$\widehat{\mathcal{D}}$-MODULES ON RIGID ANALYTIC SPACES I

KONSTANTIN ARDAKOV AND SIMON WADSLEY

Abstract. We introduce a sheaf of infinite order differential operators $\widehat{\mathcal{D}}$ on smooth rigid analytic spaces that is a rigid analytic quantisation of the cotangent bundle. We show that the sections of this sheaf over sufficiently small affinoid varieties are Fréchet-Stein algebras, and use this to define co-admissible sheaves of $\widehat{\mathcal{D}}$-modules. We prove analogues of Cartan's Theorems A and B for co-admissible $\widehat{\mathcal{D}}$-modules.

Contents

1. Introduction

1.1. **Background and motivation.** The theory of $\mathcal{D}$-modules goes back over forty years to the work of Sato and Kashiwara for $\mathcal{D}$-modules on manifolds [19] and to the work of Bernstein for $\mathcal{D}$-modules on algebraic varieties [9]. Originally introduced as a framework for the algebraic study of partial differential equations there have also been fundamental applications in the studies of harmonic analysis, algebraic geometry, Lie groups and representation theory. In this paper we attempt to initiate a new theory of $\mathcal{D}$-modules for rigid analytic spaces in the sense of Tate [30].

In their seminal paper [5], Beilinson and Bernstein explained how to study representations of a complex semi-simple Lie algebra $\mathfrak{g}$ via twisted $\mathcal{D}$-modules on the flag variety $\mathcal{B}$ of the corresponding algebraic group. In particular they established an equivalence between the category of finitely generated modules over the enveloping algebra $U(\mathfrak{g})$ with a fixed regular infinitesimal central character χ and the category of coherent modules for the sheaf of χ-twisted differential operators on $\mathcal{B}$.

Our primary motivation for this work is to establish a rigid analytic version of the Beilinson-Bernstein equivalence in order to understand the representation theory of the Arens–Michael envelope $\widehat{U(\mathfrak{g})}$ of the universal enveloping algebra of a semi-simple Lie algebra $\mathfrak{g}$ over a complete discretely valued field K. The Arens–Michael envelope is the completion of $U(\mathfrak{g})$ with respect to all submultiplicative seminorms on $U(\mathfrak{g})$ that extend the norm on K; when $\mathfrak{g}$ is the Lie algebra of a p-adic Lie group and K is a p-adic field, $\widehat{U(\mathfrak{g})}$ occurs as the algebra of locally analytic K-valued distributions on this group supported at the identity. It is therefore of interest in the theory of locally analytic representations of p-adic groups, developed by Schneider and Teitelbaum [25]. We delay the proof of our version of the Beilinson-Bernstein equivalence to a later paper, but see Theorem E below for a precise statement. Here we construct the sheaf $\widehat{\mathcal{D}}$ on a general smooth rigid analytic space over K, and establish some of its basic properties.

1.2. **Rigid analytic quantisation.** In our earlier work [1] we proved an analogous theorem for certain Banach completions of $\widehat{U(\mathfrak{g})}$ localising onto a smooth formal model $\widehat{\mathcal{B}}$ of the flag variety. In this new programme we extend that work in two directions. In the base direction, by working on the rigid analytic flag variety $\mathcal{B}^{an}$ which has a finer topology than a fixed formal model $\widehat{\mathcal{B}}$, the localisation is more refined and the geometry is more flexible[1]. In the cotangent direction, we no longer fix a level n as we did in [1], and instead work simultaneously with all n. This involves using Schneider and Teitelbaum's notions of Fréchet–Stein algebras and co-admissible modules introduced in [26].

The definition of a Fréchet–Stein algebra is modelled around key properties of Stein algebras; these latter arise as rings of functions on Stein spaces in (complex) analytic geometry. There is a well-behaved abelian category of co-admissible modules defined for each Fréchet–Stein algebra; in the case when the algebra in question is the ring of global rigid analytic functions on a quasi-Stein rigid analytic space, this category is naturally equivalent to the category of coherent sheaves on this space. It is known [24] that $\widehat{U(\mathfrak{g})}$ is a Fréchet-Stein algebra. We view $\widehat{U(\mathfrak{g})}$ as a quantisation of the algebra of rigid analytic functions on $\mathfrak{g}^*$ in much the same way that $U(\mathfrak{g})$ can be viewed as a quantisation of the algebra of polynomial functions on $\mathfrak{g}^*$. This is the starting point for our work: our Beilinson–Bernstein style equivalence should have the co-admissible modules for a central reduction of $\widehat{U(\mathfrak{g})}$ on one side.

1.3. **Lie algebroids and completed enveloping algebras.** When working with smooth algebraic varieties in characteristic zero, one can view classical sheaves of differential operators as special cases of sheaves of enveloping algebras of Lie algebroids; this is the approach taken in [6]. We adopt this more general framework here partly for convenience at certain points of our presentation and partly for the sake of flexibility in future work; in particular we will use it to define sheaves of twisted differential operators in [3]. In section 9 below for each Lie algebroid $\mathscr{L}$ on a rigid analytic space X we construct a sheaf $\widehat{\mathscr{U}(\mathscr{L})}$ of completed universal enveloping algebras on X. When X is smooth we then define $\widehat{\mathcal{D}} := \widehat{\mathscr{U}(\mathcal{T})}$. These sheaves $\widehat{\mathscr{U}(\mathscr{L})}$ may be viewed as quantisations of the total space of the vector bundle $\mathscr{L}^*$. In particular, in this picture, $\widehat{\mathcal{D}}$ is a quantisation of T^*X.

[1]In fact with a little extra effort our construction can be localised to the rigid étale site but we do not provide the details of that here.

One difficulty with extending the classical work on $\mathcal{D}$-modules to the rigid analytic setting is that the notion of quasi-coherent sheaves on rigid analytic spaces is problematic: see [14, §2.1] for a basic treatment together with some instructive examples of what can go wrong due to Gabber. See also [7] for some recent work in this direction. We resolve this difficulty by avoiding it: we restrict ourselves to the study of 'coherent' modules for our sheaves of rings. Because our sheaves of rings $\widehat{\mathscr{U}(\mathscr{L})}$ are not themselves coherent the usual notion of coherent sheaves of modules is not appropriate. However, the sections of our structure sheaves $\widehat{\mathscr{U}(\mathscr{L})}$ over sufficiently small affinoid subdomains turn out to be Fréchet–Stein, so Schneider and Teitelbaum's work shows us how to proceed: we replace the notion of 'locally finitely generated' by 'locally co-admissible'. When looked at through a particular optic, these 'locally co-admissible' sheaves do deserve to be seen as if they were coherent sheaves of modules over some non-commutative structure sheaf on the cotangent bundle T^*X. However, in order to make this interpretation precise, it seems to be necessary to fully develop a theory of micro-local sheaves in our context.

1.4. **Main results.** Our first main result is a non-commutative version of Tate's Acyclicity Theorem [30, Theorem 8.2].

Theorem A. *Suppose that X is a smooth K-affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module. Then*

$$\widehat{\mathcal{D}}(Y) := \widehat{U(\mathcal{T}(Y))}$$

defines a sheaf $\widehat{\mathcal{D}}$ of Fréchet-Stein algebras on affinoid subdomains of X with vanishing higher Čech cohomology groups.

Here $\widehat{U(\mathcal{T}(Y))}$ can be concisely defined as the completion of the enveloping algebra $U(\mathcal{T}(Y))$ with respect to all submultiplicative seminorms that extend the supremum seminorm on $\mathcal{O}(Y)$; see Section 6.2 below for a more algebraic definition. Our next result involves an appropriate version of completed tensor product $\widehat{\otimes}$, which we develop in Section 7.

Theorem B. *Suppose that X is a smooth K-affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module. Then*

$$\mathrm{Loc}(M)(U) := \widehat{\mathcal{D}}(U) \underset{\widehat{\mathcal{D}}(X)}{\widehat{\otimes}} M$$

defines a full exact *embedding $M \mapsto \mathrm{Loc}(M)$ of the category of co-admissible $\widehat{\mathcal{D}}(X)$-modules into the category of sheaves of $\widehat{\mathcal{D}}$-modules on affinoid subdomains of X, with vanishing higher Čech cohomology groups.*

We can extend $\widehat{\mathcal{D}}$ to a sheaf defined on general smooth rigid analytic varieties. Then we prove the following analogue of Kiehl's Theorem [20] for coherent sheaves of $\mathcal{O}$-modules on rigid analytic spaces.

Theorem C. *Suppose that X is a smooth analytic variety over K. Let $\mathcal{M}$ be a sheaf of $\widehat{\mathcal{D}}$-modules on X. Then the following are equivalent.*

(a) There is an admissible affinoid covering $\{X_i\}_{i\in I}$ of X such that $\mathcal{T}(X_i)$ is a free $\mathcal{O}(X_i)$-module, $\mathcal{M}(X_i)$ is a co-admissible $\widehat{\mathcal{D}}(X_i)$-module and the restriction of $\mathcal{M}$ to the affinoid subdomains of X_i is isomorphic to $\mathrm{Loc}(\mathcal{M}(X_i))$ for each $i \in I$.

(b) For every affinoid subdomain U of X such that $\mathcal{T}(U)$ is a free $\mathcal{O}(U)$-module, $\mathcal{M}(U)$ is a co-admissible $\widehat{\mathcal{D}}(U)$-module and $\mathcal{M}(V) \cong \widehat{\mathcal{D}}(V) \underset{\widehat{\mathcal{D}}(U)}{\widehat{\otimes}} \mathcal{M}(U)$ for every affinoid subdomain V of U.

We call a sheaf of $\widehat{\mathcal{D}}$-modules that satisfies the equivalent conditions of Theorem C *co-admissible*. Theorems B and C immediately give the following

Corollary. *Suppose X is a smooth K-affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module. Then* Loc *is an equivalence of abelian categories*

$$\left\{ \begin{array}{c} co\text{–}admissible \\ \widehat{\mathcal{D}}(X)\text{–}modules \end{array} \right\} \cong \left\{ \begin{array}{c} co\text{–}admissible\ sheaves\ of \\ \widehat{\mathcal{D}}\text{–}modules\ on\ X \end{array} \right\}.$$

In fact we prove each of these statements in greater generality with $\mathcal{T}$ replaced by any Lie algebroid on any rigid analytic space over K, and for right modules as well as left modules.

1.5. **Future and related work.** We plan to explain in the future how parts of the vast classical theory of $\mathcal{D}$-modules generalise to our setting with the results contained in this work being merely the leading edge of what is to come. In particular in [2] we will prove the following analogue of Kashiwara's equivalence.

Theorem D. *Let Y be a smooth closed analytic subvariety of a smooth rigid analytic variety X. There is a natural equivalence of categories*

$$\left\{ \begin{array}{c} co\text{–}admissible\ sheaves\ of \\ \widehat{\mathcal{D}}\text{–}modules\ on\ Y \end{array} \right\} \cong \left\{ \begin{array}{c} co\text{–}admissible\ sheaves\ of \\ \widehat{\mathcal{D}}\text{–}modules\ on\ X\ supported\ on\ Y \end{array} \right\}.$$

In future work [3], we will prove an analogue of Beilinson and Bernstein's localisation theorem of [5] for twisted $\widehat{\mathcal{D}}$-modules on $\mathcal{B}^{\mathrm{an}}$. For the sake of brevity, we will only state the version of this result for un-twisted $\widehat{\mathcal{D}}$-modules here.

Theorem E. *Let $\mathbf{G}$ be a connected split reductive group over K with Lie algebra $\mathfrak{g}$, let $\mathcal{B}^{\mathrm{an}}$ be the rigid analytic flag variety and let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$. Then there is an equivalence of abelian categories*

$$\left\{ \begin{array}{c} co\text{–}admissible \\ \widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g})} K\text{–}modules \end{array} \right\} \cong \left\{ \begin{array}{c} co\text{–}admissible\ sheaves\ of \\ \widehat{\mathcal{D}}\text{–}modules\ on\ \mathcal{B}^{\mathrm{an}} \end{array} \right\}.$$

We hope, perhaps even expect, that this work will have wider applications. Certainly it seems likely that the study of p-adic differential equations will be synergetic with our work. Also, much as the theory of algebraic $\mathcal{D}$-modules was influential for the field of non-commutative algebraic geometry, this work might point towards a non-commutative rigid analytic geometry (see also [27]).

It is appropriate to mention here the body of work by Berthelot and others begun in [10] that considers sheaves of arithmetic differential operators on smooth formal schemes $\mathcal{X}$ over $W(k)$. There are points of connection between our work and Berthelot's but the differences are substantial. We also note that Patel, Schmidt and Strauch have begun a programme [22] of localising locally analytic representations of non-compact semi-simple p-adic Lie groups onto Bruhat-Tits buildings. Whilst their motivation is similar to ours there are again significant differences between our approach and theirs.

1.6. **A brief summary of our constructions.** In order to construct our sheaves $\widehat{\mathscr{U}(\mathscr{L})}$ we first define some intermediate objects that may well prove to be of interest in their own right. Our definitions are heavily dependent on the notion of the enveloping algebra $U(L)$ of a Lie-Rinehart algebra L; we give a brief overview of this theory in Section 2 below.

Let $\mathcal{R}$ denote the ring of integers of our ground field K, and fix a non-zero non-unit $\pi \in \mathcal{R}$. Let X be an *affinoid* variety over K. Given an affine formal model $\mathcal{A}$ in $\mathcal{O}(X)$ and a Lie-Rinehart $(\mathcal{R},\mathcal{A})$-Lie algebra $\mathcal{L}$ we define a G-topology $X_w(\mathcal{L})$ on X consisting of those affinoid subdomains Y of X such that $\mathcal{O}(Y)$ has an affine formal model $\mathcal{B}$ with the property that the unique extension of the natural action of $\mathcal{L}$ on $\mathcal{O}(X)$ to an action on $\mathcal{O}(Y)$ preserves $\mathcal{B}$. We call these affinoid subdomains $\mathcal{L}$-*admissible*.

For example if $X = \operatorname{Sp} K\langle x\rangle$, $\mathcal{A} = \mathcal{R}\langle x\rangle$, and $\mathcal{L} = \mathcal{A}\partial_x$ then the closed disc $Y \subset X$ of radius $|p|^{1/p}$ centred at zero is $\mathcal{L}$-admissible because $\mathcal{R}\langle x, x^p/p\rangle$ is an $\mathcal{L}$-stable affine formal model in $\mathcal{O}(Y)$. The smaller closed disc of radius $|p|$ is not $\mathcal{L}$-admissible, however it *is* $p\mathcal{L}$-admissible.

