our methods in Section 5, and compute the invariant u for uniform pro-p groups of Chevalley type in §5.2.

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2. Microlocalisation, characteristic support and purity

2.1. **Frobenius pairs.** Throughout, k will denote an arbitrary base field of characteristic p. Let B be a commutative k-algebra. The Frobenius map $x \mapsto x^p$ is a ring endomorphism of B and gives a ring isomorphism of B onto its image

$$B^{[p]} := \{b^p : b \in B\}$$

in B provided that B is reduced.

Let t be a positive integer. Whenever $\{y_1, \ldots, y_t\}$ is a t-tuple of elements of B and $\alpha = (\alpha_1, \ldots, \alpha_t)$ is a t-tuple of nonnegative integers, we define

$$\mathbf{y}^\alpha = y_1^{\alpha_1} \cdots y_t^{\alpha_t}.$$

Let $[p-1]$ denote the set $\{0, 1, \ldots, p-1\}$ and let $[p-1]^t$ be the product of t copies of $[p-1]$.

Definition. [2, Definition 2.2] *Let A be a complete filtered k-algebra and let A_1 be a subalgebra of A. We always view A_1 as a filtered subalgebra of A, equipped with the subspace filtration $F_nA_1 := F_nA \cap A_1$. We say that (A, A_1) is a* Frobenius pair *if the following axioms are satisfied:*

(i) A_1 is closed in A,
(ii) $\operatorname{gr} A$ is a commutative noetherian domain, and we write $B = \operatorname{gr} A$,
(iii) the image B_1 of $\operatorname{gr} A_1$ in B satisfies $B^{[p]} \subseteq B_1$, and
(iv) there exist homogeneous elements $y_1, \ldots, y_t \in B$ such that

$$B = \bigoplus_{\alpha \in [p-1]^t} B_1 \mathbf{y}^\alpha.$$

2.2. **Microlocalisation.** As in [2], one of our main tools is *microlocalisation.* We refer the reader to [2, §4] for the necessary background.

Lemma. *Let (A, A_1) be a Frobenius pair, let T be a homogeneous multiplicatively closed set in B and let $T_1 := T \cap B_1$. Then*
(a) $(Q_T(A), Q_{T_1}(A_1))$ is also a Frobenius pair, and
(b) $Q_T(A) = Q_{T_1}(A)$.

Proof. For any $s \in A$, we let $\operatorname{gr} s$ denote its principal symbol in B. Recall [2, §4.2] that the microlocalisation $Q_T(A)$ is the completion of the localisation A_S of A at the Ore set S in A defined as follows:

$$S = \{s \in A : \operatorname{gr} s \in T\}.$$

The microlocalisation $Q_{T_1}(A_1)$ is defined similarly. Because the proof of part (a) is very similar to that of [2, Proposition 5.1(a)], we will omit the details. For part (b), note that by [2, Lemma 2.3] we can find a finite set X that simultaneously generates A as a (free) A_1-module, and also $Q_T(A)$ as a (free) $Q_{T_1}(A_1)$-module:

$$A = X \cdot A_1 \quad \text{and} \quad Q_T(A) = X \cdot Q_{T_1}(A_1).$$

Now $Q_{T_1}(A) = A \otimes_{A_1} Q_{T_1}(A_1)$ by definition of the microlocalisation of a module, so

$$Q_{T_1}(A) = X \cdot Q_{T_1}(A_1) = Q_T(A)$$

as required. □

Corollary. *Let M be a finitely generated A-module. Then $Q_T(M) \cong Q_{T_1}(M)$ as $Q_{T_1}(A_1)$-modules.*

Proof. This is a direct consequence of part (b) of the lemma:

$$Q_T(M) = M \otimes_A Q_T(A) = M \otimes_A A \otimes_{A_1} Q_{T_1}(A_1) = M \otimes_{A_1} Q_{T_1}(A_1) = Q_{T_1}(M),$$

and these identifications respect the right $Q_{T_1}(A_1)$-module structures. □

2.3. **Control of ideals.** Quite generally, if A_1 is a subring of the ring A, and I is a right ideal of A, we will write I_1 for the right ideal $I \cap A_1$ of A_1, and say that I is *controlled by* A_1 if and only if $I = I_1 \cdot A$. Controlled ideals are in some sense “understood”, as they “come from” the smaller ring A_1.

Frequently we will be able to prove that the microlocalisation of I is controlled by the microlocalisation of A_1. The next result tells us how to “lift” this information back from the microlocalisation.

Lemma. *Let (A, A_1) be a Frobenius pair, let T be a homogeneous multiplicatively closed set in B and let I be a right ideal of A. Then $Q_T(I)$ is controlled by $Q_{T_1}(A_1)$ if and only if the "failure of control" module $F := I/I_1A$ satisfies $(\operatorname{gr} F)_T = 0$.*

Proof. The ideas in this proof were already present in the proof of [2, Theorem 5.2], but we include the details for the convenience of the reader.

Write $A' = Q_T(A)$, $A'_1 = Q_{T_1}(A_1)$ and $I' = Q_T(I) = I \cdot A'$. Note that I' can be identified with a right ideal of A', by [2, Lemma 4.4(c)]. Since microlocalisation preserves pullbacks [2, Lemma 4.4(e)], $Q_{T_1}(I) \cap Q_{T_1}(A_1) = Q_{T_1}(I \cap A_1)$, so

$$(I \cdot A'_1) \cap A'_1 = (I \cap A_1) \cdot A'_1.$$

By Corollary 2.2, $I' = I \cdot A'_1$, so

$$(I' \cap A'_1) \cdot A' = (I \cap A_1) \cdot A'_1 \cdot A' = (I \cap A_1)A \cdot A'.$$

This shows that I' is controlled by A'_1 if and only if $I \cdot A' = I_1 A \cdot A'$. Since microlocalisation is exact in our setting by [2, Proposition 4.3(d)], this is equivalent to $Q_T(F) = 0$. But $\operatorname{gr} Q_T(F) = (\operatorname{gr} F)_T$, so the result follows from the fact that $Q_T(A)$ is a complete filtered ring. □

2.4. **The characteristic support.** Let A be a filtered ring whose associated graded ring $B = \operatorname{gr} A$ is commutative noetherian. Let $\operatorname{Spec}_{\mathrm{gr}}(B)$ denote the set of all graded prime ideals of B.

Definition. *Let M be a finitely generated A-module. The* characteristic support *of M is the following subset of* $\operatorname{Spec}_{\mathrm{gr}}(B)$*:*

$$\operatorname{Ch}(M) := \{P \in \operatorname{Spec}_{\mathrm{gr}}(B) : \operatorname{Ann}(\operatorname{gr} M) \subseteq P\}.$$

Lemma. $\operatorname{Ch}(M)$ *is independent of the choice of good filtration on M that defines the associated graded module* $\operatorname{gr} M$*.*

Proof. This is well-known; see, for example, [12, Chapter III, Lemma 4.1.9]. □

Any graded prime ideal P of B gives rise to the homogeneous multiplicatively closed set T_P, which consists of all homogeneous elements of B *not* in P. We can then form the localisation B_{T_P} of B and the microlocalisation $Q_{T_P}(A)$ of A. By abuse of notation, we will always write $B_P := B_{T_P}$ and $A_P := Q_{T_P}(A)$ in this case. Furthermore, if M is a finitely generated A-module, then we will write $M_P := Q_{T_P}(M)$ for the microlocalisation of M at T_P.

