Iwasawa Algebras Examples Sheet 3

1. Suppose that I is a pre-ordered set and $\mathcal{C}$ is **Grp** or **Ring** and $C = (C_a, c_{bc})$ is an inverse system in $\mathcal{C}$. Show that
$$\varprojlim_I C \cong \{(x_a) \in \prod_{a \in I} C_a \mid c_{bc}(x_c) = x_b \text{ whenever } b \leq c\}$$
together with the projection maps $\pi_a((x_a)_{a\in I}) = x_a$ is the inverse limit of C.

2. Suppose I is a pre-ordered set and $C = (C_a, c_{bc})$ is an inverse system of shape I in some category $\mathcal{C}$.

   (a) Show that, if it exists, $\varprojlim_I C$ together with the family of morphisms $\pi_a \colon \varprojlim_I C \to C_a$ is uniquely determined up to unique isomorphism.

   (b) Show that if I has a largest element t (i.e. $a \leq t$ for all $a \in I$) then $\varprojlim_I C = C_t$ and $\pi_a = c_{at}$ for all $a \in I$.

   (c) More generally suppose that I is directed and $J \subset I$ such that for all $a \in I$ there is $j \in J$ with $a \leq j$ and consider the restriction of C to J i.e. the subfamily of objects $(C_j)_{j\in J}$ and morphisms $(c_{jk} \colon C_k \to C_j)_{j \leq k \in J}$. Show that if $\varprojlim_J C$ exists then so does $\varprojlim_I C$ and there is a canonical isomorphism $\varprojlim_J C \to \varprojlim_I C$.

3. Suppose that I and J are directed pre-ordered sets. Show that $I \times J$ is a directed pre-ordered set with respect to the relation $(i,j) \leq (i',j')$ precisely if $i \leq i'$ and $j \leq j'$. Assuming that all relevant inverse limits exist, show that if C is a diagram of shape $I \times J$ then $(\varprojlim_{\{i\}\times J} C)_{i\in I}$ has canonical maps making it a diagram of shape I and $(\varprojlim_{I\times\{j\}} C)_{j\in J}$ has canonical maps making it a diagram of shape J. Finally show that
$$\varprojlim_I \left(\varprojlim_{\{i\}\times J} C \right) \cong \varprojlim_{I\times J} C \cong \varprojlim_J \left(\varprojlim_{I\times\{j\}} C \right).$$

4. Suppose that R is a ring with an ascending $\mathbf{N}_0$-filtration $(F_n R)_{n\in\mathbf{N}_0}$. Show that

   (a) if $gr\, R$ is Noetherian then R is Noetherian and;

   (b) if $gr\, R$ is a domain then R is a domain.

5. Show that if G is a complete p-valued group of finite rank and $\mathbf{Z}_p[G]$ has its usual filtration v then $v(rs) = v(r) + v(s)$ for all $r, s \in \mathbf{Z}_p[G]$.

6. Show that if (G,ω) is a complete p-valued group of rank $d < \infty$ with ordered basis $(g_1,\ldots,g_d)$ then there is an $\mathbf{F}_p$-linear isomorphism $\mathbf{F}_p G \cong \mathbf{F}_p[[T_1,\ldots,T_d]]$ with
$$(g_1 - 1)^{n_1} \cdots (g_d - 1)^{n_d} \mapsto T_1^{n_1} \cdots T_d^{n_d}.$$

7. Let p be an odd prime and suppose that (G,ω) is a complete p-valued group of rank $d < \infty$ with ordered basis $(g_1,\ldots,g_d)$. Suppose moreover that $\omega(g_i) = 1$ for each $1 \leq i \leq d$ and that in $gr\, G$
$$[\sigma(g_i), \sigma(g_j)] = \sum_{k=1}^{d} t\lambda_{ijk}\sigma(g_k) \text{ with } \lambda_{ijk} \in \mathbf{F}_p.$$
Writing $b_i = g_i - 1 \in \mathbf{Z}_p[G]$ as usual show that
$$b_i b_j - b_j b_i + \mathbf{Z}_p[G]_3 = \sum_{k=1}^{d} p\tilde{\lambda}_{ijk} b_k \mod \mathbf{Z}_p[G]_3$$
with respect to the usual filtration on $\mathbf{Z}_p[G]$ where each $\tilde{\lambda}_{ijk}$ is a lift of λ_{ijk} in $\mathbf{Z}_p$.

Show that if instead we work in $\mathbf{F}_p[G]$ then

$$b_ib_j - b_jb_i + J^{p+1} = \sum_{i=1}^{d} \lambda_{ijk} b_k^p + J^{p+1}$$

where $J = \ker(\mathbf{F}_p[G] \to \mathbf{F}_p; g \mapsto 1)$.

8. Suppose that $\mathbf{Z}_pG$ is an Iwasawa algebra. Show that $\mathbf{F}_pG \cong \mathbf{Z}_pG/p\mathbf{Z}_pG$.

   Let M be a finitely generated $\mathbf{Z}_pG$-module and write $M[p^k] = \{m \in M \mid p^k m = 0\}$. Show that there is some $n \geq 0$ such that $M[p^k] = M[p^n]$ for all $k \geq n$, that $M[p^k]/M[p^{k-1}]$ is a naturally a finitely generated $\mathbf{F}_pG$-module for each $1 \leq k \leq n$ and that $M/M[p^n]$ has no p-torsion elements.

9. Suppose that a profinite group G acts on a finite set X equipped with the discrete topology. Show that the action map $G \times X \to X$ is continuous if and only if each stabiliser $\mathrm{Stab}_G(x)$ is an open subgroup of G.

10. Suppose that X is a set with a G-action and R is a commutative ring. Writing $R[X]$ for the permutation representation show that $R[X]^G$ is spanned by the sums $[\mathcal{O}] = \sum_{x \in \mathcal{O}} x \in R[X]$ as $\mathcal{O}$ ranges over the finite G-orbits of X.

11. Show that given any set X there is a free magma $M(X)$ with respect to the forgetful functor $\mathbf{Mag} \to \mathbf{Set}$. An $\mathbf{N}$-graded magma is a magma M with a decomposition as a disjoint union $M = \bigcup_{n \geq 1} M_n$ such that $M_nM_m \subset M_{n+m}$ for all $n, m \in \mathbf{N}$. Show that there is a unique $\mathbf{N}$-grading on $M(X)$ such that the image of X in $M(X)$ lies in $M(X)_1$.

    Deduce that for any commutative ring k the free (non-unital non-associative) k-algebra $k\langle\langle X\rangle\rangle$ on X with respect to the forgetful functor $\mathbf{Alg}_k \to \mathbf{Set}$ exists and is naturally isomorphic as a k-module to the free k-module $k[M(X)]$ on X and that there is a unique way to view $k\langle\langle X\rangle\rangle$ as a graded k-algebra such that the image of X in $k\langle\langle X\rangle\rangle$ lies in degree 1. Use this to construct the free Lie algebra $L(X)$ on X with respect to the forgetful functor $\mathbf{Lie}_k \to \mathbf{Set}$ and show that there is a unique way to make $L(X)$ a graded Lie algebra such that the image of X in $L(X)$ is contained in $L(X)_1$.

    Show that the natural isomorphism $k\langle X\rangle \to U(L_X)$ from the free (unital and) associative algebra on X to the universal enveloping algebra $U(L_X)$ is an isomorphism of graded algebras.