A key result due to Rinehart [23, Theorem 3.1] that underlies much of our work can be viewed as a generalisation of the Poincaré-Birkhoff-Witt Theorem to the setting of $(\mathcal{R},\mathcal{A})$-algebras. To apply this theorem directly to an enveloping algebra $U(\mathcal{L})$, the $(\mathcal{R},\mathcal{A})$-algebra $\mathcal{L}$ is required to be finitely generated and projective as an $\mathcal{A}$-module: then Rinehart's Theorem states that $U(\mathcal{L})$ carries a natural positive filtration whose associated graded ring is isomorphic to the symmetric algebra $\operatorname{Sym}_{\mathcal{A}}(\mathcal{L})$. Under these assumptions on $\mathcal{L}$, we construct a sheaf of Noetherian Banach algebras $\widehat{\mathscr{U}(\mathcal{L})_K}$ on the $\mathcal{L}$-admissible affinoid subdomains Y of X as follows:

$$\widehat{\mathscr{U}(\mathcal{L})_K}(Y) := \widehat{U(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L})} \otimes_{\mathcal{R}} K$$

where $\mathcal{B}$ is any $\mathcal{L}$-stable affine formal model contained in $\mathcal{O}(Y)$, and $\widehat{-}$ denotes π-adic completion. It is not hard to check that $\widehat{\mathscr{U}(\mathcal{L})_K}(Y)$ does not depend on the choice of $\mathcal{B}$ up to a canonical isomorphism.

We would have liked to prove that the restriction maps $\widehat{\mathscr{U}(\mathcal{L})_K}(Y) \to \widehat{\mathscr{U}(\mathcal{L})_K}(Z)$ are flat whenever $Z \subseteq Y$ are $\mathcal{L}$-admissible affinoid subdomains of X. Because we were unable to do this, we instead define a weaker $\mathcal{L}$-*accessible* G-topology $X_{ac}(\mathcal{L})$ on X, and prove that if $Z \subseteq Y$ are $\mathcal{L}$-accessible then $\widehat{\mathscr{U}(\mathcal{L})_K}(Z)$ is a flat $\widehat{\mathscr{U}(\mathcal{L})_K}(Y)$-module on both sides. Since every affinoid subdomain of X is $\pi^n\mathcal{L}$-accessible for sufficiently large n, this turns out to be sufficient for our purposes.

Now, the $X_w(\pi^n\mathcal{L})$ form an increasing chain of G-topologies on X and every affinoid subdomain Y of X lives is $X_w(\pi^n\mathcal{L})$ for sufficiently large n, so the formula

$$\widehat{\mathscr{U}(L)}(Y) := \varprojlim \widehat{\mathscr{U}(\pi^n\mathcal{L})_K}(Y)$$

defines a presheaf of K-algebras $\widehat{\mathscr{U}(L)}$ on all affinoid subdomains of X, which only depends on the $(K,\mathcal{O}(X))$-Lie algebra $L := K\otimes_R \mathcal{L}$. We show that this presheaf is actually a sheaf, and that its sections over affinoid subdomains Y are Fréchet–Stein in the sense of [26] with respect to the family $(\widehat{\mathscr{U}(\pi^n\mathcal{L})_K}(Y))_{n\gg 0}$. The sheaf $\widehat{\mathcal{D}}$ is obtained in the special case where $L = \mathcal{T}(X)$. Given a Lie algebroid $\mathscr{L}$ on a general rigid analytic space X, we then use a version of the Comparison Lemma to glue the sheaves $\widehat{\mathscr{U}(\mathscr{L}(Y))}$ on the affinoid subdomains Y of X to a sheaf $\widehat{\mathscr{U}(\mathscr{L})}$ on X.

1.7. **Structure of this paper.** The main body of the paper begins in Section 3 where we define and study the G-topology $X_w(\mathcal{L})$ associated to a K-affinoid variety X with an affine formal model $\mathcal{A}$ and an $(\mathcal{R}, \mathcal{A})$-Lie algebra $\mathcal{L}$ as explained above. The main result of that section is that the presheaf $\widehat{\mathscr{U}(\mathcal{L})_K}$ on $X_w(\mathcal{L})$ defined therein is a sheaf with vanishing higher cohomology. In Section 4 we prove that the continuous K-algebra homomorphisms that arise as restriction maps in the sheaves $\widehat{\mathscr{U}(\mathcal{L})_K}$ on $X_{ac}(\mathcal{L})$ are flat. In Section 5 we prepare the way for Theorems B and C by proving preliminary versions for the sheaves $\widehat{\mathscr{U}(\mathcal{L})_K}$ on $X_{ac}(\mathcal{L})$.

In Section 6 we begin our study of Fréchet–Stein algebras. In particular we give a functorial construction that associates a Fréchet–Stein algebra $\widehat{U(L)}$ to each (K, A)-Lie algebra L with A an affinoid algebra and L finitely generated as an A-module. We do this via a more general construction that associates a Fréchet–Stein algebra to every deformable $\mathcal{R}$-algebra with commutative Noetherian associated graded ring. Then in Section 7 we define a base change functor $\widehat{\otimes}$ between categories of co-admissible modules over Fréchet–Stein algebras U and V that possess a suitable $U - V$-bimodule.

In Sections 8 and 9, we put all of this together in order to prove Theorems A–C. More precisely, Theorems A and B are special cases of Theorems 8.1 and 8.2, whereas Theorem C and its Corollary are special cases of Theorem 9.4 and Theorem 9.5, respectively.

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1.9. **Conventions.** Throughout this paper K will denote a complete discrete valuation field with valuation ring $\mathcal{R}$ and residue field k. We fix a non-zero non-unit element π in $\mathcal{R}$. If $\mathcal{M}$ is an $\mathcal{R}$-module, then $\widehat{\mathcal{M}}$ denotes the π-adic completion of $\mathcal{M}$. The term "module" will mean *left* module, unless explicitly stated otherwise.

2. Enveloping algebras of Lie–Rinehart algebras

2.1. **Lie–Rinehart algebras.** Let R be a commutative base ring, and let A be a commutative R-algebra. A *Lie–Rinehart algebra*, or more precisely, an (R, A)-*Lie algebra* is a pair (L, ρ) where

- L is an R-Lie algebra and an A-module, and
- $\rho\colon L \to \operatorname{Der}_R(A)$ is an A-linear Lie algebra homomorphism

called the *anchor map*, such that $[x, ay] = a[x, y] + \rho(x)(a)y$ for all $x, y \in L$ and $a \in A$; see [23]. We will frequently abuse notation and simply denote (L, ρ) by L whenever the anchor map ρ is understood.

For every (R,A)-Lie algebra L there is an associative R-algebra $U(L)$ called the *enveloping algebra* of L, which comes equipped with canonical homomorphisms

$$i_A\colon A \to U(L) \quad \text{and} \quad i_L : L \to U(L)$$

of R-algebras and R-Lie algebras respectively, satisfying

$$i_L(ax) = i_A(a)i_L(x) \quad \text{and} \quad [i_L(x), i_A(a)] = i_A(\rho(x)(a)) \quad \text{for all} \quad a \in A, x \in L.$$

The enveloping algebra $U(L)$ enjoys the following universal property: whenever $j_A\colon A \to S$ is an R-algebra homomorphism and $j_L : L \to S$ is an R-Lie algebra homomorphism such that

$$j_L(ax) = j_A(a)j_L(x) \quad \text{and} \quad [j_L(x), j_A(a)] = j_A(\rho(x)(a)) \quad \text{for all} \quad a \in A, x \in L,$$

there exists a unique R-algebra homomorphism $\varphi\colon U(L) \to S$ such that

$$\varphi \circ i_A = j_A \quad \text{and} \quad \varphi \circ i_L = j_L.$$

It is easy to show [23, §2] that $i_A\colon A \to U(L)$ is always injective, and we will always identify A with its image in $U(L)$ via i_A.

If $(L,\rho), (L',\rho')$ are two (R,A)-Lie algebras then a *morphism of* (R,A)*-Lie algebras* is an A-linear map $f : L \to L'$ that is also a morphism of R-Lie algebras and satisfies $\rho' f = \rho$.

A morphism of (R,A)-Lie algebras $f\colon L \to L'$ induces an R-algebra homomorphism $U(f)\colon U(L) \to U(L')$ via $U(f)(a) = a$ for $a \in A$ and $U(f)(i_L(x)) = i_{L'}(f(x))$ for $x \in L$. So in this way, U defines a functor from (R,A)-Lie algebras to associative R-algebras.

Definition. We say that an (R,A)-Lie algebra L is *coherent* if it is coherent as an A-module. We say that L is *smooth* if in addition it is projective as a A-module.

2.2. **Base extensions of Lie–Rinehart algebras.** Let A and B be commutative R-algebras and let $\varphi\colon A \to B$ be an R-algebra homomorphism. If L is an (R,A)-Lie algebra, the B-module $B \otimes_A L$ will not be an (R,B)-Lie algebra, in general. However this is true in many interesting situations.

Lemma. *Suppose that the anchor map $\rho\colon L \to \mathrm{Der}_R(A)$ lifts to an A-linear Lie algebra homomorphism $\sigma\colon L \to \mathrm{Der}_R(B)$ in the sense that*

$$\sigma(x) \circ \varphi = \varphi \circ \rho(x) \quad \textit{for all} \quad x \in L.$$

Then $(B \otimes_A L, 1 \otimes \sigma)$ with the natural B-linear structure is an (R,B)-Lie algebra in a unique way.

Proof. Write $x \cdot b := \sigma(x)(b)$ and $bx := b \otimes x$ for all $x \in L$ and $b \in B$. Following [23, (3.5)], we define a bracket operation on $B \otimes_A L$ in the only possible way as follows:

$$[bx, b'x'] := bb'[x,x'] - b'(x' \cdot b)x + b(x \cdot b')x'$$

for all $b, b' \in B$ and $x, x' \in L$. It is straightforward to verify that this bracket is well-defined, skew-symmetric, and satisfies

$$[bx, c(b'x')] = c[bx, b'x'] + (1 \otimes \sigma)(bx)(c)b'x'$$

for all $b, b', c \in B$ and $x, x' \in L$. Note that if $x, y, z \in L$ and $b \in B$ then

$$[[1x, 1y], bz] + [[1y, bz], 1x] + [[bz, 1x], 1y] = ([x,y] \cdot b - x \cdot (y \cdot b) + y \cdot (x \cdot b))z$$

so the condition that $\sigma : L \to \mathrm{Der}_k(B)$ is a Lie homomorphism is necessary for the Jacobi identity to hold. A longer, but still straightforward, computation shows that this condition is also sufficient. $\square$

Corollary. *Suppose that $\psi\colon \mathrm{Der}_R(A) \to \mathrm{Der}_R(B)$ is an A-linear homomorphism of R-Lie algebras such that $\psi(u)\circ\varphi = \varphi\circ u$ for each $u \in \mathrm{Der}_R(A)$. There is a natural functor $B\otimes_A -$ from (R,A)-Lie algebras to (R,B)-Lie algebras sending (L,ρ) to $(B\otimes_A L, 1\otimes\psi\rho)$.*

Proof. Suppose that (ρ, L) and (ρ', L') are (R,A)-Lie algebras and $f\colon L \to L'$ is a morphism of (R,A)-Lie algebras. Give $(B\otimes_A L, 1\otimes\psi\rho)$ and $(B\otimes_A L', 1\otimes\psi\rho')$ the structures of (R,B)-Lie algebras guaranteed by the Lemma; we have to show that $1\otimes f : B\otimes_A L \to B\otimes_A L'$ is then a morphism of (R,B)-Lie algebras.

It is B-linear and satisfies $(1\otimes\psi\rho')\circ(1\otimes f) = 1\otimes\psi\rho$ because $\rho' f = \rho$. Now if $b, c \in B$ and $x, y \in L$ then

$$\begin{aligned}[][(1\otimes f)(bx), (1\otimes f)(cy)] &= bc[f(x), f(y)] - c(y\cdot b)f(x) + b(x\cdot c)f(y)\\ &= (1\otimes f)([bx, cy]).\end{aligned}$$

Thus $1\otimes f$ is an R-Lie algebra homomorphism. $\square$

2.3. **Rinehart's Theorem.** Let $\mathrm{Sym}(L)$ denote the symmetric algebra of the A-module L. Rinehart proved [23, Theorem 3.1] that there is always a surjection

$$\mathrm{Sym}(L) \twoheadrightarrow \mathrm{gr}\, U(L)$$

which is even an isomorphism whenever L is smooth. Therefore $U(L)$ is a (left and right) Noetherian ring whenever A is Noetherian and L is a finitely generated A-module; we will use this basic fact without further mention in what follows.

Proposition. *Let $\varphi\colon A \to B$ be a homomorphism of commutative R-algebras and let (L,ρ) be a smooth (R,A)-Lie algebra. Suppose that $\rho\colon L \to \mathrm{Der}_R(A)$ lifts to an A-linear Lie algebra homomorphism $\sigma\colon L \to \mathrm{Der}_R(B)$. Then there are natural isomorphisms*

$$B\otimes_A U(L) \to U(B\otimes_A L) \quad \textit{and} \quad U(L)\otimes_A B \to U(B\otimes_A L)$$

of filtered left B-modules and filtered right B-modules, respectively.

Proof. The pair $(B\otimes_A L, 1\otimes\sigma)$ is an (R,B)-Lie algebra by Lemma 2.2. The universal property of $U(L)$ induces a homomorphism of filtered R-algebras

$$U(\varphi)\colon U(L) \to U(B\otimes_A L)$$

such that $U(\varphi)(i_L(x)) = i_{B\otimes_A L}(1\otimes x)$ for all $x \in L$. Since $U(\varphi)$ is left A-linear, we obtain a filtered left B-linear homomorphism

$$1\otimes U(\varphi)\colon B\otimes_A U(L) \longrightarrow U(B\otimes_A L).$$

By [23, Theorem 3.1], its associated graded can be identified with the natural map

$$B\otimes_A \mathrm{Sym}(L) \longrightarrow \mathrm{Sym}(B\otimes_A L)$$

which is an isomorphism by [13, Chapter III, §6, Proposition 4.7]. The isomorphism $U(\varphi)\otimes 1\colon U(L)\otimes_A B \to U(B\otimes_A L)$ of right B-modules is established similarly. $\square$

2.4. **Lifting derivations of affinoid algebras.** Recall [8, §3.3], that if $A \to B$ is a morphism of affinoid algebras, then there is a finitely generated B-module

$$\Omega_{B/A}$$

such that for any Banach B-module M there is a natural isomorphism

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \cong \mathrm{Der}^b_A(B, M)$$

where $\mathrm{Der}^b_A(B, M)$ denotes the set of A-linear *bounded* derivations from B to M. Note that every K-linear derivation from B to a *finitely generated* B-module M is automatically bounded; this follows from the proof of [15, Theorem 3.6.1]. So in particular, $\mathrm{Der}^b_K(B, B) = \mathrm{Der}_K(B)$.