Proposition. *For any finitely generated A-module M, we have*

$$\operatorname{Ch}(M) = \{P \in \operatorname{Spec}_{\mathrm{gr}}(B) : M_P \neq 0\}.$$

Proof. Let $P \in \operatorname{Spec}_{\mathrm{gr}}(B)$ and let $N = \operatorname{gr} M$; then $M_P = 0$ if and only if $N_P = 0$. Now N is a quotient of a direct sum of copies of $B/\operatorname{Ann}(N)$, and $B/\operatorname{Ann}(N)$ is a submodule of a direct sum of copies of N, so $N_P = 0$ if and only if $(B/\operatorname{Ann}(N))_P = 0$. However the last condition is easily seen to be equivalent to $\operatorname{Ann}(N) \nsubseteq P$, and the result follows. □

Corollary. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated A-modules, then* $\operatorname{Ch}(M) = \operatorname{Ch}(L) \cup \operatorname{Ch}(N)$*.*

Proof. Microlocalisation is exact in our setting: see [2, Proposition 4.3(d)]. □

2.5. **Characteristic support and Frobenius pairs.** Let (A, A_1) be a Frobenius pair. The inclusion $\iota : B_1 \hookrightarrow B$ induces a map $\iota_* : \operatorname{Spec}_{\mathrm{gr}}(B) \to \operatorname{Spec}_{\mathrm{gr}}(B_1)$, given by $\iota_*(P) = P \cap B_1$.

Proposition. *The map ι_* is a bijection. Let I be a right ideal of A, and let $I_1 = I \cap A_1$. Then* $\operatorname{Ch}(A/I) = \operatorname{Ch}(A/I_1A) = \iota_*^{-1}(\operatorname{Ch}(A_1/I_1))$.

Proof. Since B is a finitely generated B_1-module by definition, B is integral over B_1. Hence ι_* is surjective. Because $b^p \in B_1$ for all $b \in B$, we see that ι_* is injective. In fact, it is easy to see that the inverse of ι_* is given explicitly by the formula

$$\iota_*^{-1}(\mathfrak{p}) = \sqrt{\mathfrak{p}B},$$

for any $\mathfrak{p} \in \operatorname{Spec}_{\mathrm{gr}}(B_1)$.

Let $P \in \operatorname{Spec}_{\mathrm{gr}}(B)$ and write $P_1 = \iota_* P$. Since $B_1 \backslash P_1 = (B \backslash P) \cap B_1$, Corollary 2.2 implies that $M_P = M_{P_1}$ for any finitely generated A-module M. We will use Proposition 2.4 without further mention in what follows.

Since A/I is a quotient of A/I_1A, $\operatorname{Ch}(A/I) \subseteq \operatorname{Ch}(A/I_1A)$ by Corollary 2.4. Now if $P_1 \notin \operatorname{Ch}(A_1/I_1)$, then $(A_1/I_1)_{P_1} = 0$, so

$$(A/I_1A)_P = (A_1/I_1) \otimes_{A_1} A \otimes_A A_P = (A_1/I_1) \otimes_{A_1} A_P = 0$$

whence $P \notin \operatorname{Ch}(A/I_1A)$. This shows that $\operatorname{Ch}(A/I_1A) \subseteq \iota_*^{-1} \operatorname{Ch}(A_1/I_1)$.

Finally, if $P \notin \operatorname{Ch}(A/I)$ then $(A/I)_P = 0$ and hence $(A/I)_{P_1} = 0$. Since A_1/I_1 is an A_1-submodule of A/I, Corollary 2.4 implies that $P \notin \iota_*^{-1} \operatorname{Ch}(A_1/I_1)$. Hence $\iota_*^{-1} \operatorname{Ch}(A_1/I_1) \subseteq \operatorname{Ch}(A/I)$ and the proof is complete. $\square$

2.6. **Purity of modules.** Let A be an Auslander-Gorenstein ring and let M be a finitely generated A-module. Recall that M has a *grade* $j_A(M)$ defined by the formula

$$j_A(M) = \min\{j : \operatorname{Ext}_A^j(M, A) \neq 0\}.$$

Recall that if $0 \to L \to M \to N \to 0$ is a short exact sequence, then we have

$$j_A(M) = \min\{j_A(L), j_A(N)\}.$$

M is said to be *pure* if $j_A(N) = j_A(M)$ for all non-zero submodules N of M; clearly any submodule of a pure module is itself pure.

Lemma. *Let (A, A_1) be a Frobenius pair such that B and B_1 are Gorenstein, and let M be a finitely generated A-module.*

(a) A and A_1 are also Auslander-Gorenstein.

(b) The grade of M equals the codimension of the characteristic support of M:

$$j_A(M) = j_B(\operatorname{gr} M) = \min\{\operatorname{ht} P : P \in \operatorname{Ch}(M)\}.$$

(c) $j_A(M) = j_{A_1}(M|_{A_1})$.

(d) M is pure if and only if $M|_{A_1}$ is pure.

Proof. (a) Use [7, Theorem 3.9].

(b) For the first equality, use [8, Remark 5.8]. For the second equality, see [5].

(c) The map $\iota_* : \operatorname{Spec}_{\mathrm{gr}}(B) \to \operatorname{Spec}_{\mathrm{gr}}(B_1)$ extends to an order preserving bijection between the usual spectra $\operatorname{Spec}(B)$ and $\operatorname{Spec}(B_1)$; this shows that $\operatorname{ht} \iota_* P = \operatorname{ht} P$ for any $P \in \operatorname{Spec}_{\mathrm{gr}}(B)$. Since M is finitely generated, M is a finite extension of cyclic A-modules; using Corollary 2.4 and Proposition 2.5, we see that

$$\iota_* \operatorname{Ch}(M) = \operatorname{Ch}(M|_{A_1}).$$

Part (c) now follows from part (b).

(d) ($\Leftarrow$) This is easy, given part (c).

($\Rightarrow$) Let N be a non-zero A_1-submodule of M. Since $N \otimes_{A_1} A$ surjects onto the non-zero A-submodule $N \cdot A$ of M and since M is pure, we have

$$j_{A_1}(M) = j_A(M) = j_A(N \cdot A) \geqslant j_A(N \otimes_{A_1} A) = j_{A_1}(N) \geqslant j_{A_1}(M)$$

Here we have used the fact that A is a free A_1-module [2, Lemma 2.3]. Hence $j_{A_1}(N) = j_{A_1}(M)$ and $M|_{A_1}$ is pure. □

Proposition. *Let (A, A_1) be a Frobenius pair such that B and B_1 are Gorenstein. Let I be a right ideal of A, let $I_1 = I \cap A_1$ and suppose that A/I is pure. Then either I/I_1A is zero, or it is pure of the same grade as A/I.*

Proof. By part (d) of the lemma, A_1/I_1 is pure, being an A_1-submodule of the pure A_1-module A/I. It will be enough to show that $A/I_1A \cong (A_1/I_1) \otimes_{A_1} A$ is also pure, and has the same grade as A/I.

Recall [8, Theorem 2.12] that a finitely generated module M over an Auslander-Gorenstein ring R is pure if and only if $\mathrm{Ext}^i_R(\mathrm{Ext}^i_R(M, R), R) = 0$ for all $i > j_R(M)$. Hence $\mathrm{Ext}^i_{A_1}(\mathrm{Ext}^i_{A_1}(A_1/I_1, A_1), A_1) = 0$ for all $i > j_{A_1}(A_1/I_1)$.

