Iwasawa Algebras Examples Sheet 4

1. Recall that the free associative k-algebra on a set X, has a unique $\mathbf{N}_0$-grading as a k-algebra
$$k\langle X\rangle = \bigoplus_{n\geq 0} k\langle X\rangle^{(n)}$$
such that the image of X in $k\langle X\rangle$ has degree 1. Show that the completion of $k\langle X\rangle$ with respect to the filtration
$$v\left(\sum_{n\geq 0} r_n\right) = \inf\{n\in\mathbf{N}_0 \mid r_n\neq 0\} \text{ for } r_n\in k\langle X\rangle^{(n)}$$
is isomorphic to $\prod_{n\geq 0} k\langle X\rangle^{(n)}$.
Similarly prove that if the free k-Lie algebra L_X on X is given its unique $\mathbf{N}$-grading such that the image of X in L_X is in degree 1 then the completion of L_X with respect to the filtration
$$v\left(\sum_{n\geq 0} x_n\right) = \inf\{n\in\mathbf{N}_0 \mid x_n\neq 0\} \text{ for } x_n\in L_X$$
is isomorphic to $\prod_{n\geq 0} L_X^{(n)}$.

2. Show that the pair of functions $\exp\colon T\mathbf{Q}[[T]]\to 1+T\mathbf{Q}[[T]]$ and $\log\colon 1+T\mathbf{Q}[[T]]$ are mutual inverses.

3. Compute directly the terms $\Phi_1(U,V)$, $\Phi_2(U,V)$ and $\Phi_3(U,V)$ of the Hausdorff series. Then compute them again using Dynkin's formula for Φ.

4. Suppose that (A,w) is a Banach $\mathbf{Q}_p$-algebra. Show that for all $x,y\in A$ with $w(x),w(y)>\frac{1}{p-1}$:
   (a) $\exp(\log(1+x)=1+x$;
   (b) $\log(\exp(x))=x$ and
   (c) $\Phi(x,y)=\log(\exp(x)\exp(y))$.

5. Show that there is a canonical functor from the category of complete p-valued groups of finite rank to the category of $\mathbf{Q}_p$-Banach algebras that sends G to $\widehat{\mathbf{Q}_pG}$ and such that each natural diagram
$$\begin{array}{ccc} H & \longrightarrow & G \\ \downarrow & & \downarrow \\ \widehat{\mathbf{Q}_pH} & \longrightarrow & \widehat{\mathbf{Q}_pG} \end{array}$$
commutes.

6. Show that if (G,ω) is a complete p-valued group then we can equip $G\times G$ with a p-filtration $\omega_{G\times G}$ such that
$$\omega_{G\times G}\left((g,h)\right) = \min\left\{\omega(g),\omega(h)\right\}$$
so that $(G\times G,\omega_{G\times G})$ is a complete p-valued group and $gr(G\times G)\cong gr\,G\times gr\,G$.

7. Show that $\mathcal{G}(\widehat{KG})$ is a subgroup of $\widehat{KG}^{\times}$ containing the image of G in $\widehat{KG}$ and that $\mathcal{P}(\widehat{KG})$ is a Lie K-subalgebra of $\widehat{KG}$ equipped with its commutator bracket. Finally show that exp restricts to a bijection
$$\mathcal{P}(\widehat{KG})\cap\widehat{KG}_{\frac{1}{p-1}+}\to\mathcal{G}(\widehat{KG})\cap\left(1+\widehat{KG}_{\frac{1}{p-1}+}\right)$$
with inverse log.

8. Show that if p is odd and $GL_n^1(\mathbf{Z}_p)$ has its usual p-valuation ω then G is p-saturated and has an ordered basis $(g_1, \ldots, g_{d^2})$ with $\omega(g_i) = 1$ for each i.

9. Suppose that $p = 2$, G is p-saturated, and that $\omega(g_i) = 2$ for $i = 1, \ldots, d$. Let

$$\mathfrak{g} = \{x \in \mathcal{P}(\widehat{KG}) \mid w(x) \geq 0\}.$$

Show that $\mathfrak{g}$ is an $\mathcal{O}$-Lie algebra free of finite rank over $\mathcal{O}$ and there is an isomorphism of Banach algebras

$$\widehat{U(\mathfrak{g}_K)} \xrightarrow{\sim} \widehat{KG}.$$

Show that these conditions are satisfied for $GL_n^2(\mathbf{Z}_2)$ with its usual p-valuation.

10. + Show that if G is any complete p-valued group of rank d then the filtration ω on

$$\mathcal{G}(\widehat{KG})_{\frac{1}{p-1}+} = \mathcal{G}(\widehat{KG}) \cap (1 + \widehat{KG}_{\frac{1}{p-1}+})$$

given by $\omega(g) = \widehat{w}(g-1)$ makes $\mathcal{G}(\widehat{KG})_{\frac{1}{p-1}+}$ into a complete p-valued group of rank d and that G is isomorphic to an open subgroup of it.