Lemma. *Let $\varphi\colon A \to B$ be an étale morphism of K-affinoid algebras. Then there is a unique A-linear map*

$$\psi : \mathrm{Der}_K(A) \to \mathrm{Der}_K(B)$$

such that $\psi(u) \circ \varphi = \varphi \circ u$ for each $u \in \mathrm{Der}_K(A)$. Moreover ψ is a homomorphism of K-Lie algebras.

Proof. By [11, Corollary 2.1.8/3 and Theorem 6.1.3/1], $\varphi : A \to B$ is bounded. Hence, composition with φ induces A-linear maps

$$\mathrm{Der}_K(A) \xrightarrow{\alpha} \mathrm{Der}^b_K(A, B) \xleftarrow{\beta} \mathrm{Der}_K(B).$$

Since $A \to B$ is étale, [8, Proposition 3.5.3(i)] guarantees that the natural morphism $B \otimes_A \Omega_{A/K} \to \Omega_{B/K}$ is an isomorphism. Taking B-linear duals shows that the restriction map

$$\beta : \mathrm{Der}_K(B) \to \mathrm{Der}^b_K(A, B)$$

is also an isomorphism and therefore every K-linear derivation of B is determined by its restriction to A. We therefore obtain a unique A-linear map

$$\psi := \beta^{-1} \circ \alpha : \mathrm{Der}_K(A) \to \mathrm{Der}_K(B)$$

such that $\psi(u) \circ \varphi = \varphi \circ u$ for all $u \in \mathrm{Der}_K(A)$. If $u, v \in \mathrm{Der}_K(A)$ then the K-linear derivations $\psi([u, v])$ and $[\psi(u), \psi(v)]$ of B agree on the image of A in B and therefore are equal. Hence ψ is a Lie homomorphism. $\square$

Combining the Lemma with Corollary 2.2 gives the following

Corollary. *Let $A \to B$ be an étale morphism of K-affinoid algebras and let L be a (K, A)-Lie algebra. Then there is a unique structure of a (K, B)-Lie algebra on $B \otimes_A L$ with its natural B-module structure such that the natural map $L \to B \otimes_A L$ is a K-Lie algebra homomorphism and the diagram*

$$\begin{tikzcd} L \arrow[r, "\rho_L"] \arrow[d] & \mathrm{Der}_K(A) \arrow[d, "\psi"] \\ B \otimes_A L \arrow[r, "\rho_{B\otimes_A L}"'] & \mathrm{Der}_K(B) \end{tikzcd}$$

commutes. Moreover this defines a canonical functor $L \mapsto B \otimes_A L$ from (K, A)-Lie algebras to (K, B)-Lie algebras.

2.5. **Torsion in $U(\mathcal{L})$.** Let $\mathcal{A}$ be a commutative Noetherian $\mathcal{R}$-algebra, and let $\mathcal{L}$ be a coherent $(\mathcal{R},\mathcal{A})$-Lie algebra. Let $\overline{\mathcal{L}}$ denote the image of $\mathcal{L}$ in $\mathcal{L}\otimes_{\mathcal{R}} K$; this is again an $(\mathcal{R},\mathcal{A})$-Lie algebra which is now flat as an $\mathcal{R}$-module.

Let $\overline{U(\mathcal{L})}$ denote the image of $U(\mathcal{L})$ in $U(\mathcal{L})\otimes_{\mathcal{R}} K$; note that unless $\mathcal{L}$ happens to be smooth, the π-torsion submodule of $U(\mathcal{L})$ may well be non-zero. In any case, it is easy to see that there is a commutative diagram of $\mathcal{R}$-algebra homomorphisms with surjective arrows

$$\begin{tikzcd} U(\mathcal{L}) \arrow[r, two heads] \arrow[d, two heads] & U(\overline{\mathcal{L}}) \arrow[d, two heads] \\ \overline{U(\mathcal{L})} \arrow[r, dotted, two heads] & \overline{U(\overline{\mathcal{L}})}. \end{tikzcd}$$

Note that because $U(\mathcal{L}\otimes_{\mathcal{R}} K)\cong U(\mathcal{L})\otimes_{\mathcal{R}} K$, the bottom arrow in this diagram is actually an isomorphism.

Lemma. *The functor $X\mapsto \widehat{X_K}=\widehat{X}\otimes_{\mathcal{R}} K$ transforms each arrow in the above diagram into an isomorphism.*

Proof. The kernel of $U(\mathcal{L})\to\overline{U(\mathcal{L})}$ is a finitely generated left ideal T in $U(\mathcal{L})$ since $U(\mathcal{L})$ is Noetherian. Since $T\otimes_{\mathcal{R}} K=0$ by construction, we see that $\pi^n\cdot T=0$ for some $n\geqslant 0$. The sequence $0\to\widehat{T}\to\widehat{U(\mathcal{L})}\to\widehat{\overline{U(\mathcal{L})}}\to 0$ is exact by [10, §3.2.3(ii)], and $\pi^n\cdot\widehat{T}=0$, so $\widehat{U(\mathcal{L})_K}\to\widehat{\overline{U(\mathcal{L})}_K}$ is an isomorphism. This deals with the vertical arrows, and the result follows. $\square$

3. Tate's Acyclicity Theorem for $\widehat{\mathscr{U}(\mathcal{L})_K}$

3.1. **$\mathcal{L}$-stable affine formal models.** Recall [12, §1] that an *admissible $\mathcal{R}$-algebra* is a commutative $\mathcal{R}$-algebra which is topologically of finite type and flat over $\mathcal{R}$.

If A is a K-affinoid algebra and $\mathcal{A}$ is an admissible $\mathcal{R}$-algebra then we say that $\mathcal{A}$ is an *affine formal model in A* if $A\cong\mathcal{A}\otimes_{\mathcal{R}} K$.

Lemma. *Let $\mathcal{A},\mathcal{B}$ be two affine formal models in the K-affinoid algebra A. Then their product $\mathcal{A}\mathcal{B}$ is another formal model in A, and $\mathcal{A}\mathcal{B}$ is finitely generated as a module over $\mathcal{A}$ and $\mathcal{B}$.*

Proof. It follows from [11, 6.1.3/1, 2.1.8/3, 1.2.5/4] that $f(A^\circ)\subseteq B^\circ$ whenever $f:A\to B$ is a K-algebra homomorphism between two K-affinoid algebras. Because $\mathcal{A}$ is the image of $\mathcal{R}\langle x_1,\ldots,x_n\rangle$ under some homomorphism $K\langle x_1,\ldots,x_n\rangle\to A$, it follows that $\mathcal{A}$ and $\mathcal{B}$ are both contained in A°. Now if S is a finite topological generating set for $\mathcal{B}$ as an $\mathcal{R}$-algebra, then the subalgebra $\mathcal{A}[S]$ of A generated by $\mathcal{A}$ and S is finitely generated as an $\mathcal{A}$-module by [12, Lemma 4.5], and is therefore contained in $\frac{1}{\pi^n}\mathcal{A}$ for some $n\geqslant 0$. This subalgebra is therefore closed, and hence contains $\mathcal{B}$. Thus $\mathcal{A}\cdot\mathcal{B}=\mathcal{A}[S]$ and the result follows. $\square$

Definition. Let $\sigma:A\to B$ be an étale morphism of K-affinoid algebras and let $\mathcal{A}$ be an affine formal model in A. Let $\mathcal{L}$ be an $(\mathcal{R},\mathcal{A})$-Lie algebra and let $\mathcal{B}$ be an affine formal model in B. We say that $\mathcal{B}$ is *$\mathcal{L}$-stable* if $\sigma(\mathcal{A})\subset\mathcal{B}$ and the action of $\mathcal{L}$ on $\mathcal{A}$ lifts to $\mathcal{B}$. We say that $\sigma:A\to B$ is *$\mathcal{L}$-admissible* if there exists at least one $\mathcal{L}$-stable affine formal model $\mathcal{B}$ in B.

Note that because $\sigma : A \to B$ is étale, the action of $L := \mathcal{L} \otimes_{\mathcal{R}} K$ on A lifts automatically to B by Corollary 2.4, making this definition meaningful.

Corollary. *Let $\sigma : A \to B$ be an $\mathcal{L}$-admissible morphism of K-affinoid algebras, let $\mathcal{A}$ be an affine formal model in A and let $\mathcal{L}$ be an $(\mathcal{R}, \mathcal{A})$-Lie algebra. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two $\mathcal{L}$-stable formal models in B then there is a third $\mathcal{L}$-stable formal model $\mathcal{B}_3$ in B containing both $\mathcal{B}_1$ and $\mathcal{B}_2$, and $t_1, t_2 \geqslant 0$ such that $\pi^{t_i}\mathcal{B}_3 \subseteq \mathcal{B}_i$ for $i = 1, 2$.*

Proof. The product $\mathcal{B}_3 := \mathcal{B}_1\mathcal{B}_2$ is again an affine formal model by the Lemma, and it is $\mathcal{L}$-stable because $\mathcal{L}$ acts on B by $\mathcal{R}$-linear derivations. Now $\mathcal{B}_3$ is a finitely generated $\mathcal{B}_i$-module by the Lemma, and $\mathcal{B}_3 \otimes_{\mathcal{R}} K = \mathcal{B}_i \otimes_{\mathcal{R}} K$ for $i = 1, 2$. $\square$

3.2. **The $\mathcal{L}$-admissible G-topology.** Let X be a K-affinoid variety. Recall from [11, §9.1.4] the *strong G-topology* X_{rig} consisting of the *admissible open subsets* of X and admissible coverings, and the *weak G-topology* X_w on X consisting of the *affinoid subdomains* of X and *finite* coverings by affinoid subdomains.

Definition. Let X be a K-affinoid variety, and let $\mathcal{L}$ be an $(\mathcal{R}, \mathcal{A})$-Lie algebra for some affine formal model $\mathcal{A}$ in $\mathcal{O}(X)$. We say that an affinoid subdomain Y of X is $\mathcal{L}$-*admissible* if the pullback on functions $\mathcal{O}(X) \to \mathcal{O}(Y)$ is $\mathcal{L}$-admissible. We will denote the full subcategory of X_w consisting of the $\mathcal{L}$-admissible affinoid subdomains by $X_w(\mathcal{L})$. We define an $\mathcal{L}$-*admissible covering* of an $\mathcal{L}$-admissible affinoid subdomain of X to be a finite covering by objects in $X_w(\mathcal{L})$.

The following Lemma now shows that $X_w(\mathcal{L})$ is a G-topology on X in the sense of [11, Definition 9.1.1/1].

Lemma. *Let X be a K-affinoid variety, and let $\mathcal{L}$ be an $(\mathcal{R}, \mathcal{A})$-Lie algebra for some affine formal model $\mathcal{A}$ in $\mathcal{O}(X)$. Then $X_w(\mathcal{L})$ is stable under finite intersections.*

Proof. Let $Y, Z \in X_w(\mathcal{L})$ have $\mathcal{L}$-stable affine formal models $\mathcal{B}$ and $\mathcal{C}$ respectively, and let $\overline{\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}}$ denote the image of $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}$ in $\mathcal{O}(Y \cap Z) = \mathcal{O}(Y)\widehat{\otimes}_{\mathcal{O}(X)}\mathcal{O}(Z)$. Then the fibre product $\mathrm{Spf}(\mathcal{B}) \times_{\mathrm{Spf}(\mathcal{A})} \mathrm{Spf}(\mathcal{C})$ *in the category of admissible formal schemes* is $\mathrm{Spf}(\overline{\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}})$ by definition, see [12, p. 298]. Therefore $\overline{\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}}$ is an affine formal model in $\mathcal{O}(Y \cap Z) = \mathcal{O}(Y)\widehat{\otimes}_{\mathcal{O}(X)}\mathcal{O}(Z)$ by [12, Corollary 4.6]. It contains the $\mathcal{L}$-stable subalgebra generated by the images of $\mathcal{B}\widehat{\otimes}1$ and $1\widehat{\otimes}\mathcal{C}$ as a dense subspace, and therefore is $\mathcal{L}$-stable because every K-linear derivation of the K-affinoid algebra $B\widehat{\otimes}_A C$ is automatically continuous. $\square$

3.3. **The functor $\widehat{\mathscr{U}(\mathcal{L})_K}$.** Let V and W be two K-Banach spaces. Recall that a K-linear map $f : V \to W$ is an *isomorphism* if it is bounded, and has a bounded K-linear inverse.

Proposition. *Let $\sigma : A \to B$ be a map of K-affinoid algebras, let $\mathcal{A}$ be an affine formal model in A and let $\mathcal{B} \subseteq \mathcal{B}'$ be affine formal models in B containing $\sigma(\mathcal{A})$.*

(a) Let Q be a flat $\mathcal{A}$-module. Then there is an isomorphism of Banach B-modules

$$\widehat{\mathcal{B} \otimes_{\mathcal{A}} Q} \otimes_{\mathcal{R}} K \xrightarrow{\cong} \widehat{\mathcal{B}' \otimes_{\mathcal{A}} Q} \otimes_{\mathcal{R}} K.$$

(b) Suppose that σ is étale, let $\mathcal{L}$ be a smooth $(\mathcal{R}, \mathcal{A})$-Lie algebra, and suppose that $\mathcal{B}$ and $\mathcal{B}'$ are $\mathcal{L}$-stable. Then there is an isomorphism of K-Banach algebras

$$\widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L})} \otimes_{\mathcal{R}} K \xrightarrow{\cong} \widehat{U(\mathcal{B}' \otimes_{\mathcal{A}} \mathcal{L})} \otimes_{\mathcal{R}} K.$$

Proof. (a) $\mathcal{B}'$ is a finitely generated $\mathcal{B}$-module by Lemma 3.1, so $\pi^a\mathcal{B}' \subseteq \mathcal{B}$ for some integer a. Since Q is flat, we obtain $\mathcal{B}$-module embeddings

$$\mathcal{B} \otimes_{\mathcal{A}} Q \hookrightarrow \mathcal{B}' \otimes_{\mathcal{A}} Q \hookrightarrow \frac{1}{\pi^a}\mathcal{B} \otimes_{\mathcal{A}} Q$$

that induce the required isomorphism $\widehat{\mathcal{B} \otimes_A Q} \otimes_R K \xrightarrow{\cong} \widehat{\mathcal{B}' \otimes_A Q} \otimes_{\mathcal{R}} K$ after completing and inverting π.