Since A is a free right and left A_1-module by [2, Lemma 2.3], [2, Proposition 1.2] implies that $j_A(A/I_1A) = j_{A_1}(A/I_1)$ and also that

$$\mathrm{Ext}^i_A(\mathrm{Ext}^i_A(A/I_1A, A), A) \cong \mathrm{Ext}^i_{A_1}(\mathrm{Ext}^i_{A_1}(A_1/I_1, A_1), A_1) \otimes_{A_1} A = 0$$

for all $i > j_A(A/I_1A)$. So A/I_1A is pure, and our result follows. □

3. A control theorem for $\mathcal{S}$-invariant right ideals

3.1. **Inducing derivations on** $\operatorname{gr} A$. Let A be a filtered ring with associated graded ring B and let α be a ring endomorphism of A. Suppose that there is an integer $m_\alpha \geqslant 1$ such that

$$(\alpha - 1)(F_nA) \subseteq F_{n-m_\alpha}A$$

for all $n \in \mathbb{Z}$. This induces additive maps

$$\begin{array}{rccc} d_\alpha : & \frac{F_nA}{F_{n-1}A} & \to & \frac{F_{n-m_\alpha}A}{F_{n-m_\alpha-1}A} \\ & & & \\ & x + F_{n-1}A & \mapsto & \alpha(x) - x + F_{n-m_\alpha-1}A \end{array}$$

for each $n \in \mathbb{Z}$, which patch together to give a graded endomorphism d_α of the abelian group B.

Lemma. *d_α is a graded derivation of B of degree m_α.*

Proof. Let $x \in F_mA$ and $y \in F_nA$, so that $X = x + F_{m-1}A$ and $Y = y + F_{n-1}A$ are homogeneous elements of B of degree m and n respectively. Then

$$\begin{array}{rcll} d_\alpha(X \cdot Y) & = & \alpha(xy) - xy + F_{m+n-m_\alpha-1}A, & \\ d_\alpha(X) \cdot Y & = & \alpha(x)y - xy + F_{m+n-m_\alpha-1}A, & \text{and} \\ X \cdot d_\alpha(Y) & = & x\alpha(y) - xy + F_{m+n-m_\alpha-1}A. & \end{array}$$

Because $m_\alpha \geqslant 1$, $(\alpha(x) - x)(\alpha(y) - y) \in F_{m+n-2m_\alpha}A \subseteq F_{m+n-m_\alpha-1}A$. Hence

$$\alpha(xy) - xy \equiv \alpha(x)y - xy + x\alpha(y) - xy \mod F_{m+n-m_\alpha-1}A$$

and the result follows. □

3.2. **New sources of derivations.** We now introduce a new notion of "source of derivations" for a Frobenius pair, which is slightly different from the one introduced in [2, §3.3]. We hope that the inconsistency in terminology will not cause any confusion; it is just a matter of language. After reading this paper, the reader may get the feeling that the "real" source of our derivations — at least for our applications — is the Lie algebra $\mathfrak{g}$ defined below in §4.2.

Definition. *A* source of derivations *for a Frobenius pair* (A, A_1) *is a set* $\mathbf{a} = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ *of endomorphisms of* A *such that there exist functions* $\theta, \theta_1 : \mathbf{a} \to \mathbb{N}$ *satisfying the following conditions:*

(i) $(\alpha_r - 1)F_n A \subseteq F_{n-\theta(\alpha_r)}A$ *for all* $r \geqslant 0$ *and all* $n \in \mathbb{Z}$
(ii) $(\alpha_r - 1)F_n A_1 \subseteq F_{n-\theta_1(\alpha_r)}A$ *for all* $r \geqslant 0$ *and all* $n \in \mathbb{Z}$,
(iii) $\theta_1(a_r) - \theta(\alpha_r) \to \infty$ *as* $r \to \infty$.

As in [2, §5], we will need to know that sources of derivations are compatible with microlocalisations. The next result shows that this is indeed the case.

Proposition. *Let* (A, A_1) *be a Frobenius pair, let* T *be a homogeneous multiplicatively closed set in* B, *and let* $T_1 = B_1 \cap T$. *Then*
(a) each source of derivations $\mathbf{a}$ *of* (A, A_1) *induces a source of derivations* $\mathbf{a}_T$ *of* $(Q_T(A), Q_{T_1}(A_1))$,
(b) the derivations of B_T *induced by* $\mathbf{a}_T$ *coincide with the extensions to* B_T *of the derivations of* B *induced by* $\mathbf{a}$.

Proof. Let α be a ring endomorphism of A such that $(\alpha - 1)(F_n A) \subseteq F_{n-m}A$ for some integer $m \geqslant 1$, for all $n \in \mathbb{Z}$. Let $x \in A$; since $\deg(\alpha(x) - x) < \deg x$, we have $\operatorname{gr} \alpha(x) = \operatorname{gr} x$ and in particular $\deg \alpha(x) = \deg x$. Hence $\alpha(S) \subseteq S$, where S denotes the Ore subset of A determined by T, see §2.2. Thus α extends to an endomorphism α of the Ore localisation A_S.

Now if $r \in A$ and $s \in S$, then the formula

$$\alpha(rs^{-1}) - rs^{-1} = (\alpha(r) - r)\alpha(s)^{-1} - rs^{-1}(\alpha(s) - s)\alpha(s)^{-1}$$

together with the explicit description of the filtration on A_S given in [2, Lemma 4.2] shows that

$$(\alpha - 1)(F_n A_S) \subseteq F_{n-m}A_S$$

for all $n \in \mathbb{Z}$. Because $m \geqslant 1$, it follows from this that α preserves the filtration on A_S and hence extends to a ring endomorphism α of the completion $Q_T(A)$ such that $(\alpha - 1)F_n Q_T(A) \subseteq F_{n-m}Q_T(A)$ for all $n \in \mathbb{Z}$. Similarly, if $(\alpha - 1)F_n A_1 \subseteq F_{n-m}A$ then $(\alpha - 1)F_n Q_T(A_1) \subseteq F_{n-m}Q_T(A)$. Part (a) follows, and part (b) is clear. $\square$

3.3. **The delta function.** Let (A, A_1) be a Frobenius pair and n be an integer. Each filtered part $F_n A_1$ is closed in A_1 by definition of the filtration topology, and A_1 is closed in A by assumption. Hence $F_n A_1$ is closed in A, which can be expressed as follows:

$$F_n A_1 = \bigcap_{m \geqslant 0} (F_n A_1 + F_{n-m}A) \, .$$

We can now define a key invariant of elements of $A \backslash A_1$:

Definition. *For any* $w \in A \backslash A_1$, *let* $n = \deg w$ *and define*

$$\delta(w) := \max\{m : w \in F_n A_1 + F_{n-m}A\}.$$

Clearly $\delta(w) \geqslant 0$. Note that if $w \in F_n A \backslash A_1$, then $w \notin F_n A_1 + F_{n-m} A$ for some $m \geqslant 0$ by the above remarks, so the definition makes sense. The number $\delta(w)$ measures how closely the element w can be approximated by elements of A_1. It should be remarked that $\delta(w) > 0$ if and only if $\operatorname{gr} w \in B_1$, since both conditions are equivalent to $w \in F_n A_1 + F_{n-1} A$.