(b) Since $\mathcal{B} \subseteq \mathcal{B}'$ are $\mathcal{L}$-stable affine formal models, the natural inclusion $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L} \to \mathcal{B}' \otimes_{\mathcal{A}} \mathcal{L}$ is a homomorphism of $(\mathcal{R}, \mathcal{B})$-Lie algebras, which induces an $\mathcal{R}$-algebra homomorphism $U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}) \to U(\mathcal{B}' \otimes_{\mathcal{A}} \mathcal{L})$, and a K-algebra homomorphism

$$\widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L})} \otimes_{\mathcal{R}} K \to \widehat{U(\mathcal{B}' \otimes_{\mathcal{A}} \mathcal{L})} \otimes_{\mathcal{R}} K$$

by functoriality. Now $U(\mathcal{L})$ is a projective (hence flat) $\mathcal{A}$-module by [23, Theorem 3.1], so in view of Proposition 2.3, this homomorphism is an isomorphism of Banach B-modules by part (a). Hence it is also a K-Banach algebra isomorphism. $\square$

Definition. For any $\mathcal{L}$-admissible affinoid subdomain Y of X and any $\mathcal{L}$-stable affine formal model $\mathcal{B}$ in $\mathcal{O}(Y)$, we define

$$\widehat{\mathscr{U}(\mathcal{L})_K}(Y) := \widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L})} \otimes_{\mathcal{R}} K.$$

Note that $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ is an $(\mathcal{R}, \mathcal{B})$-Lie algebra by Lemma 2.2, so $\widehat{\mathscr{U}(\mathcal{L})_K}(Y)$ is an associative K-Banach algebra, which does not depend on the choice of $\mathcal{L}$-stable affine formal model $\mathcal{B}$ in $\mathcal{O}(Y)$ by Corollary 3.1 and Proposition 3.3(b).

Suppose now that Z and Y are $\mathcal{L}$-admissible affinoid subdomains of X with $Z \subseteq Y$. Choose an $\mathcal{L}$-stable affine formal model $\mathcal{B}$ in $\mathcal{O}(Y)$, and an $\mathcal{L}$-stable affine formal model $\mathcal{C}'$ in $\mathcal{O}(Z)$. Let $\mathcal{C}$ be the product of $\mathcal{C}'$ with the image of $\mathcal{B}$ in $\mathcal{O}(Z)$; then $\mathcal{C}$ is again an $\mathcal{L}$-stable affine formal model in $\mathcal{O}(Z)$. In this way we obtain $\mathcal{R}$-algebra homomorphisms $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ which produce the restriction maps $\mathcal{O}(X) \to \mathcal{O}(Y) \to \mathcal{O}(Z)$ after applying $-\otimes_{\mathcal{R}} K$. The universal property of the enveloping algebra of a Lie–Rinehart algebra now yields $\mathcal{R}$-algebra homomorphisms

$$U(\mathcal{L}) \to U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}) \to U(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{L})$$

and we see that $\widehat{\mathscr{U}(\mathcal{L})_K}$ is a functor from $X_w(\mathcal{L})$ to K-Banach algebras. We have thus defined a presheaf $\widehat{\mathscr{U}(\mathcal{L})_K}$ on $X_w(\mathcal{L})$.

3.4. **Rig-affinoid formal schemes.** Recall the *rigid generic fibre* functor $\mathcal{X} \mapsto \mathcal{X}_{\text{rig}}$ from admissible formal $\mathcal{R}$-schemes to rigid K-analytic varieties [12, §4]. We say that a quasi-compact and admissible (not necessarily affine) formal $\mathcal{R}$-scheme $\mathcal{X}$ is a *formal model* for the quasi-compact, quasi-separated rigid K-analytic variety X if $X = \mathcal{X}_{\text{rig}}$. We say that a quasi-compact admissible formal $\mathcal{R}$-scheme $\mathcal{X}$ is *rig-affinoid* if $\mathcal{X}_{\text{rig}}$ is affinoid.

Let $\mathcal{BT}$ be the category of π-torsion $\mathcal{R}$-modules which are *bounded*, i.e. killed by some power of π. This is a Serre subcategory of the abelian category $\mathcal{R}$–mod of $\mathcal{R}$-modules, and we will work in the quotient abelian category

$$\mathcal{Q} := \mathcal{R}\text{–mod}/\mathcal{BT}.$$

Let $q : \mathcal{R}\text{–mod} \to \mathcal{Q}$ denote the natural exact quotient functor.

On several occasions we will need the Čech-to-derived functor spectral sequence

$$E_2^{ij} = \check{H}^i(\mathcal{U}, \mathcal{H}^j(\mathcal{F})) \Rightarrow H^{i+j}(X, \mathcal{F}) \tag{1}$$

from [29, Theorem 3.4.4]; see also [28, Lemma 21.11.6]. Here X is a site, $\mathcal{U}$ is a covering of X, $\mathcal{F}$ is an abelian sheaf on X and $\mathcal{H}^j(\mathcal{F})$ is the abelian presheaf on X given by $U \mapsto H^j(U, \mathcal{F})$ for any object U of X.

Lemma. *Let $\mathcal{X}$ be a quasi-compact admissible formal $\mathcal{R}$-scheme, let $\tau : \mathcal{Y} \to \mathcal{X}$ be an admissible blow-up and let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathcal{Y}}$-module.*
(a) $R^i\tau_*(\mathcal{F})$ is killed by a power of π for all $i > 0$.
(b) The map $\tau^\sharp : \mathcal{O}_{\mathcal{X}} \to \tau_*\mathcal{O}_{\mathcal{Y}}$ is injective, and $\operatorname{coker}(\tau^\sharp)$ *is killed by a power of π.*
(c) $qH^i(\mathcal{X}, \mathcal{O}) = qH^i(\mathcal{Y}, \mathcal{O})$ for all $i \geqslant 0$.

Proof. (a) Because $\mathcal{X}$ is quasi-compact and the formation of admissible blow-ups is local on the base, we may assume that $\mathcal{X}$ is affine, so that $\mathcal{A} := \mathcal{O}(\mathcal{X})$ is an admissible $\mathcal{R}$-algebra. Now $R^i\tau_*(\mathcal{F})$ is the coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules associated to $H^i(\mathcal{Y}, \mathcal{F})$ by [16, Chapitre III, 3.4.5.1], so it is sufficient to show that $H^i(\mathcal{Y}, \mathcal{F})$ is killed by a power of π for all $i > 0$. Furthermore, since τ is proper, $H^i(\mathcal{Y}, \mathcal{F})$ a finitely generated $\mathcal{A}$-module by [16, Chapitre III, Corollaire 3.4.4]. It will therefore be enough to show that $H^i(\mathcal{Y}, \mathcal{F}) \otimes_{\mathcal{R}} K = 0$ for all $i > 0$.

Choose an open affine covering $\mathcal{U} = \{\mathcal{Y}_1, \ldots, \mathcal{Y}_n\}$ of $\mathcal{Y}$. Note that $\mathcal{Y}$ is separated because it is projective over the affine formal scheme $\mathcal{X}$. Thus every finite intersection of members of $\mathcal{U}$ is again affine, and hence the coherent sheaf $\mathcal{F}$ has vanishing higher cohomology on these intersections by [17, Theorem II.9.7]. Hence the spectral sequence (1) for the covering $\mathcal{U}$ of $\mathcal{Y}$ collapses on page 2, and induces isomorphisms $H^i(\mathcal{Y}, \mathcal{F}) \cong \check{H}^i(\mathcal{U}, \mathcal{F})$ for all $i \geqslant 0$.

On the other hand, we can localise $\mathcal{F}$ to a coherent sheaf $\mathcal{F}_{\mathrm{rig}}$ of $\mathcal{O}_{\mathcal{X}_{\mathrm{rig}}}$-modules as in [12, p. 315], and it follows from [12, Corollary 4.6] that $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_{\mathcal{R}} K = C^\bullet(\mathcal{U}_{\mathrm{rig}}, \mathcal{F}_{\mathrm{rig}})$. Hence $\check{H}^i(\mathcal{U}, \mathcal{F}) \otimes_{\mathcal{R}} K \cong \check{H}^i(\mathcal{U}_{\mathrm{rig}}, \mathcal{F}_{\mathrm{rig}})$ for all $i \geqslant 0$. However $\mathcal{X}_{\mathrm{rig}}$ is affinoid, so these cohomology groups vanish by the theorems of Kiehl and Tate [11, Theorem 9.4.3/3 and Corollary 8.2.1/5] whenever $i > 0$.

(b) Again we may assume that $\mathcal{X}$ is affine with $\mathcal{A} := \mathcal{O}(\mathcal{X})$, and we choose an open affine covering $\mathcal{U} = \{\mathcal{Y}_1, \ldots, \mathcal{Y}_n\}$ of $\mathcal{Y}$. Then we want to show that

$$\tau^\sharp(\mathcal{X}) : \mathcal{A} \to \mathcal{O}(\mathcal{Y}) = \check{H}^0(\mathcal{U}, \mathcal{O})$$

is injective and that its cokernel is killed by a power of π. But

$$\tau^\sharp(\mathcal{X}) \otimes_{\mathcal{R}} K : \mathcal{O}(\mathcal{X}_{\mathrm{rig}}) \to \mathcal{O}(\mathcal{Y}_{\mathrm{rig}}) = \check{H}^0(\mathcal{U}_{\mathrm{rig}}, \mathcal{O}_{\mathrm{rig}})$$

is an isomorphism by [11, Theorem 8.2.1/1] and $\mathcal{A}$ embeds into $\mathcal{O}(\mathcal{X}_{\mathrm{rig}}) = \mathcal{A} \otimes_{\mathcal{R}} K$ because $\mathcal{O}(\mathcal{X})$ is an admissible $\mathcal{R}$-algebra. Hence $\tau^\sharp(\mathcal{X})$ is injective and its cokernel is π-torsion. Since this cokernel is also a finitely generated $\mathcal{A}$-module by part (a), the result follows.

(c) We have the convergent Leray spectral sequence [31, Theorem 5.8.6]:

$$E_2^{ij} = H^i(\mathcal{X}, R^j\tau_*\mathcal{O}) \Rightarrow H^{i+j}(\mathcal{Y}, \mathcal{O})$$

whose terms E_2^{ij} lie in $\mathcal{BT}$ whenever $j > 0$ by part (a). Hence the image of this spectral sequence in the quotient category $\mathcal{Q} = \mathcal{R}-\mathrm{mod}/\mathcal{BT}$ collapses on page 2 and the edge maps induce canonical isomorphisms $qH^n(\mathcal{X}, \tau_*\mathcal{O}) \cong qH^n(\mathcal{Y}, \mathcal{O})$ for all $n \geqslant 0$. Now a long exact sequence together with part (b) implies that $qH^n(\mathcal{X}, \tau_*\mathcal{O}) \cong qH^n(\mathcal{X}, \mathcal{O})$ for all $n \geqslant 0$, and the result follows. $\square$

Corollary. *If the quasi-compact admissible formal $\mathcal{R}$-scheme $\mathcal{X}$ is rig-affinoid, then $H^i(\mathcal{X}, \mathcal{O}) \in \mathcal{BT}$ for all $i > 0$.*

Proof. By part (d) of the proof of Raynaud's Theorem [12, Theorem 4.1], we can find an affine admissible formal $\mathcal{R}$-scheme $\mathcal{X}'$ together with a diagram $\mathcal{X} \leftarrow \mathcal{Y} \rightarrow \mathcal{X}'$, where both arrows are admissible blow-ups. Now Lemma 3.4(c) implies that

$$qH^i(\mathcal{X}, \mathcal{O}) = qH^i(\mathcal{Y}, \mathcal{O}) = qH^i(\mathcal{X}', \mathcal{O}) \quad \text{for all} \quad i \geqslant 0.$$

But $H^i(\mathcal{X}', \mathcal{O}) = 0$ for $i > 0$ by [17, Theorem II.9.7] because $\mathcal{X}'$ is affine. $\square$

Proposition. *Let $\mathcal{U} = \{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$ be an open covering of a quasi-compact admissible formal $\mathcal{R}$-scheme $\mathcal{X}$. Suppose that $\mathcal{X}$ and each $\mathcal{X}_i$ are rig-affinoid. Then $\check{H}^n(\mathcal{U}, \mathcal{O}) \in \mathcal{BT}$ for all $n > 0$.*

Proof. Consider the spectral sequence (1) for the covering $\mathcal{U}$ of $\mathcal{X}$ and the sheaf $\mathcal{O}$. Since $\mathcal{X}_{\rm rig}$ is affinoid, it is separated; because each $\mathcal{X}_{i,\rm rig}$ is assumed to be affinoid, it now follows from [12, Proposition 4.7] that every finite intersection $\mathcal{Z}$ of members of $\mathcal{U}$ is rig-affinoid. Hence $\mathcal{H}^j(\mathcal{O})(\mathcal{Z}) \in \mathcal{BT}$ for all $j > 0$ by the Corollary, which implies that $\check{H}^i(\mathcal{U}, \mathcal{H}^j(\mathcal{O})) \in \mathcal{BT}$ for all $i \geqslant 0$ and all $j > 0$. Hence the image of this spectral sequence in $\mathcal{Q}$ collapses on page 2 and the edge maps induce canonical isomorphisms $q\check{H}^n(\mathcal{U}, \mathcal{O}) \cong qH^n(\mathcal{X}, \mathcal{O})$ for all $n \geqslant 0$. However $H^n(\mathcal{X}, \mathcal{O}) \in \mathcal{BT}$ for all $n > 0$ because $\mathcal{X}$ is rig-affinoid, again by the Corollary. $\square$

3.5. **An analogue of Tate's Acyclicity Theorem.** Whenever $\mathcal{X}$ is a formal model for X, we will identify the $\mathcal{R}$-algebra $\mathcal{O}(\mathcal{X})$ with its natural image inside the K-algebra $\mathcal{O}(X)$.

Lemma. *Let X be a K-affinoid variety and let $\mathcal{X}, \mathcal{X}'$ be two formal models for X. Then there exist integers $a, b \geqslant 0$ such that $\pi^a \mathcal{O}(\mathcal{X}') \subseteq \mathcal{O}(\mathcal{X}) \subseteq \pi^{-b}\mathcal{O}(\mathcal{X}')$.*

Proof. By part (d) of the proof of Raynaud's Theorem [12, Theorem 4.1], we can find a diagram $\mathcal{X} \xleftarrow{\tau} \mathcal{Y} \rightarrow \mathcal{X}'$ where both arrows are admissible formal blow-ups. By Lemma 3.4(b), there exists some $a \geqslant 0$ such that $\pi^a \tau_* \mathcal{O}_{\mathcal{Y}} \subseteq \tau^\sharp(\mathcal{O}_{\mathcal{X}}) \subseteq \tau_*\mathcal{O}_{\mathcal{Y}}$. Applying $\Gamma(\mathcal{X}, -)$ produces inclusions $\pi^a\mathcal{O}(\mathcal{Y}) \subseteq \mathcal{O}(\mathcal{X}) \subseteq \mathcal{O}(\mathcal{Y})$ inside $\mathcal{O}(X)$. By symmetry, we also obtain $\pi^b\mathcal{O}(\mathcal{Y}) \subseteq \mathcal{O}(\mathcal{X}') \subseteq \mathcal{O}(\mathcal{Y})$ for some $b \geqslant 0$. Therefore

$$\pi^a\mathcal{O}(\mathcal{X}') \subseteq \pi^a\mathcal{O}(\mathcal{Y}) \subseteq \mathcal{O}(\mathcal{X}) \subseteq \mathcal{O}(\mathcal{Y}) \subseteq \pi^{-b}\mathcal{O}(\mathcal{X}')$$

as required. $\square$

We can now state and prove the main result of Section 3.

Theorem. *Let X be a K-affinoid variety, and let $\mathcal{L}$ be a smooth $(\mathcal{R}, \mathcal{A})$-Lie algebra for some affine formal model $\mathcal{A}$ in $\mathcal{O}(X)$. Then every $\mathcal{L}$-admissible covering of X is $\widehat{\mathscr{U}(\mathcal{L})}_K$-acyclic.*

Proof. Let $\mathcal{U} := \{X_1, \ldots, X_n\}$ be the given $\mathcal{L}$-admissible covering, and choose some $\mathcal{L}$-stable affine formal model $\mathcal{A}_i$ in $\mathcal{O}(X_i)$ for each $i = 1, \ldots, n$. For every non-empty subset S of $\{1, \ldots, n\}$, write $X_S = \bigcap_{i \in S} X_i$, and let $\mathcal{A}_S$ be the closed $\mathcal{R}$-subalgebra of $\mathcal{O}(X_S)$ generated by the images of $\mathcal{A}_i \subset \mathcal{O}(X_i)$ in $\mathcal{O}(X_S)$ under the natural restriction maps. Then $\mathcal{A}_S$ is an $\mathcal{L}$-stable affine formal model in $\mathcal{O}(X_S)$ for every S by Lemma 3.2, and moreover we have a natural subcomplex

$$\mathcal{C}^\bullet := [0 \rightarrow \mathcal{A} \rightarrow \prod_{i=1}^n \mathcal{A}_i \rightarrow \prod_{|S|=2} \mathcal{A}_S \rightarrow \cdots \rightarrow \mathcal{A}_{\{1,\ldots,n\}} \rightarrow 0]$$

of the augmented Čech complex

$$C^\bullet_{\mathrm{aug}}(\mathcal{U},\mathcal{O}) = [0 \to \mathcal{O}(X) \to \prod_{i=1}^n \mathcal{O}(X_i) \to \prod_{|S|=2} \mathcal{O}(X_S) \to \cdots \to \mathcal{O}(X_{\{1,\ldots,n\}}) \to 0].$$

Now by [12, Lemma 4.4], we can find an admissible formal blow-up $\mathcal{X}' \to \mathcal{X} := \mathrm{Spf}(\mathcal{A})$ together with an open covering $\mathcal{U}' = \{\mathcal{X}'_1,\ldots,\mathcal{X}'_n\}$ of $\mathcal{X}'$ such that $X_i = \mathcal{X}'_{i,\mathrm{rig}}$ for all $i = 1,\ldots,n$. Writing $\mathcal{X}'_S := \bigcap_{i\in S}\mathcal{X}'_i$ and $\mathcal{B}_S := \mathcal{O}(\mathcal{X}'_S)$ for every non-empty subset S of $\{1,\ldots,n\}$, we can consider the augmented Čech complex

$$\mathcal{D}^\bullet := C^\bullet_{\mathrm{aug}}(\mathcal{U}',\mathcal{O}) := [0 \to \mathcal{A} \to \prod_{i=1}^n \mathcal{B}_i \to \prod_{|S|=2} \mathcal{B}_S \to \cdots \to \mathcal{B}_{\{1,\ldots,n\}} \to 0].$$

Applying the Lemma to each term of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ produces embeddings of complexes

$$\mathcal{C}^\bullet \hookrightarrow C^\bullet_{\mathrm{aug}}(\mathcal{U},\mathcal{O}) \hookleftarrow \mathcal{D}^\bullet$$

together with integers $a,b \geqslant 0$ such that $\pi^a\mathcal{D}^\bullet \subseteq \mathcal{C}^\bullet \subseteq \pi^{-b}\mathcal{D}^\bullet$. Since $\mathcal{X}$ and each $\mathcal{X}'_i$ are rig-affinoid, we see that $\mathcal{D}^\bullet$ has *bounded* π-torsion cohomology by Lemma 3.4(b) and Proposition 3.4, and the same is true for the isomorphic sub-complex $\pi^a\mathcal{D}^\bullet$. Since $\mathcal{C}^\bullet/\pi^a\mathcal{D}^\bullet$ is killed by π^{a+b} by construction, a long exact sequence shows that $\mathcal{C}^\bullet$ also has bounded π-torsion cohomology.

Because $\mathcal{L}$ is smooth, it follows from [23, Theorem 3.1] that $U(\mathcal{L})$ is a projective, and hence flat, $\mathcal{A}$-module. Hence $U(\mathcal{L})\otimes_{\mathcal{A}}\mathcal{C}^\bullet$ also has bounded π-torsion cohomology. Therefore $\widehat{U(\mathcal{L})\otimes_{\mathcal{A}}\mathcal{C}^\bullet}\otimes_{\mathcal{R}} K$ is exact by the elementary Lemma 3.6 below. But it follows from Proposition 2.3 that $\widehat{U(\mathcal{L})\otimes_{\mathcal{A}}\mathcal{C}^\bullet}\otimes_{\mathcal{R}} K \cong C^\bullet_{\mathrm{aug}}(\mathcal{U},\widehat{\mathscr{U}(\mathcal{L})_K})$. □

3.6. **Lemma.** Let $C^\bullet$ be a complex of flat $\mathcal{R}$-modules with bounded torsion cohomology. Then $H^q(\widehat{C^\bullet}) \cong H^q(C^\bullet)$ for all q, and $\widehat{C^\bullet}\otimes_{\mathcal{R}} K$ is exact.

Proof. Since $C^\bullet$ has no π-torsion by assumption, for each $n,m \geqslant 0$ we have a commutative diagram of complexes of $\mathcal{R}$-modules with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet \arrow[r, "\pi^{n+m}"] \arrow[d, "\pi^m"] & C^\bullet \arrow[r] \arrow[d, equal] & C^\bullet/\pi^{n+m}C^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^\bullet \arrow[r, "\pi^n"'] & C^\bullet \arrow[r] & C^\bullet/\pi^nC^\bullet \arrow[r] & 0. \end{tikzcd}$$

Now fix q, choose $N \geqslant 0$ such that $H^q(C^\bullet)$ and $H^{q+1}(C^\bullet)$ are killed by π^N and let $n,m \geqslant N$. Applying the long exact sequence of cohomology produces another commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & H^q(C^\bullet) \arrow[r] \arrow[d, equal] & H^q(C^\bullet/\pi^{n+m}C^\bullet) \arrow[r] \arrow[d] & H^{q+1}(C^\bullet) \arrow[r] \arrow[d, "\pi^m"] & 0 \\ 0 \arrow[r] & H^q(C^\bullet) \arrow[r] & H^q(C^\bullet/\pi^nC^\bullet) \arrow[r] & H^{q+1}(C^\bullet) \arrow[r] & 0. \end{tikzcd}$$

Consider this diagram as a short exact sequence of towers of $\mathcal{R}$-modules. Since the vertical arrow on the right is zero for $m \geqslant N$ by assumption, and the vertical arrow on the left is an isomorphism, the long exact sequence associated to the inverse limit functor $\varprojlim$ shows that

$$H^q(C^\bullet) \cong \varprojlim H^q(C^\bullet/\pi^n C^\bullet) \quad \text{and} \quad \varprojlim{}^1 H^q(C^\bullet/\pi^n C^\bullet) = 0 \quad \text{for all} \quad q.$$

Because the maps in the tower of complexes $(C^\bullet/\pi^n C^\bullet)_n$ are surjective, this tower satisfies the Mittag-Leffler condition. The cohomological variant of [31, Theorem 3.5.8] implies that

$$\varprojlim H^q(C^\bullet/\pi^n C^\bullet) \cong H^q(\widehat{C^\bullet}) \quad \text{for all} \quad q.$$

Therefore $\widehat{C^\bullet}$ has π-torsion cohomology. $\square$

4. Exactness of localisation

4.1. **One-variable Tate extensions.** Let A be a (not necessarily commutative) Banach K-algebra. Then the *free Tate algebra in one variable t over A* is

$$A\langle t\rangle := \left\{ \sum_{i=0}^{\infty} t^i a_i \in A[[t]] : a_i \to 0 \quad \text{as} \quad i \to \infty \right\}.$$

Similarly we can define $M\langle t\rangle$ for a Banach A-module M, and it is readily checked that $M\langle t\rangle$ is naturally a Banach $A\langle t\rangle$-module.

We will soon need to understand certain torsion submodules of $M\langle t\rangle$.

Lemma. *Let $f \in A$.*

(a) $M\langle t\rangle$ is $(ft-1)$-torsion-free.
(b) If $(t-f)\cdot\left(\sum_{j=0}^{\infty} t^j m_j\right) = 0$ then $fm_0 = 0$ and $fm_j = m_{j-1}$ for all $j \geqslant 1$.
(c) If f is central in A and M is Noetherian, then $M\langle t\rangle$ is $(t-f)$-torsion-free.

Proof. (a) If $(ft-1)\sum_{j=0}^{\infty} t^j m_j = 0$ then $m_0 = 0$ and $fm_j = m_{j+1}$ for all $j \geqslant 0$. Hence $m_j = 0$ for all $j \geqslant 0$ by induction.

(b) This is a direct calculation, similar to part (a).

(c) Since f acts by A-linear endomorphisms of M and M is Noetherian, the ascending chain of A-submodules $0 \subseteq \ker f \subseteq \ker f^2 \subseteq \cdots$ in M must terminate at $\ker f^r$, say. Suppose that $(t-f)\cdot\left(\sum_{j=0}^{\infty} t^j m_j\right) = 0$. Then $f^{j+1}m_j = fm_0 = 0$ for all $j \geqslant 0$ by part (b), so $m_j \in \ker f^{j+1} = \ker f^r$ for $j \geqslant r$. Hence $0 = f^r m_{i+r} = m_i$ for all $i \geqslant 0$ and $\sum_{i=0}^{\infty} t^i m_i = 0$. $\square$

4.2. **Lifting derivations from $\mathcal{A}$ to $\mathcal{A}\langle t\rangle$.** We begin with an elementary result.

Lemma. *Let $\mathcal{A}$ be a π-adically complete $\mathcal{R}$-algebra, let u be an $\mathcal{R}$-linear derivation of $\mathcal{A}$ and let $b \in \mathcal{A}\langle t\rangle$. Then u extends uniquely to an $\mathcal{R}$-linear derivation v of $\mathcal{A}\langle t\rangle$ such that $v(t) = b$.*

Proof. There is a unique $\mathcal{R}$-linear derivation $v_0 : \mathcal{A}[t] \to \mathcal{A}\langle t\rangle$ extending $u : \mathcal{A} \to \mathcal{A}$ such that $v_0(t) = b$. Since $\mathcal{A}$ is π-adically complete, so is $\mathcal{A}\langle t\rangle$, and $v_0 : \mathcal{A}[t] \to \mathcal{A}\langle t\rangle$ is π-adically continuous, being $\mathcal{R}$-linear. Since $\mathcal{A}[t]$ is dense in $\mathcal{A}\langle t\rangle$, v_0 extends uniquely to an $\mathcal{R}$-linear derivation v of $\mathcal{A}\langle t\rangle$. $\square$

Proposition. *Let $\mathcal{A}$ be an affine formal model in a K-affinoid algebra A and let $\mathcal{L}$ be an $(\mathcal{R},\mathcal{A})$-Lie algebra with anchor map $\rho : \mathcal{L} \to \operatorname{Der}_{\mathcal{R}}(\mathcal{A})$. Write $x \cdot a = \rho(x)(a)$ for $x \in \mathcal{L}$ and $a \in \mathcal{A}$. Let $f \in A$ be such that $\mathcal{L} \cdot f \subset \mathcal{A}$. Then there are two lifts*

$$\sigma_1, \sigma_2 : \mathcal{L} \to \operatorname{Der}_{\mathcal{R}}(\mathcal{A}\langle t\rangle)$$

of the action of $\mathcal{L}$ on $\mathcal{A}$ to $\mathcal{A}\langle t\rangle$, such that

$$\sigma_1(x)(t) = x \cdot f \quad \textit{and} \quad \sigma_2(x)(t) = -t^2(x \cdot f) \quad \textit{for all} \quad x \in \mathcal{L}.$$

Proof. The $\mathcal{R}$-algebra $\mathcal{A}$ is admissible and is therefore π-adically complete. By the Lemma, for any $x \in \mathcal{L}$ there is a unique $\mathcal{R}$-linear derivation $\sigma_1(x)$ of $\mathcal{A}\langle t\rangle$ such that $\sigma_1(x)(t) = x \cdot f$. The map $\sigma_1 : \mathcal{L} \to \mathrm{Der}_{\mathcal{R}}(\mathcal{A}\langle t\rangle)$ obtained in this way is $\mathcal{A}$-linear because ρ is $\mathcal{A}$-linear. Let $x, y \in \mathcal{L}$; then

$$\sigma_1([x,y])(t) = [x,y] \cdot f = x \cdot (y \cdot f) - y \cdot (x \cdot f) = (\sigma_1(x)\sigma_1(y) - \sigma_1(y)\sigma_1(x))\,(t)$$

so the derivation $\sigma_1([x,y]) - [\sigma_1(x), \sigma_1(y)]$ is identically zero on $\mathcal{A}[t]$. Since $\mathcal{A}\langle t\rangle$ is π-adically complete and $\mathcal{A}[t]$ is dense in $\mathcal{A}\langle t\rangle$, σ_1 is a Lie homomorphism.