Now suppose that $w \in A \backslash A_1$. By the definition of $\delta := \delta(w)$, we can find elements $x \in F_n A_1$ and $y \in F_{n-\delta} A$ such that $w = x + y$; if $\delta = 0$ we take x to be zero. Note that $y \notin F_{n-\delta-1} A$ by the maximality of δ and hence

$$Y_w := \operatorname{gr} y = y + F_{n-\delta-1} A.$$

In view of our assumption on x, we have $Y_w = \operatorname{gr} w$ when $\delta = 0$.

3.4. **$\mathcal{S}$-closures.** Let (A, A_1) be a Frobenius pair and let $\mathcal{S}$ be a fixed set of sources of derivations of A. If I is a right ideal of A, we say that I is *$\mathcal{S}$-invariant* if for all $\mathbf{a} \in \mathcal{S}$, $\alpha_r(I) \subseteq I$ for all $r \gg 0$.

Definition. *Let (A, A_1) be a Frobenius pair, let $\mathcal{S}$ be a set of sources of derivations of A and let J be a graded ideal of B. The $\mathcal{S}$-*closure *$J^{\mathcal{S}}$ of J is defined to be*

$$J^{\mathcal{S}} := \{Y \in B : \quad \forall \mathbf{a} \in \mathcal{S}, \quad d_{\alpha_r}(Y) \in J \quad \textit{for all} \quad r \gg 0.\}$$

Because d_{α_r} is a B_1-linear derivation of B for large enough r, we see that $J^{\mathcal{S}}$ is a B_1-submodule of B containing B_1. It is in fact a graded B_1-submodule.

Proposition. *Let I be an $\mathcal{S}$-invariant right ideal of A and write $J := \operatorname{gr} I$. Then for any $w \in I \backslash A_1$, $Y_w \in J^{\mathcal{S}}$.*

Proof. Let us write $w = x + y$ as in the previous subsection and let $\mathbf{a} \in \mathcal{S}$. We can find an integer $r_0 \geqslant 1$ such that $\theta_1(\alpha_r) - \theta(\alpha_r) > \delta := \delta(w)$ for all $r \geqslant r_0$. Therefore

$$\begin{aligned} &\alpha_r(x) - x \in F_{n-\theta_1(\alpha_r)} A \subseteq F_{n-\delta-\theta(\alpha_r)-1} A \quad \text{and} \\ &\alpha_r(y) - y \ \in F_{n-\delta-\theta(\alpha_r)} A, \end{aligned}$$

for all $r \geqslant r_0$. Hence

$$\begin{aligned} &\alpha_r(w) - w \ \in F_{n-\delta-\theta(\alpha_r)} A, \quad \text{and} \\ &\alpha_r(w) - w \ \equiv \alpha_r(y) - y \quad \operatorname{mod} F_{n-\delta-\theta(\alpha_r)-1} A \end{aligned}$$

for all $r \geqslant r_0$. We can rewrite the above as follows:

$$\alpha_r(w) - w + F_{n-\delta-\theta(\alpha_r)-1} A = \alpha_r(y) - y + F_{n-\delta-\theta(\alpha_r)-1} A = d_{\alpha_r}(Y_w)$$

for $r \geqslant r_0$. Since $w \in I$ and I is $\mathcal{S}$-invariant, $d_{\alpha_r}(Y_w)$ must lie in the ideal $J = \operatorname{gr} I$ of B for $r \gg 0$, and hence $Y_w \in J^{\mathcal{S}}$ as required. □

Let $\mathcal{D}$ denote the set of all B_1-linear derivations of B.

Corollary. *Suppose that $\mathcal{D}(J^{\mathcal{S}}) \subseteq J$. Then J is controlled by B_1: $J = (J \cap B_1)B$.*

Proof. By [2, Proposition 2.4(d)], it is enough to show that $\mathcal{D}(J) \subseteq J$. So let $X \in J$ be a homogeneous element. If $X \in B_1$ then $\mathcal{D}(X) = 0 \in J$, so assume $X \notin B_1$. Choose $w \in I$ such that $X = \operatorname{gr} w$; then $\delta(w) = 0$ by definition and $X = Y_w$. Hence $X \in J^{\mathcal{S}}$ by the Proposition and hence $\mathcal{D}(X) \subseteq J$ by the assumption on J. The result follows. □

3.5. **The control theorem.** We can now state and prove our first main result, a control theorem for $\mathcal{S}$-invariant ideals. It should be viewed as a generalization of the control theorem for normal elements [2, Theorem 3.1].

Theorem. *Let (A, A_1) be a Frobenius pair, let $\mathcal{S}$ be a set of sources of derivations, let I be a $\mathcal{S}$-invariant right ideal of A and let $J := \operatorname{gr} I$. If $\mathcal{D}(J^{\mathcal{S}}) \subseteq J$ then I is controlled by A_1:*

$$I = (I \cap A_1) \cdot A.$$

Proof. We will first show that $J \cap B_1 \subseteq \operatorname{gr}(I \cap A_1)$. Let $X \in J \cap B_1$ be homogeneous of degree n say, and choose $w \in I$ such that $\operatorname{gr} w = X$. If $w \in A_1$ then $X = \operatorname{gr} w \in \operatorname{gr}(I \cap A_1)$ as required, so assume that $w \notin A_1$. Write $w = x + y$ as in §3.3; by Proposition 3.4, $Y := Y_w = \operatorname{gr} y \in J^{\mathcal{S}}$, so $\mathcal{D}(Y) \subseteq J$ by assumption on J.

Since J is controlled by B_1 by Corollary 3.4, applying [2, Proposition 2.4(c)] to the image of Y in B/J shows that $Y \in J + B_1$.

Write $\delta = \delta(w)$, so that $\deg y = n - \delta$. Note that $\delta > 0$ because $X \in B_1$. We can find some $s \in I \cap F_{n-\delta}A$, $z \in F_{n-\delta}A_1$ and $\epsilon \in F_{n-\delta-1}A$ such that $y = s + z + \epsilon$. Then $w' := w - s \in I$ and $\operatorname{gr} w' = \operatorname{gr} w = X$ because $s \in F_{n-\delta}A$ and $\delta > 0$. Moreover, $w' = x + z + \epsilon \in F_nA_1 + F_{n-\delta-1}A$, so that $\delta(w') > \delta(w)$.

Iterating the above argument, we can construct a sequence $w_1, w_2, w_3, \ldots$ of elements of I having the following properties:

- $\operatorname{gr} w_i = X$,
- $w_i \notin A_1$,
- $w_{i+1} \equiv w_i \mod F_{n-\delta(w_i)}A$, and
- $\delta(w_{i+1}) > \delta(w_i)$

for all $i \geqslant 1$. Note that we may always assume that $w_i \notin A_1$, because if any w_i does happen to lie in A_1 then $X = \operatorname{gr} w_i \in \operatorname{gr}(I \cap A_1)$ and we're done.

This sequence converges to an element $u \in A$ such that $\operatorname{gr} u = X$. Since the filtration on A is complete and since $B = \operatorname{gr} A$ is noetherian, I is closed in the filtration topology by [12, Chapter II, Theorem 2.1.2(6)] so $u \in I$. Given an integer $m > 0$, $\delta(w_i) > m$ and $u - w_i \in F_{n-m}A$ for sufficiently large i, so

$$u = w_i + (u - w_i) \in (F_nA_1 + F_{n-\delta(w_i)}A) + F_{n-m}A \subseteq F_nA_1 + F_{n-m}A$$

for all $m > 0$. Because A_1 is closed in A, $u \in A_1$ and therefore $X = \operatorname{gr} u \in \operatorname{gr}(I \cap A_1)$.