Similarly we can construct an $\mathcal{A}$-linear map $\sigma_2 : \mathcal{L} \to \mathrm{Der}_{\mathcal{R}}(\mathcal{A}\langle t\rangle)$ extending ρ such that $\sigma_2(x)(t) = -t^2 x \cdot f$ for all $x \in \mathcal{L}$. Let $x, y \in \mathcal{L}$ and write $x \cdot b = \sigma_2(x)(b)$ for $b \in \mathcal{A}\langle t\rangle$. Then $x \cdot (y \cdot t) = x \cdot (-t^2(y \cdot f)) = 2t^3(x \cdot f)(y \cdot f) - t^2 x \cdot (y \cdot f)$. Therefore $x \cdot (y \cdot t) - y \cdot (x \cdot t) = -t^2[x,y] \cdot f = [x,y] \cdot t$ because $f \in \mathcal{A}$ and ρ is a Lie homomorphism. Hence σ_2 is also a Lie homomorphism. □

4.3. **$\mathcal{L}$-stable affine formal models for Weierstrass and Laurent domains.** Let A be a K-affinoid algebra and fix $f \in A$. Let $\mathcal{A}$ be an affine formal model for A, and choose $a \in \mathbb{N}$ such that $\pi^a f \in \mathcal{A}$. Define

$$u_1 = \pi^a t - \pi^a f \quad \text{and} \quad u_2 := \pi^a f t - \pi^a \in \mathcal{A}\langle t\rangle.$$

Let $X := \mathrm{Sp}(A)$ and let $C_i = A\langle t\rangle / u_i A\langle t\rangle$ be the K-affinoid algebras corresponding to the Weierstrass and Laurent subdomains

$$X_1 := X(f) = \mathrm{Sp}(C_1) \quad \text{and} \quad X_2 := X(1/f) = \mathrm{Sp}(C_2)$$

of X, respectively.

Let $\mathcal{L}$ be an $(\mathcal{R}, \mathcal{A})$-Lie algebra such that $\mathcal{L} \cdot f \subset \mathcal{A}$. Then by Proposition 4.2, the action of $\mathcal{L}$ on $\mathcal{A}$ lifts to $\mathcal{A}\langle t\rangle$ in two different ways σ_1 and σ_2, and $\mathcal{L}_i := \mathcal{A}\langle t\rangle \otimes_{\mathcal{A}} \mathcal{L}$ becomes an $(\mathcal{R}, \mathcal{A}\langle t\rangle)$-Lie algebra by Lemma 2.2, with anchor map $1 \otimes \sigma_i$.

Lemma. *Let $\mathcal{L}$ be an $(\mathcal{R}, \mathcal{A})$-Lie algebra and let $f \in A$ be a non-zero element such that $\mathcal{L} \cdot f \subset \mathcal{A}$. Then the affinoid subdomains X_i of X are $\mathcal{L}$-admissible.*

Proof. Let $\mathcal{C}_i := \mathcal{A}\langle t\rangle / u_i \mathcal{A}\langle t\rangle$. A direct calculation shows that

$$\sigma_1(x)(u_1) = 0 \quad \text{and} \quad \sigma_2(x)(u_2) = -(x \cdot f) t u_2 \quad \text{for all} \quad x \in \mathcal{L}.$$

It follows that $u_i \mathcal{A}\langle t\rangle$ is a $\sigma_i(\mathcal{L})$-stable ideal of $\mathcal{A}\langle t\rangle$, and therefore the image $\overline{\mathcal{C}_i}$ of $\mathcal{C}_i$ in C_i is an $\mathcal{L}$-stable affine formal model in C_i. Hence X_i is $\mathcal{L}$-admissible. □

Proposition. *Let $\mathcal{L}$ be a smooth $(\mathcal{R}, \mathcal{A})$-Lie algebra and let $f \in A$ be a non-zero element such that $\mathcal{L} \cdot f \subset \mathcal{A}$.*

(a) $U(\mathcal{L}_i)/\pi U(\mathcal{L}_i)$ is isomorphic to $(U(\mathcal{L})/\pi U(\mathcal{L}))\,[t]$ as a $\mathcal{A}\langle t\rangle$-module.

(b) $\widehat{U(\mathcal{L}_i)_K}$ is a flat $\widehat{U(\mathcal{L})_K}$-module on both sides.

(c) There is a short exact sequence

$$0 \to \widehat{U(\mathcal{L}_i)_K} \xrightarrow{u_i \cdot} \widehat{U(\mathcal{L}_i)_K} \to \widehat{\mathscr{U}(\mathcal{L})_K}(X_i) \to 0$$

of right $\widehat{U(\mathcal{L}_i)_K}$-modules, and a short exact sequence

$$0 \to \widehat{U(\mathcal{L}_i)_K} \xrightarrow{\cdot u_i} \widehat{U(\mathcal{L}_i)_K} \to \widehat{\mathscr{U}(\mathcal{L})_K}(X_i) \to 0$$

of left $\widehat{U(\mathcal{L}_i)_K}$-modules.

Proof. (a) By Proposition 2.3, there is a $\mathcal{A}\langle t\rangle$-module isomorphism $U(\mathcal{L}_i) = U(\mathcal{A}\langle t\rangle \otimes_{\mathcal{A}} \mathcal{L}) \cong \mathcal{A}\langle t\rangle \otimes_{\mathcal{A}} U(\mathcal{L})$. It induces $\mathcal{A}\langle t\rangle$-module isomorphisms

$$\frac{U(\mathcal{L}_i)}{\pi U(\mathcal{L}_i)} \cong \frac{\mathcal{A}\langle t\rangle}{\pi \mathcal{A}\langle t\rangle} \otimes_{\frac{\mathcal{A}}{\pi\mathcal{A}}} \frac{U(\mathcal{L})}{\pi U(\mathcal{L})} \cong k[t] \otimes_k \frac{U(\mathcal{L})}{\pi U(\mathcal{L})}.$$

(b) The associated graded ring $\operatorname{gr} \widehat{U(\mathcal{L}_i)_K}$ with respect to the π-adic filtration is $k[t] \otimes_k \operatorname{gr} \widehat{U(\mathcal{L})_K}$, which is flat over $\operatorname{gr} \widehat{U(\mathcal{L})_K}$. Now apply [26, Proposition 1.2].

(c) By symmetry, it is sufficient to prove the first statement. By definition, the sequence $0 \to \mathcal{A}\langle t\rangle \xrightarrow{u_i \cdot} \mathcal{A}\langle t\rangle \to \mathcal{C}_i \to 0$ is exact. Tensor it on the right with the flat left $\mathcal{A}\langle t\rangle$-module $U(\mathcal{L}_i)$ and apply Proposition 2.3 to get a short exact sequence of right $U(\mathcal{L}_i)$-modules $0 \to U(\mathcal{L}_i) \xrightarrow{u_i \cdot} U(\mathcal{L}_i) \to U(\mathcal{C}_i \otimes_{\mathcal{A}} \mathcal{L}) \to 0$. Since $U(\mathcal{L}_i)$ is Noetherian, π-adic completion is exact on finitely generated $U(\mathcal{L}_i)$-modules by [10, §3.2.3(ii)]. Hence $0 \to \widehat{U(\mathcal{L}_i)_K} \xrightarrow{u_i \cdot} \widehat{U(\mathcal{L}_i)_K} \to \widehat{U(\mathcal{C}_i \otimes_{\mathcal{A}} \mathcal{L})_K} \to 0$ is exact. Now by Lemma 2.5, there is a natural isomorphism

$$\widehat{U(\mathcal{C}_i \otimes_{\mathcal{A}} \mathcal{L})_K} \xrightarrow{\cong} \widehat{U(\overline{\mathcal{C}_i} \otimes_{\mathcal{A}} \mathcal{L})_K}$$

and $\widehat{U(\overline{\mathcal{C}_i} \otimes_{\mathcal{A}} \mathcal{L})_K} \cong \widehat{\mathscr{U}(\mathcal{L})_K}(X_i)$ as $\overline{\mathcal{C}_i}$ is an $\mathcal{L}$-stable affine formal model in C_i. □

Remark. *It follows from part (c) of the Proposition that the image of $\widehat{\mathscr{U}(\mathcal{L})_K}(X)$ in $\widehat{\mathscr{U}(\mathcal{L})_K}(X_1)$ is dense since it also contains the image of t.*

4.4. **Towards flatness.** We keep the notation from the previous Subsection.

Lemma. *Let M be a finitely generated $\widehat{U(\mathcal{L})_K}$-module. Then there is a natural isomorphism of Banach $A\langle t\rangle$-modules $\eta_M \colon M\langle t\rangle \xrightarrow{\cong} \widehat{U(\mathcal{L}_i)_K} \otimes_{\widehat{U(\mathcal{L})_K}} M$.*

Similarly, if N is a finitely generated right $\widehat{U(\mathcal{L})_K}$-module there is a natural isomorphism of Banach $A\langle t\rangle$-modules $\eta_N \colon N\langle t\rangle \xrightarrow{\cong} N \otimes_{\widehat{U(\mathcal{L})_K}} \widehat{U(\mathcal{L}_i)_K}$.

Proof. Choose a finitely generated $\widehat{U(\mathcal{L})}$-submodule $\mathcal{M}$ in M which generates M as a K-vector space. Then

$$\widehat{U(\mathcal{L}_i)_K} \otimes_{\widehat{U(\mathcal{L})_K}} M \cong \left(\widehat{U(\mathcal{L}_i)} \otimes_{\widehat{U(\mathcal{L})}} \mathcal{M}\right) \otimes_{\mathcal{R}} K.$$

The finitely generated $\widehat{U(\mathcal{L}_i)}$-module $\widehat{U(\mathcal{L}_i)} \otimes_{\widehat{U(\mathcal{L})}} \mathcal{M}$ is π-adically complete by [10, §3.2.3(v)] because $\widehat{U(\mathcal{L}_i)}$ is Noetherian. Therefore, for any sequence of elements $m_j \in M$ tending to zero, the series $\sum_{j=0}^{\infty} t^j \otimes m_j$ converges to a unique element $\eta_M\left(\sum_{j=0}^{\infty} t^j m_j\right)$ in $\widehat{U(\mathcal{L}_i)_K} \otimes_{\widehat{U(\mathcal{L})_K}} M$. Because t commutes with A, it is straightforward to see that η_M is $A\langle t\rangle$-linear. It follows from Proposition 4.3(a) that $\eta_{\widehat{U(\mathcal{L})_K}}$ is an isomorphism. We may now view η as a natural transformation between two right exact functors and use the Five Lemma to conclude that $\eta_{\mathcal{M}}$ is always an isomorphism. The proof of the right module version is similar. □

Our proof of the exactness of the localisation functor rests on the following elementary

Proposition. *Let $S \to T$ be a ring homomorphism. Let $u \in T$ be a left regular element and suppose that*

(a) T is a flat right S-module,
(b) $T \otimes_S M$ is u-torsion-free for all finitely generated left S-modules M.
Then $W := T/uT$ is also a flat right S-module.

Proof. Let M be a finitely generated S-module and pick a projective resolution $P_\bullet \twoheadrightarrow M$ of M. Since T_S is flat, $T \otimes_S P_\bullet \twoheadrightarrow T \otimes_S M$ is a projective resolution so

$$\mathrm{Tor}_1^S(W, M) = H_1(W \otimes_S P_\bullet) = H_1(W \otimes_T (T \otimes_S P_\bullet)) \cong \mathrm{Tor}_1^T(W, T \otimes_S M).$$

The short exact sequence $0 \to T \xrightarrow{u\cdot} T \to W \to 0$ induces the long exact sequence

$$0 = \mathrm{Tor}_1^T(T, T \otimes_S M) \to \mathrm{Tor}_1^T(W, T \otimes_S M) \to T \otimes_S M \xrightarrow{u\cdot} T \otimes_S M,$$

so $\mathrm{Tor}_1^S(W, M) = \mathrm{Tor}_1^T(W, T \otimes_S M)$ vanishes by assumption (b). Hence W is a flat right S-module by [31, Proposition 3.2.4]. $\square$

4.5. **Flatness for Weierstrass and Laurent embeddings.** Here is the first main result of Section 4.

Theorem. *Let X be a K-affinoid variety and let $f \in \mathcal{O}(X)$ be non-zero. Let $\mathcal{A}$ be an affine formal model in $\mathcal{O}(X)$ and let $\mathcal{L}$ be a smooth $(\mathcal{R}, \mathcal{A})$-Lie algebra such that $\mathcal{L} \cdot f \subseteq \mathcal{A}$. Let $X_1 = X(f)$ and $X_2 = X(1/f)$. Then $\widehat{\mathscr{U}(\mathcal{L})_K}(X_i)$ is a flat $\widehat{\mathscr{U}(\mathcal{L})_K}(X)$-module on both sides for $i = 1$ and $i = 2$.*

Proof. We know that $T_i := \widehat{U(\mathcal{L}_i)_K}$ is a flat right $S := \widehat{U(\mathcal{L})_K}$-module, and that $\widehat{\mathscr{U}(\mathcal{L})_K}(X_i) \cong T_i/u_iT_i$ as a right T_i-module by Proposition 4.3. Let M be a finitely generated S-module. By Lemma 4.4 and Proposition 4.4, to prove that $\widehat{\mathscr{U}(\mathcal{L})_K}(X_i)$ is a flat right S-module it will be enough to show that the $A\langle t\rangle$-module $M\langle t\rangle$ is u_i-torsion-free. The case $i = 2$ follows immediately from Lemma 4.1(a). Since $u_1 = \pi^a(t - f)$, we just have to show that $M\langle t\rangle$ is $(t - f)$-torsion-free.