Thus $J \cap B_1 \subseteq \operatorname{gr}(I \cap A_1)$ as claimed. Because $\operatorname{gr}(I \cap A_1)$ is obviously contained in $J \cap B_1$, we have the equality $J \cap B_1 = \operatorname{gr}(I \cap A_1)$. Now

$$\operatorname{gr}((I \cap A_1)A) = \operatorname{gr}(I \cap A_1) \cdot \operatorname{gr} A = (J \cap B_1) \cdot B = J = \operatorname{gr} I$$

and therefore $I = (I \cap A_1)A$ by [12, Chapter II, Lemma 1.2.9]. □

4. Iwasawa algebras

4.1. **Uniform Γ-actions.** Let p be an odd prime and let Γ and G be uniform pro-p groups. We assume that Γ acts on G by group automorphisms and that the action is *uniform*:

$$\gamma \cdot g \equiv g \mod G^p$$

for all $\gamma \in \Gamma$ and $g \in G$. Let $\tau : \Gamma \to \operatorname{Aut}(G)$ be the associated group homomorphism; note that τ is automatically continuous by [10, Corollary 1.21(i)]. Let L_G denote the $\mathbb{Z}_p$-Lie algebra of G — this is a free $\mathbb{Z}_p$-module of rank $d = \dim G$.

Any automorphism of G gives rise to an automorphism of L_G: this gives rise to a natural injection

$$\iota : \mathrm{Aut}(G) \hookrightarrow \mathrm{GL}(L_G).$$

Clearly Γ acts uniformly on G if and only if the image of $\iota\tau$ is contained in the first congruence subgroup $\Gamma_1(\mathrm{GL}(L_G)) := \ker(\mathrm{GL}(L_G) \to \mathrm{GL}(L_G/pL_G))$ of $\mathrm{GL}(L_G)$. Since $\Gamma_1(\mathrm{GL}(L_G))$ has finite index in $\mathrm{GL}(L_G)$, we see that if Γ is any pro-p group of finite rank acting on G by group automorphisms, then Γ always has a uniform pro-p subgroup Γ_1 of finite index that acts uniformly. Of course this last fact is also implied by the finiteness of the group G/G^p.

4.2. **Some Lie theory.** The category of uniform pro-p groups is isomorphic to the category of powerful Lie algebras by [10, Theorem 9.10], so the homomorphism $\iota\tau$ gives rise to a Lie algebra homomorphism

$$\sigma = \log \circ \iota\tau \circ \exp : L_\Gamma \to p\,\mathrm{End}_{\mathbb{Z}_p}(L_G)$$

since $p\,\mathrm{End}_{\mathbb{Z}_p}(L_G)$ is the $\mathbb{Z}_p$-Lie algebra of $\Gamma_1(\mathrm{GL}(L_G))$. In other words, L_G is naturally a L_Γ-module, acting by derivations and moreover

$$x \cdot L_G \subseteq pL_G \quad \text{for all} \quad x \in L_\Gamma.$$

Let $N_\Gamma = \{x \in \mathbb{Q}_p L_\Gamma : x \cdot L_G \subseteq L_G\}$ be the inverse image of $\mathrm{End}_{\mathbb{Z}_p}(L_G)$ under the homomorphism

$$\sigma : \mathbb{Q}_p L_\Gamma \to \mathrm{End}_{\mathbb{Q}_p}(\mathbb{Q}_p L_G).$$

Note that $\frac{1}{p}L_\Gamma \subseteq N_\Gamma$. N_Γ also contains $\ker\sigma$ and $N_\Gamma/\ker\sigma$ is a finitely generated $\mathbb{Z}_p$-module. Hence

$$\mathfrak{g} := N_\Gamma/pN_\Gamma$$

is a finite dimensional $\mathbb{F}_p$-Lie algebra. Define

$$V := L_G/pL_G,$$

an $\mathbb{F}_p$-vector space of dimension d. Letting $\overline{} : L_G \twoheadrightarrow V$ and $\overline{} : N_\Gamma \twoheadrightarrow \mathfrak{g}$ denote the natural surjections, V becomes a $\mathfrak{g}$-module via the rule

$$\overline{x} \cdot \overline{y} = \overline{x \cdot y}$$

for all $x \in N_\Gamma$ and $y \in L_G$. Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be the associated homomorphism.

4.3. **The derivation hypothesis.** As explained in [3, §1.3], every endomorphism φ of V extends to a derivation of $\mathrm{Sym}(V)$ and for each $r \geqslant 0$ we also have the deformed derivations $\varphi^{[p^r]}$ of $\mathrm{Sym}(V)$, defined by the rule

$$\varphi^{[p^r]}(v) = \varphi(v)^{p^r} \quad \text{for all} \quad v \in V.$$

Let $B = \mathrm{Sym}(V \otimes k)$ and let P be a graded prime ideal of B. Let $\mathcal{D} = \mathrm{Der}_k(B)$ denote the space of all k-linear derivations of B; in fact, $\mathcal{D}$ is also the space of all B_1-linear derivations of B if we set $B_1 = kB^{[p]}$. Also, $\mathcal{D}$ is canonically isomorphic to $B \otimes_{\mathbb{F}_p} V^*$ and $\mathrm{Der}_k(B_P)$ is canonically isomorphic to $B_P \otimes_{\mathbb{F}_p} V^* = \mathcal{D}_P$ (see §2.4 for the notation). The derivations $\varphi^{[p^r]}$ extend to B_P, so we can also think of them as lying in $\mathcal{D}_P$.

We can view V as a $\mathfrak{g}$-module via ρ; in this way, V^* is also naturally a $\mathfrak{g}$-module. The next result gives a sufficient condition that ensures that a local analogue of the derivation hypothesis of [2, §3.5] holds.

Proposition. *Let P be a graded prime ideal of B which does not contain $\mathfrak{g}.v$ for any $v \in V\backslash 0$. Let J be a graded ideal of B_P and let $Y \in B_P$ be such that for all $x \in \mathfrak{g}$, we have*
$$\rho(x)^{[p^r]}(Y) \in J \quad \textit{for all} \quad r \gg 0.$$
Then $\mathcal{D}_P(Y) \subseteq J$.

Proof. Fix $x \in \mathfrak{g}$ and $f \in V^*$. By [3, Proposition 1.4], some B-linear combination of the derivations $\rho(x)^{[p^r]}$ equals $U(x \cdot f)$, where U is some product of elements u, each lying in $x.V \backslash \ker f$. If $f(P \cap V) = 0$ then every such u lies outside of P and is hence a unit in B_P. Let $W = (P \cap V)^{\perp}$ be the annihilator of $P \cap V$ in V^*; then
$$(x \cdot f)(Y) \in J \quad \text{for all} \quad x \in \mathfrak{g} \quad \text{and all} \quad f \in W.$$
Now if $\mathfrak{g} \cdot W < V^*$ then there exists $v \in V\backslash 0$ such that $(\mathfrak{g} \cdot W)(v) = 0$. But then $W(\mathfrak{g}.v) = 0$, which forces $\mathfrak{g}.v \subseteq P \cap V$ and hence contradicts our assumption on P.

Hence $\mathfrak{g}{\cdot}W = V^*$; however $\mathcal{D}_P$ is generated by V^* as a B_P-module, so $\mathcal{D}_P(Y) \subseteq J$ as required. $\square$

Note that for "most" P, the intersection $P \cap V$ will be zero and then P can only contain $\mathfrak{g}.v$ if v lies in the space of $\mathfrak{g}$-invariants $V^{\mathfrak{g}}$ of V. Since $V^{\mathfrak{g}}$ can be arranged to be zero in many interesting cases, this means that the condition on P imposed above is not very strong.

4.4. **Derivations for Iwasawa algebras.** Let $A = kG$ and $A_1 = kG^p$ be the completed group algebras of G and G^p, with coefficients in our ground field k. As usual, we equip A with the $\mathfrak{m}$-adic filtration, where $\mathfrak{m} := (G-1)kG$ is the augmentation ideal of kG:
$$F_nA := \begin{cases} \mathfrak{m}^{-n} & \text{if} \quad n \leqslant 0 \\ A & \text{otherwise.} \end{cases}$$
It is not hard to see that this is the same filtration as the one considered in [2, §6.6]. Now by [2, Proposition 6.6], (A, A_1) is a Frobenius pair, and by [2, Lemma 6.2(d) and Proposition 6.4], there is a canonical isomorphism
$$\operatorname{Sym}(V \otimes_{\mathbb{F}_p} k) \xrightarrow{\cong} \operatorname{gr} A.$$
Compare the following result with [3, Proposition 3.3].

Proposition. *Let $x \in \mathfrak{g}$ be non-zero, and choose a lift $\tilde{x}$ of x in $N_\Gamma \backslash pN_\Gamma$. Let $m \geqslant 1$ be such that $p^m\tilde{x} \in L_\Gamma$. Let $\alpha = \tau(\exp(p^m\tilde{x})) \in \operatorname{Aut}(G)$ and view α as an algebra endomorphism of $A = kG$. Then*
(a) $(\alpha - 1)F_nA \subseteq F_{n-p^m+1}A$, for all $n \in \mathbb{Z}$,
(b) $(\alpha - 1)F_nA_1 \subseteq F_{n-p^{m+1}+p}A$ for all $n \in \mathbb{Z}$, and
(c) $d_\alpha = \rho(x)^{[p^m]}$ as derivations of $\operatorname{gr} A$.

Proof. Let C be the procyclic subgroup of Γ generated by $\gamma := \exp(p^m\tilde{x}) \in \Gamma$. Let H be the semidirect product of G and C with γ acting on G by the automorphism α. So inside this new group H, we have the relation
$$\gamma g \gamma^{-1} = \gamma \cdot g$$
for all $g \in G$. The group H is uniform and L_H is the semidirect product of L_G and $L_C = p^m\tilde{x}\mathbb{Z}_p$, with $p^m\tilde{x}$ acting via the derivation $\sigma(p^m\tilde{x}) : L_G \to L_G$:
$$L_H = L_G \rtimes L_C.$$

Because L_C is abelian and $\tilde{x} \cdot L_G \subseteq L_G$, $[p^m \tilde{x}, L_H] \subseteq p^m L_H$. Hence by [2, Proposition 6.7], the following relations hold in kH for all $n \in \mathbb{Z}$:

$$\begin{array}{rcl} [\gamma, F_n kH] & \subseteq & F_{n-p^m+1} kH \\ [\gamma, F_n kH^p] & \subseteq & F_{n-p^{m+1}+p} kH. \end{array}$$

Now $(\alpha - 1)(b) = \gamma b \gamma^{-1} - b = [\gamma, b]\gamma^{-1}$ for all $b \in kG$ and

$$F_n kH \cap kG = F_n kG$$

for all $n \in \mathbb{Z}$. Parts (a) and (b) follow.

Finally, part (c) follows from [2, Theorem 6.8]: one only needs to note that d_α coincides with the restriction to $\operatorname{gr} kG$ of the derivation $\{\gamma, -\}_{p^m-1}$ of $\operatorname{gr} kH$. □

4.5. **The support of the "failure of control" module.** We can now put the main pieces together and prove a refined version of [2, Theorem 5.2]. The theorem below places a severe restriction on the characteristic support of the failure of control module of any Γ-invariant right ideal of kG; see §2.4 for the notation.

Theorem. *Let I be a Γ-invariant right ideal of kG and let $F = I/(I \cap kG^p)kG$ be the failure of control module. Then for any $P \in \operatorname{Ch}(F)$ there exists $v \in V \backslash 0$ such that $\mathfrak{g}.v \subseteq P$.*

Proof. Suppose for a contradiction that P does not contain any subspace of V of the form $\mathfrak{g}.v$ for $v \in V\backslash 0$. Let $P_1 = \iota_* P$; by Lemma 2.2, $(A_P, (A_1)_{P_1})$ is a Frobenius pair and we plan to apply the Control Theorem, Theorem 3.5 to it.

Let $x \in \mathfrak{g}$ and choose a lift $\tilde{x} \in N_\Gamma$ for x. There exists $m_x \geqslant 1$ such that $p^{m_x}\tilde{x} \in L_\Gamma$; then $\gamma_x := \exp(p^{m_x}\tilde{x})$ lies in Γ, so

$$\mathbf{a}(x) = \{\tau(\gamma_x), \tau(\gamma_x)^p, \tau(\gamma_x)^{p^2}, \ldots\}$$

is a source of derivations of (A, A_1) by Proposition 4.4(a) and (b). We're only interested in the derivations of B induced by $\mathbf{a}(x)$; by Proposition 4.4(c) these derivations do not depend on the choice of $\tilde{x}$, being precisely the $\rho(x)^{[p^r]}$ for $r \geqslant m_x$.

Let $\mathcal{S} := \{\mathbf{a}(x) : x \in \mathfrak{g}\}$; then $\mathcal{S}_P := \{\mathbf{a}(x)_{T_P} : x \in \mathfrak{g}\}$ is a set of sources of derivations of $(A_P, (A_1)_{P_1})$ by Proposition 3.2(a).

Since I is Γ-invariant, I is clearly $\mathcal{S}$-invariant in the sense of §3.4, and the definition of $\mathbf{a}(x)_{T_P}$ shows that the microlocalised right ideal I_P of A_P is $\mathcal{S}_P$-invariant.

Let $J = \operatorname{gr} I_P$. In view of Proposition 4.4(c) and Proposition 3.2(b), if $Y \in B_P$ lies in the $\mathcal{S}_P$-closure $J^{\mathcal{S}_P}$ of J, then for all $x \in \mathfrak{g}$,

$$\rho(x)^{[p^r]}(Y) \in J \quad \text{for all} \quad r \gg 0.$$

It now follows from Proposition 4.3 that $\mathcal{D}_P(Y)$ must be contained in J, and hence all the conditions of Theorem 3.5 are satisfied. We can therefore deduce from Theorem 3.5 that I_P is controlled by $(A_1)_{P_1}$. Now Lemma 2.3 implies that $(\operatorname{gr} F)_P = 0$ and hence $P \notin \operatorname{Ch}(F)$, a contradiction. □

4.6. **Another control theorem.** Recall the definition of purity from §2.6.