Suppose now that the element $\sum_{j=0}^\infty t^j m_j \in M\langle t\rangle$ is killed by $t - f$. Then setting $m_{-1} := 0$, we have the equations $fm_j = m_{j-1}$ for all $j \geqslant 0$ from Lemma 4.1(b), and $\lim\limits_{j\to\infty} m_j = 0$. We consider the S-submodule N of M generated by the m_j. Since M is Noetherian, N must be generated by $m_0, \ldots, m_d$ for some $d \geqslant 0$, say.

Let $\mathcal{M}$ be a finitely generated $\mathcal{S} := \widehat{U(\mathcal{L})}$-submodule of M which generates M as a K-vector space, and let $\mathcal{N} := \sum_{i=0}^d \mathcal{S}m_i$. Since $\mathcal{S}$ is Noetherian, $\mathcal{M} \cap N$ is a finitely generated $\mathcal{S}$-submodule of N which generates N as a K-vector space, so the $\mathcal{S}$-modules $\mathcal{M} \cap N$ and $\mathcal{N}$ contain π-power multiples of each other. So for all $n \geqslant 0$ we can find $j_n \geqslant 0$ such that $m_j \in \pi^n\mathcal{N}$ for all $j \geqslant j_n$, because $\lim\limits_{j\to\infty} m_j = 0$.

Since $U(\mathcal{L})$ is generated by $\mathcal{A} + \mathcal{L}$ as an $\mathcal{R}$-algebra and $[f, \mathcal{A} + \mathcal{L}] \subseteq \mathcal{L} \cdot f \subset \mathcal{A}$ we see that $[f, U(\mathcal{L})] \subseteq U(\mathcal{L})$ and consequently $[f, \mathcal{S}] \subseteq \mathcal{S}$. Because

$$f \sum_{j=0}^d s_j m_j = \sum_{j=0}^d [f, s_j]m_j + s_j m_{j-1} \in \mathcal{N} \quad \text{for all} \quad s_0, \ldots, s_d \in \mathcal{S}$$

we see that $f^i\mathcal{N} \subseteq \mathcal{N}$ for all $i \geqslant 0$. Therefore for any $j, n \geqslant 0$ we have

$$m_j = f^{j_n} m_{j+j_n} \in f^{j_n}\pi^n\mathcal{N} \subseteq \pi^n\mathcal{N}.$$

Hence $m_j \in \bigcap_{n=0}^\infty \pi^n\mathcal{N} = 0$ for all $j \geqslant 0$ and $\sum_{j=0}^\infty t^j m_j = 0$, so T_i is a flat right S-module as claimed. The same argument for finitely generated right S-modules M also shows that T_i is a flat left S-module. $\square$

4.6. **$\mathcal{L}$-accessible rational subdomains.**
Until the end of Section 4, we will fix the following notation.

- X is a K-affinoid variety,
- $\mathcal{A}$ is an affine formal model in $\mathcal{O}(X)$,
- $\mathcal{L}$ is a smooth $(\mathcal{R}, \mathcal{A})$-Lie algebra,
- $\mathcal{S} := \widehat{\mathscr{U}(\mathcal{L})_K}$.

We start with the following Lemma, which tells us that our sheaves $\widehat{\mathscr{U}(\mathcal{L})_K}$ behave well with respect to restriction.

Lemma. *Let Y be an $\mathcal{L}$-admissible affinoid subdomain of X, let $\mathcal{B}$ be an $\mathcal{L}$-stable affine formal model in $\mathcal{O}(Y)$ and let $\mathcal{L}' = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$. Then an affinoid subdomain Z of Y is $\mathcal{L}$-admissible if and only if it is $\mathcal{L}'$-admissible, and there is a natural isomorphism $\widehat{\mathscr{U}(\mathcal{L}')_K}(Z) \xrightarrow{\cong} \widehat{\mathscr{U}(\mathcal{L})_K}(Z)$ for every $Z \in Y_w(\mathcal{L}')$.*

Proof. If Z is $\mathcal{L}$-admissible with $\mathcal{L}$-stable affine formal model $\mathcal{C}$, then the closed $\mathcal{R}$-subalgebra of $\mathcal{O}(Z)$ generated by $\mathcal{C}$ and the image of $\mathcal{B}$ in $\mathcal{O}(Z)$ is $\mathcal{L}'$-stable, so Z is $\mathcal{L}'$-admissible. The converse is clear.

Now choose an $\mathcal{L}'$-stable affine formal model $\mathcal{C}$ in $\mathcal{O}(Z)$. Then $\mathcal{C}$ is also $\mathcal{L}$-stable, and $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{L}' = \mathcal{C} \otimes_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}) \cong \mathcal{C} \otimes_{\mathcal{A}} \mathcal{L}$ as $(\mathcal{R}, \mathcal{C})$-Lie algebras. Hence $\widehat{\mathscr{U}(\mathcal{L}')_K}(Z) \cong \widehat{U(\mathcal{C} \otimes_{\mathcal{B}} \mathcal{L}')_K} \cong \widehat{U(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{L})_K} \cong \widehat{\mathscr{U}(\mathcal{L})_K}(Z)$. $\square$

We would like to prove that every $\mathcal{L}$-admissible étale morphism of affinoids $Y \to X$ has the property that $\mathcal{S}(Y)$ is a flat right and left $\mathcal{S}(X)$-module. Unfortunately we cannot do this at the moment and we introduce a new notion, that of $\mathcal{L}$-*accessibility*, as a consequence of this.

Definition. (a) Let $Y \subseteq X$ be a rational subdomain. If $Y = X$, we say that it is $\mathcal{L}$-*accessible in* 0 *steps*. Inductively, if $n \geqslant 1$ then we say that it is $\mathcal{L}$-*accessible in* n *steps* if there exists a chain $Y \subseteq Z \subseteq X$, such that
- $Z \subseteq X$ is $\mathcal{L}$-accessible in $(n-1)$ steps,
- $Y = Z(f)$ or $Z(1/f)$ for some non-zero $f \in \mathcal{O}(Z)$,
- there is an $\mathcal{L}$-stable affine formal model $\mathcal{C} \subseteq \mathcal{O}(Z)$ such that $\mathcal{L} \cdot f \subseteq \mathcal{C}$.

(b) A rational subdomain $Y \subseteq X$ is said to be $\mathcal{L}$-*accessible* if it is $\mathcal{L}$-accessible in n steps for some $n \in \mathbb{N}$.

Proposition. *Let $Y \subseteq X$ be an $\mathcal{L}$-accessible rational subdomain. Then it is $\mathcal{L}$-admissible and $\mathcal{S}(Y)$ is a flat $\mathcal{S}(X)$-module on both sides.*

Proof. Assume that $Y \subseteq X$ is $\mathcal{L}$-accessible in n steps, and proceed by induction on n. The statement is vacuous when $n = 0$, so assume $n \geqslant 1$. Choose a chain $Y \subseteq Z \subseteq X$ where $Z \subseteq X$ is $\mathcal{L}$-accessible in $n-1$ steps, assume that $Y = Z(f)$ or $Z(1/f)$ for some non-zero $f \in \mathcal{O}(Z)$ and let $\mathcal{C} \subseteq \mathcal{O}(Z)$ be an $\mathcal{L}$-stable affine formal model such that $\mathcal{L} \cdot f \subseteq \mathcal{C}$. Then $\mathcal{L}' := \mathcal{C} \otimes_{\mathcal{A}} \mathcal{L}$ is an $(\mathcal{R}, \mathcal{C})$-Lie algebra and $\mathcal{L}' \cdot f \subseteq \mathcal{C}$, so $Y \subseteq Z$ is $\mathcal{L}'$-admissible by Lemma 4.3. Hence $Y \subseteq X$ is also $\mathcal{L}$-admissible. Now $\mathcal{L}'$ is also smooth, so $\widehat{\mathscr{U}(\mathcal{L}')_K}(Y)$ is a flat $\widehat{\mathscr{U}(\mathcal{L}')_K}(Z)$-module on both sides by Theorem 4.5, whereas $\widehat{\mathscr{U}(\mathcal{L}')_K}(Y) \cong \mathcal{S}(Y)$ and $\widehat{\mathscr{U}(\mathcal{L}')_K}(Z) \cong \mathcal{S}(Z)$ by the Lemma. Since $\mathcal{S}(Z)$ is a flat $\mathcal{S}(X)$-module on both sides by induction, $\mathcal{S}(Y)$ is also a flat $\mathcal{S}(X)$-module on both sides. $\square$

4.7. **Proposition.** Let Y be a rational subdomain of X, $\mathcal{L}$-accessible in n steps.

(a) Let U be an $\mathcal{L}$-admissible affinoid subdomain of X, and let $\mathcal{B}$ be an $\mathcal{L}$-stable affine formal model in U. Then $U \cap Y$ is a rational subdomain of U which is $\mathcal{L}' := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$-accessible in n steps.

(b) Let $\mathcal{B}$ be an $\mathcal{L}$-stable affine formal model in $\mathcal{O}(Y)$, and let Z be a rational subdomain of Y which is $\mathcal{L}' := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$-accessible in m steps. Then Z is a rational subdomain of X which is $\mathcal{L}$-accessible in $(n+m)$ steps.

Proof. (a) Proceed by induction on n, and suppose that $n \geqslant 1$ as the case when $n = 0$ is trivial. We have a commutative pullback diagram

$$\begin{tikzcd}
Y \arrow[r] & Z \arrow[r] & X \\
U \cap Y \arrow[r] \arrow[u] & U \cap Z \arrow[r] \arrow[u] & U \arrow[u]
\end{tikzcd}$$

where $Z \subseteq X$ is $\mathcal{L}$-accessible in $(n-1)$ steps, $Y = Z(f)$ or $Z(1/f)$ for some $f \in \mathcal{O}(Z)$ and $\mathcal{L} \cdot f \subseteq \mathcal{C}$ for some $\mathcal{L}$-stable affine formal model $\mathcal{C}$ in $\mathcal{O}(Z)$. Let $g = 1 \otimes f$ be the image of f in $\mathcal{O}(U \cap Z)$. Then $U \cap Z \subseteq U$ is $\mathcal{L}'$-accessible in $(n-1)$-steps by induction, $U \cap Y$ is either $(U \cap Z)(g)$ or $(U \cap Z)(1/g)$, and $\mathcal{L}' \cdot g \subseteq \overline{\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}}$ which is an $\mathcal{L}$-stable affine formal model in $\mathcal{O}(U \cap Z)$ by the proof of Lemma 3.2. Therefore $U \cap Y \subseteq U$ is $\mathcal{L}$-accessible in n steps.

(b) Proceed by induction on m, and assume that $m \geqslant 1$ as the case when $m = 0$ is trivial. Choose $Z \subseteq W \subseteq Y$ with $W \subseteq Y$ being $\mathcal{L}'$-accessible in $(m-1)$ steps, and $Z = W(f)$ or $W(1/f)$ for some $f \in \mathcal{O}(W)$, and $\mathcal{L}' \cdot f \subseteq \mathcal{C}$ for some $\mathcal{L}$-stable affine formal model $\mathcal{C}$ in $\mathcal{O}(W)$. Then $W \subseteq X$ is $\mathcal{L}$-accessible in $n + m - 1$ steps by induction, and $\mathcal{L} \cdot f \subseteq \mathcal{C}$, so $Z \subseteq X$ is $\mathcal{L}$-accessible in $n + m$ steps. $\square$

Corollary. *Let $Y \subseteq X$ and $U \subseteq X$ be two $\mathcal{L}$-accessible rational subdomains. Then $U \cap Y \subseteq X$ is also an $\mathcal{L}$-accessible rational subdomain.*

Proof. Choose an $\mathcal{L}$-stable affine formal model $\mathcal{B}$ in $\mathcal{O}(U)$. By part (a) of the Proposition, $U \cap Y \subseteq U$ is a rational subdomain which is $\mathcal{L}' := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$-accessible. Since $U \subseteq X$ is also $\mathcal{L}$-accessible, part (b) of the Proposition (applied to $U \cap Y \subseteq U$) gives that $U \cap Y$ is an $\mathcal{L}$-accessible rational subdomain in X. $\square$

4.8. **$\mathcal{L}$-accessible affinoid subdomains.** Recall that by the Gerritzen-Grauert Theorem [15, Theorem 4.10.4], every affinoid subdomain Y of an affinoid K-variety X is actually the union of finitely many rational subdomains in X. In view of this fact, we make the following

Definition.

(a) An affinoid subdomain Y of X is said to be $\mathcal{L}$-*accessible* if it is $\mathcal{L}$-admissible and there exists a finite covering $Y = \bigcup_{j=1}^{r} X_j$ where each X_j is an $\mathcal{L}$-accessible rational subdomain of X.

(b) A finite affinoid covering $\{X_j\}$ of X is said to be $\mathcal{L}$-*accessible* if each X_j is an $\mathcal{L}$-accessible affinoid subdomain of X.

It follows from Proposition 4.6 that every $\mathcal{L}$-accessible rational subdomain is $\mathcal{L}$-admissible, and is therefore also an $\mathcal{L}$-accessible affinoid subdomain.

Lemma. *(a) The intersection of finitely many $\mathcal{L}$-accessible affinoid subdomains is again an $\mathcal{L}$-accessible affinoid subdomain.*

(b) If $Z \subseteq Y$ are $\mathcal{L}$-accessible affinoid subdomains of X and $\mathcal{B}$ is an $\mathcal{L}$-stable affine formal model in $\mathcal{O}(Y)$, then Z is an $\mathcal{L}' := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$-accessible affinoid subdomain of Y.

Proof. (a) This follows from Corollary 4.7 together with Lemma 3.2.