Theorem. *Let I be a Γ-invariant right ideal of kG and suppose that kG/I is pure. Suppose that no minimal prime P above $\operatorname{gr} I$ contains $\mathfrak{g}.v$ for any $v \in V\backslash 0$. Then I is controlled by kG^p.*

Proof. Let $F = I/I_1A$ be the failure of control module, and let $\mathcal{X}$ denote the set of all graded prime ideals of B that contain some $\mathfrak{g}.v$ for $v \in V\backslash 0$. By Corollary 2.4, Proposition 2.5 and Theorem 4.5,

$$\mathrm{Ch}(F) \subseteq \mathrm{Ch}(A/I_1A) \cap \mathcal{X} = \mathrm{Ch}(A/I) \cap \mathcal{X}.$$

Since A/I is pure, Gabber's Purity of the Characteristic Variety theorem [8, Corollary 5.21] implies that every prime in $\min \mathrm{Ch}(A/I)$ has the same height. Because none of these primes lie in $\mathcal{X}$ by assumption, it now follows from Lemma 2.6(b) that $j(F) > j(A/I)$. Therefore $F = 0$ by Proposition 2.6, as required. □

4.7. **Consequences for** $\mathrm{Ch}(A/I)$ **when** A/I **is pure.** The group Γ acts naturally on G^p, so L_Γ acts on $L_{G^p} = pL_G$. It is easy to see that the normaliser N_Γ for this action is the same as before, meaning that $\mathfrak{g}$ is unchanged. Now $\mathfrak{g}$ also acts on $V_1 := L_{G^p}/L_{G^{p^2}} = pL_G/p^2L_G$. Clearly the map $x \mapsto px$ induces an isomorphism of $\mathfrak{g}$-modules between V and V_1; if we view V and V_1 as being embedded into B and B_1 respectively, then this isomorphism is given by $v \mapsto v^p$ for any $v \in V$.

Lemma. *Let* $\mathcal{X}_1 = \{\mathfrak{p} \in \mathrm{Spec}_{\mathrm{gr}}(B_1) : \mathfrak{p} \supseteq \mathfrak{g} \cdot v_1$ *for some* $v_1 \in V_1\backslash 0\}$. *Then*

$$\mathcal{X}_1 = \iota_*(\mathcal{X}).$$

Proof. This is immediate, using the fact that $\iota_*(P) = P \cap B_1$ for $P \in \mathrm{Spec}_{\mathrm{gr}}(B)$, and that $\iota_*^{-1}(\mathfrak{p}) = \sqrt{\mathfrak{p}B}$ for $\mathfrak{p} \in \mathrm{Spec}_{\mathrm{gr}}(B_1)$. □

Proposition. *Let I be a proper, non-zero Γ-invariant right ideal of kG such that kG/I is pure. Then there exists a minimal prime P above* $\mathrm{gr}\, I$ *and* $v \in V\backslash 0$ *such that* $\mathfrak{g} \cdot v \subseteq P$.

Proof. Suppose for a contradiction that $\min \mathrm{Ch}(A/I) \cap \mathcal{X} = \emptyset$; we will show that $I = 0$. By Theorem 4.6, I is controlled by A_1: $I = I_1A$. Since $\iota_*(\mathrm{Ch}(A/I)) = \mathrm{Ch}(A_1/I_1)$ by Proposition 2.5 and since $\iota_*(\mathcal{X}) = \mathcal{X}_1$ by the lemma, we see that

$$\min \mathrm{Ch}(A_1/I_1) \cap \mathcal{X}_1 = \emptyset;$$

moreover A_1/I_1 is pure by Lemma 2.6(c). Thus I_1 satisfies the same conditions as I. We can now apply the above argument repeatedly and deduce that I is controlled by kG^{p^n} for all $n \geqslant 0$. Since I is proper and since kG^{p^n} is scalar local, $I = (I \cap kG^{p^n})kG \subseteq (G^{p^n} - 1)kG$ for all $n \geqslant 0$. The intersection of these augmentation ideals is zero, so $I = 0$. This is the required contradiction. □

Corollary. *Let I be a proper, non-zero Γ-invariant right ideal of kG such that kG/I is pure. Then*

$$j(kG/I) \geqslant u := \min\{\dim \mathfrak{g} \cdot v : v \in V\backslash 0\}.$$

Proof. By the theorem, we can find $P \in \min \mathrm{Ch}(A/I)$ such that $P \in \mathcal{X}$. Then P contains $(\mathfrak{g} \cdot v)B$, which is clearly a prime ideal of height $\dim \mathfrak{g} \cdot v$. Hence $\mathrm{ht}\, P \geqslant \dim \mathfrak{g} \cdot v \geqslant u$. Since every prime in $\min \mathrm{Ch}(A/I)$ has the same height by Gabber's purity theorem, we can now use Lemma 2.6(b) to deduce that $j(A/I) = \mathrm{ht}\, P \geqslant u$ as required. □

In fact, the assumption that kG/I is pure is unnecessary:

4.8. **Theorem.** Let I be a proper, non-zero Γ-invariant right ideal of kG. Then there exists a minimal prime ideal P above $\operatorname{gr} I$ and $v \in V\backslash 0$ such that $\mathfrak{g} \cdot v \subseteq P$. It follows that $j(kG/I) \geqslant \min\{\dim \mathfrak{g} \cdot v : v \in V\backslash 0\}$.

Proof. Let $\overline{I}/I$ be the largest submodule of A/I of grade strictly bigger than $j(A/I)$. We claim that the right ideal $\overline{I}$ is still Γ-invariant. For any $\gamma \in \Gamma$, we saw in the proof of Proposition 3.2 that $\operatorname{gr} \gamma(x) = \operatorname{gr} x$ for any $x \in A$. This means that $\operatorname{gr} \gamma(\overline{I}) = \operatorname{gr} \overline{I}$, so

$$j_A(\gamma(\overline{I})/I) = j_B(\operatorname{gr} \gamma(\overline{I})/\operatorname{gr} I) = j_B(\operatorname{gr} \overline{I}/\operatorname{gr} I) = j_A(\overline{I}/I) > j_A(A/I)$$

by Lemma 2.6(b). Since $\gamma(\overline{I})$ contains $\gamma(I) = I$, it follows that $\gamma(\overline{I}) \subseteq \overline{I}$. Hence $\overline{I}$ is Γ-invariant as claimed, it is still proper and non-zero, but now $A/\overline{I}$ is also pure.

Let $P \in \min \operatorname{Ch}(A/\overline{I})$; then $P \in \operatorname{Ch}(A/I)$ by Corollary 2.4. If $Q \subseteq P$ also lies in $\operatorname{Ch}(A/I)$, then $\operatorname{ht} Q \geqslant j(A/I) = j(A/\overline{I}) = \operatorname{ht} P$ by Lemma 2.6(b), forcing $Q = P$. Hence

$$\min \operatorname{Ch}(A/\overline{I}) \subseteq \min \operatorname{Ch}(A/I).$$

Therefore $\min \operatorname{Ch}(A/I) \cap \mathcal{X} \neq \emptyset$ by Proposition 4.7, and $j(A/I) = j(A/\overline{I}) \geqslant u$ by Corollary 4.7. □

5. Applications

5.1. **Linear actions.** One obvious example of a situation where our results are applicable is the case of a uniform pro-p group Γ acting linearly on some free abelian pro-p group G of finite rank n. Because G is abelian, Γ automatically acts by group automorphisms, and we only have to assume that the image of the action is contained in the first congruence subgroup of $\operatorname{Aut}(G) \cong \mathrm{GL}_n(\mathbb{Z}_p)$ to ensure the action is uniform.

Theorem. *Suppose that the image of the action of* Γ *is open in* $\mathrm{GL}_n(\mathbb{Z}_p)$*. Then the only* Γ*-invariant prime ideals of* kG *are the zero ideal and the augmentation ideal.*

Proof. Without loss of generality, we may assume that the image of Γ is equal to a congruence subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$. Then it is easy to see that the normaliser Lie algebra N_Γ defined in §4.2 is the full linear Lie algebra $\mathfrak{gl}_n(\mathbb{Z}_p)$. Hence $\mathfrak{g} = N_\Gamma/pN_\Gamma = \mathfrak{gl}_n(\mathbb{F}_p)$ and $V = G/G^p = \mathbb{F}_p^n$ is the natural $\mathfrak{g}$-module. It is now clear that $\min\{\dim \mathfrak{g} \cdot v : v \in V\backslash 0\} = n$, so Theorem 4.8 implies that a non-zero Γ-invariant prime ideal I of kG must satisfy $j(kG/I) = n$. This forces I to be the augmentation ideal of kG since kG is local, and the result follows. □

An obvious modification of the above proof shows that the result is also true if we only assume that the image of Γ is an open subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$. These results provide some very weak evidence towards the following conjecture, which is inspired by Roseblade's theorem [13, Theorem D]:

Conjecture. *Let* I *be a* Γ*-invariant prime ideal of* kG *which is* faithful*, that is* $(1 + I) \cap G = 1$*. Then* I *is controlled by the subgroup of* Γ*-fixed points of* G*,* G^Γ*:*

$$I = (I \cap kG^\Gamma)kG.$$

A more conceptual proof of Roseblade's theorem can be found in [6, §1.3].

5.2. **Calculations with Chevalley Lie algebras.** Let Φ be an indecomposable root system. We fix a set of fundamental roots Π and the corresponding set of positive roots Φ^+ of Φ. Let $\rho = \frac{1}{2}\sum_{s\in\Phi^+} s$ and let θ be the highest root of Φ^+. Recall that the root system Φ is by definition a finite subset of some real Euclidean space, so the inner product $(2\rho, \theta)$ makes sense.

Let $\mathfrak{g}_k = \Phi(k)$ be the Chevalley Lie algebra over our field k of characteristic p. Recall [9, §4] that by definition, $\mathfrak{g}_k$ has a basis consisting of the root vectors e_s $(s \in \Phi)$, and the fundamental co-roots h_r, $(r \in \Pi)$. Let $\mathfrak{b}_k$ denote the (positive) Borel subalgebra of $\mathfrak{g}_k$, spanned by all of the co-roots and the e_s where $s \in \Phi^+$.

Recall [3, §0.3] that we call p a *nice prime* for Φ if $p \geqslant 5$ and if $p \nmid n+1$ when Φ is the root system A_n. We would like to thank Alexander Premet for providing the proof of the following result.

Proposition. *Suppose that p is a nice prime for Φ. Let*

$$u_k := \min\{\dim_k[\mathfrak{g}_k, x] : x \in \mathfrak{g}_k \backslash 0\}.$$

Then $u_k = (2\rho, \theta)$.

Proof. We first assume that the field k is algebraically closed. Let $\mathcal{G}$ denote the Chevalley group over k associated to Φ. This is a connected, adjoint, simple algebraic group over k, whose Lie algebra equals $\mathfrak{g}_k$. Hence $\mathcal{G}$ acts naturally on $\mathfrak{g}_k$ by Lie algebra automorphisms. For each $n \geqslant 0$, let $X(n) = \{x \in \mathfrak{g}_k : \dim[\mathfrak{g}_k, x] \leqslant n\}$. Then clearly

$$u_k = \min\{n : X(n) \neq \{0\}\}.$$

Now $X(u_k)$ is a closed subset of $\mathfrak{g}_k$ in the Zariski topology, which is moreover conical, $\mathcal{G}$-stable and contains at least one line. Let $\mathcal{B}$ be the Borel subgroup of $\mathcal{G}$ whose Lie algebra equals $\mathfrak{b}_k$. By Borel's fixed point theorem [4, Chapter III, Theorem 10.4], $X(u_k)$ contains a $\mathcal{B}$-stable line $\langle x\rangle$ for some nonzero $x \in \mathfrak{g}_k$. By our choice of $\mathcal{B}$, x is then a highest weight vector for the $\mathcal{G}$-module $\mathfrak{g}_k$, so x is also a highest weight vector for the $\mathfrak{g}_k$-module $\mathfrak{g}_k$.

Our assumption on p implies that $\mathfrak{g}_k$ is a simple Lie algebra — see [14, p. 187]. Hence this $\mathfrak{g}_k$-module is irreducible. It now follows (essentially from the PBW theorem) that x must be a non-zero scalar multiple of e_θ, where θ is the highest root of Φ^+. Following [17], we call $\mathbb{S} := \{\beta \in \Phi^+ : \theta - \beta \in \Phi\}$ the set of *special roots*. Since $p \geqslant 5$ by assumption, Chevalley's basis theorem [9, Theorem 4.2.1] implies that $[e_r, e_s] \neq 0$ whenever r, s and $r+s \in \Phi$. We can therefore compute

$$[\mathfrak{g}_k, e_\theta] = \langle e_\theta, h_\theta\rangle \oplus \langle e_{\theta-\beta} : \beta \in \mathbb{S}\rangle,$$

whence $u_k = \dim_k[\mathfrak{g}_k, e_\theta] = |\mathbb{S}| + 2$. It is shown in the proof of [17, Lemma 3] that $|\mathbb{S}| + 2 = (2\rho, \theta)$, so $u_k = (2\rho, \theta)$ when k is algebraically closed.

Returning to the general case, let $\overline{k}$ denote an algebraic closure of k. The computation of $[\mathfrak{g}_k, e_\theta]$ performed above does not require $k = \overline{k}$, so

$$(2\rho, \theta) = \dim_k[\mathfrak{g}_k, e_\theta] \geqslant u_k \geqslant u_{\overline{k}} = (2\rho, \theta)$$

and the proposition follows. $\square$

5.3. **Iwasawa algebras of Chevalley type.** Here is our main result.

Theorem. *Let p be a nice prime for Φ, let $G = \exp(p^t\Phi(\mathbb{Z}_p))$ be the uniform pro-p group of Chevalley type for some $t \geqslant 1$, and let I be a non-zero two-sided ideal of kG. Then*

$$j(kG/I) \geqslant (2\rho, \theta).$$

Proof. $\Gamma = G$ acts on G by conjugation, and this action is uniform in the sense of §4.1 because G is uniform. Clearly I is a Γ-invariant right ideal of kG. Following the proof of [3, Theorem 3.4], we see that $N_\Gamma = \Phi(\mathbb{Z}_p)$ and that $\mathfrak{g} = \Phi(\mathbb{F}_p)$. Moreover, $x \mapsto p^t x$ induces a $\mathfrak{g}$-module isomorphism between the adjoint $\mathfrak{g}$-module $\mathfrak{g}$ and the $\mathfrak{g}$-module $V = G/G^p$.

We can now use Theorem 4.8 to deduce that $j(kG/I)$ is bounded below by $\min\{\dim[\mathfrak{g}, x] : x \in \mathfrak{g}\backslash 0\}$. But this number equals $(2\rho, \theta)$ by Proposition 5.2. □

It can be shown [11, Exercise 6.2] that $(2\rho, \theta) = 2\hat{h} - 2$, where $\hat{h}$ is the *dual Coxeter number* of Φ that arises in the study of affine Lie algebras. A list of values of this invariant can be found in [11, §6.1]; we used this list to construct the table given in §1.5.

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