(b) Let $\{Z_1, \ldots, Z_n\}$ be a covering of Z by $\mathcal{L}$-accessible rational subdomains of X. By Proposition 4.7(a), each $Z_i = Y \cap Z_i$ is an $\mathcal{L}'$-accessible rational subdomain of Y. $\square$

4.9. **Theorem.**

(a) Let $Y \subseteq X$ be an $\mathcal{L}$-accessible affinoid subdomain.
Then $\mathcal{S}(Y)$ is a flat $\mathcal{S}(X)$-module on both sides.

(b) Let $\mathcal{X} = \{X_1, \ldots, X_m\}$ be an $\mathcal{L}$-accessible covering of X.
Then $\bigoplus_{i=1}^m \mathcal{S}(X_j)$ is a faithfully flat $\mathcal{S}(X)$-module on both sides.

Proof. (a) By definition, there is a finite covering $\mathcal{V} = \{X_1, \ldots, X_m\}$ of Y by $\mathcal{L}$-accessible rational subdomains X_j. Every finite intersection of these subdomains is $\mathcal{L}$-accessible by Lemma 4.8(a), so every ring appearing in $C^\bullet(\mathcal{V}, \mathcal{S})$ is flat as a $\mathcal{S}(X)$-module on both sides by Proposition 4.6.

The augmented Čech complex $C^\bullet_{\text{aug}}(\mathcal{V}, \mathcal{S})$ is acyclic by Theorem 3.5. A long exact sequence of Tor groups shows that the kernel of a surjection between two flat modules is again flat. By an induction starting with the last term, the kernel of every differential in this complex is a flat $\mathcal{S}(X)$-module on both sides. In particular, $\mathcal{S}(Y)$ is flat as $\mathcal{S}(X)$-module on both sides.

(b) By part (a), $\bigoplus_{i=1}^m \mathcal{S}(X_j)$ is a flat right $\mathcal{S}(X)$-module. By Lemma 4.8(a) and part (a), each term in the complex $C^\bullet_{\text{aug}}(\mathcal{X}, \mathcal{S})$ is a flat right $\mathcal{S}(X)$-module. Since it is acyclic by Theorem 3.5 we may view it as a flat resolution of the zero module. Let N be a left $\mathcal{S}(X)$-module. By [31, Lemma 3.2.8], $C^\bullet_{\text{aug}}(\mathcal{X}, \mathcal{S}) \otimes_{\mathcal{S}(X)} N$ computes $\operatorname{Tor}_\bullet^{\mathcal{S}(X)}(0, N)$ and is therefore acyclic. So N embeds into $\oplus_{j=1}^m \mathcal{S}(X_j) \otimes_{\mathcal{S}(X)} N$ and hence $\oplus_{j=1}^m \mathcal{S}(X_j)$ is a faithfully flat right $\mathcal{S}(X)$-module. The same proof shows that it is also a faithfully flat left $\mathcal{S}(X)$-module. $\square$

5. Kiehl's Theorem for coherent $\widehat{\mathscr{U}(\mathcal{L})_K}$-modules

In this Section we continue with the assumptions made in Subsection 4.6, namely:

- X is a K-affinoid variety,
- $\mathcal{A}$ is an affine formal model in $\mathcal{O}(X)$,
- $\mathcal{L}$ is a smooth $(\mathcal{R}, \mathcal{A})$-Lie algebra,
- $\mathcal{S} := \widehat{\mathscr{U}(\mathcal{L})_K}$.

It follows from Lemma 4.8(a) that the $\mathcal{L}$-accessible affinoid subdomains of X together with the $\mathcal{L}$-accessible coverings form a G-topology on X. We will denote this G-topology by $X_{ac}(\mathcal{L})$. Thus we have at our disposal four different G-topologies on X, represented on the level of objects as follows:

$$X_{ac}(\mathcal{L}) \quad \subseteq \quad X_w(\mathcal{L}) \quad \subset \quad X_w \quad \subseteq \quad X_{\text{rig}}.$$

5.1. **Localisation.** Suppose that Y is an $\mathcal{L}$-admissible affinoid subdomain of X. For every finitely generated $\mathcal{S}(X)$-module M, we can define a presheaf of $\mathcal{S}$-modules $\operatorname{Loc}(M)$ on $X_w(\mathcal{L})$ by setting

$$\operatorname{Loc}(M)(Y) := \mathcal{S}(Y) \otimes_{\mathcal{S}(X)} M.$$

Similarly, for every finitely generated right $\mathcal{S}(X)$-module M, we can define a presheaf of right $\mathcal{S}$-modules $\mathrm{Loc}(M)$ on $X_w(\mathcal{L})$ by setting

$$\mathrm{Loc}(M)(Y) := M \otimes_{\mathcal{S}(X)} \mathcal{S}(Y).$$

We will frequently use the fact that $\mathcal{S}(Z)$ is a flat $\mathcal{S}(Y)$-module on both sides whenever $Z \subseteq Y$ are $\mathcal{L}$-accessible affinoid subdomains of X — this follows from Theorem 4.9(a).

Proposition. Loc *is a full exact embedding of abelian categories from the category of finitely generated $\mathcal{S}(X)$-modules (respectively, right $\mathcal{S}(X)$-modules) to the category of sheaves of $\mathcal{S}$-modules (respectively, right $\mathcal{S}$-modules) on $X_{ac}(\mathcal{L})$ with vanishing higher Čech cohomology groups.*

Proof. Suppose first that $f\colon M \to N$ is a morphism of finitely generated $\mathcal{S}(X)$-modules. By the universal property of tensor product, for each Y in $X_{ac}(\mathcal{L})$ there is a unique morphism of $\mathcal{S}(Y)$-modules $\mathrm{Loc}(f)(Y) := \mathrm{id} \otimes f$ making the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ \mathcal{S}(Y) \otimes_{\mathcal{S}(X)} M & \xrightarrow[\mathrm{id}\otimes f]{} & \mathcal{S}(Y) \otimes_{\mathcal{S}(X)} N \end{array}$$

commute. It is now easy to see that Loc is full and faithful.

Next, suppose that $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of finitely generated $\mathcal{S}(X)$-modules. Since $\mathcal{S}(Y)$ is a flat right $\mathcal{S}(X)$-module for each $Y \in X_{ac}(\mathcal{L})$, each sequence

$$0 \to \mathrm{Loc}(M_1)(Y) \to \mathrm{Loc}(M_2)(Y) \to \mathrm{Loc}(M_3)(Y) \to 0$$

is exact. This suffices to see that Loc is exact.

Finally, we prove that if M is any finitely generated $\mathcal{S}(X)$-module then every $X_{ac}(\mathcal{L})$-covering $\mathcal{U} = \{U_1, \ldots, U_n\}$ of every $\mathcal{L}$-accessible affinoid subdomain Y of X is $\mathrm{Loc}(M)$-acyclic. This will imply that $\mathrm{Loc}(M)$ is a sheaf on $X_{ac}(\mathcal{L})$ with vanishing higher Čech cohomology groups.

Let $\mathcal{B}$ be an $\mathcal{L}$-stable affine formal model in Y and let $\mathcal{L}' = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$. By Lemma 4.8(b) we can view $\mathcal{U}$ as a covering of Y in $Y_{ac}(\mathcal{L}')$. Then $\mathcal{U}$ is $\mathcal{S}$-acyclic by Lemma 4.6 and Theorem 3.5. But every term in the Čech complex $C^\bullet_{\mathrm{aug}}(\mathcal{U}, \mathcal{S})$ is a flat right $\mathcal{S}(Y) = \widehat{U(\mathcal{L}')_K}$ module by Theorem 4.9(a). Therefore

$$C^\bullet_{\mathrm{aug}}(\mathcal{U}, \mathrm{Loc}(M)) \cong C^\bullet_{\mathrm{aug}}(\mathcal{U}, \mathcal{S}) \otimes_{\mathcal{S}(Y)} \mathrm{Loc}(M)(Y)$$

is also acyclic as claimed. The case of right modules is almost identical. $\square$

5.2. **$\mathcal{U}$-coherent modules.** Following [11, §9.4.3], we say that an $\mathcal{S}$-module $\mathcal{M}$ is *coherent* if there is an $X_{ac}(\mathcal{L})$-covering $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X such that, for each $1 \leqslant i \leqslant n$, $\mathcal{M}|_{U_i}$ may be presented by an exact sequence of the form

$$\mathcal{S}^{r_i}|_{U_i} \to \mathcal{S}^{s_i}|_{U_i} \to \mathcal{M}_{|U_i} \to 0.$$

Using Proposition 5.1, we note that in this situation, if we choose $\mathcal{L}$-stable affine formal models $\mathcal{B}_i$ in U_i and write $\mathcal{L}_i = \mathcal{B}_i \otimes_{\mathcal{A}} \mathcal{L}$, we may view the morphism $\mathcal{S}^{r_i}|_{U_i} \to \mathcal{S}^{s_i}|_{U_i}$ as $\mathrm{Loc}(f_i)$ for some $\mathcal{S}(U_i)$-linear map $f_i\colon \mathcal{S}(U_i)^{r_i} \to \mathcal{S}(U_i)^{s_i}$. Writing M_i for the cokernel of f_i and applying Proposition 5.1 again we see that there is an isomorphism $\mathcal{M}_{|U_i} \cong \operatorname{Loc}(M_i)$ as $\mathcal{S}|_{U_i}$-modules since both arise as the cokernel of $\operatorname{Loc}(f_i)$.

Since each ring $\mathcal{S}(U_i)$ is left Noetherian and so every finitely generated $\mathcal{S}(U_i)$-module is finitely presented, it follows from the discussion above that an $\mathcal{S}$-module $\mathcal{M}$ is coherent precisely if there is an $X_{ac}(\mathcal{L})$-covering $\{U_1, \ldots, U_n\}$ of X such that for each $1 \leqslant i \leqslant n$, $\mathcal{M}|_{U_i}$ is isomorphic to $\operatorname{Loc}(M_i)$ for some finitely generated $\mathcal{S}(U_i)$-module M_i.

Definition. Given an $X_{ac}(\mathcal{L})$-covering $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X, we say that an $\mathcal{S}$-module (respectively, right $\mathcal{S}$-module) $\mathcal{M}$ is *$\mathcal{U}$-coherent* if for each $1 \leqslant i \leqslant n$ there is a finitely generated $\mathcal{S}(U_i)$-module (respectively, right $\mathcal{S}(U_i)$-module) M_i such that $\mathcal{M}|_{U_i}$ is isomorphic to $\operatorname{Loc}(M_i)$ as a sheaf of $\mathcal{S}|_{U_i}$-modules (respectively, right $\mathcal{S}|_{U_i}$-modules).

Proposition. *Let $\mathcal{U}$ be an $X_{ac}(\mathcal{L})$-covering of X, and suppose that $\alpha\colon \mathcal{M} \to \mathcal{N}$ is a morphism of $\mathcal{U}$-coherent left or right $\mathcal{S}$-modules. Then* $\ker\alpha, \operatorname{coker}\alpha$ *and* $\operatorname{Im}\alpha$ *are each $\mathcal{U}$-coherent.*

Proof. We compute using Proposition 5.1 that $(\ker\alpha)|_{U_i} \cong \operatorname{Loc}(\ker\alpha(U_i))$, that $(\operatorname{coker}\alpha)|_{U_i} \cong \operatorname{Loc}(\operatorname{coker}\alpha(U_i))$ and that $\operatorname{Im}\alpha|_{U_i} = \operatorname{Loc}(\operatorname{Im}\alpha(U_i))$. □

5.3. **Coverings of the form $X = X(f) \cup X(1/f)$.** We generalise some technical results from [15, §4.5] to our non-commutative setting. This involves making appropriate changes to the material presented in [15, §4.5], but we repeat these proofs here nevertheless. Note that it is incorrectly asserted in the proof of [15, Lemma 4.5.4] that s_2 has dense image; in fact it is the map s_1 that has dense image.

First, we suppose that $f \in \mathcal{O}(X)$ is such that $\mathcal{L} \cdot f \subseteq \mathcal{A}$. Then

$$X_1 := X(f), \quad X_2 := X(1/f) \quad \text{and} \quad X_3 := X(f) \cap X(1/f)$$

are all $\mathcal{L}$-accessible. We write $s_i\colon \mathcal{S}(X_i) \to \mathcal{S}(X_3)$ for the canonical restriction maps $(i = 1, 2)$. We define the *norm* $||M||$ of a matrix M with entries in a K-Banach algebra to be the supremum of the norms of the entries of M.

Lemma. *There is a constant $c > 0$ such that every matrix $M \in M_n(\mathcal{S}(X_3))$ with $||M - I|| < c$ can be written as a product $M = s_1(Q_1)^{-1} \cdot s_2(Q_2)^{-1}$ for some $Q_i \in \operatorname{GL}_n(\mathcal{S}(X_i))$.*

Proof. By Theorem 3.5, the bounded K-linear map

$$s_1 - s_2 : \mathcal{S}(X_1) \oplus \mathcal{S}(X_2) \to \mathcal{S}(X_3)$$

is surjective. So, by Banach's Open Mapping Theorem there is a constant $0 < d < 1$ such that if N is any $n \times n$ matrix with entries in $\mathcal{S}(X_3)$ we can find $N_1 \in M_n(\mathcal{S}(X_1))$ and $N_2 \in M_n(\mathcal{S}(X_2))$ such that

$$N = s_1(N_1) - s_2(N_2) \quad \text{and} \quad d \cdot \sup(||N_1||, ||N_2||) \leqslant ||N||.$$

We define $c := d^3$. Suppose now that $M \in \operatorname{GL}_n(\mathcal{S}(X_3))$ satisfies $||M - I|| < c$ and let $A_1 = M - I$. We can then find $B_{1i} \in M_n(\mathcal{S}(X_i))$ of norm at most d^2 such that $A_1 = s_1(B_{11}) + s_2(B_{12})$. Then

$$\begin{aligned} A_2 &:= (I - s_1(B_{11}))(I + A_1)(I - s_2(B_{12})) - I \\ &= s_1(B_{11})s_2(B_{12}) - s_1(B_{11})A_1 - A_1 s_2(B_{12}) - s_1(B_{11}) \cdot A_1 \cdot s_2(B_{12}) \end{aligned}$$

is a matrix with coefficients in $\mathcal{O}(X_3)$ and has norm at most d^4.

Inductively, we can find sequences A_m, B_{m1}, B_{m2} of matrices with coefficients in $\mathcal{S}(X_3), \mathcal{S}(X_1)$, $\mathcal{S}(X_2)$ and norms bounded by d^{m+1}, d^m and d^m respectively such that $A_m = s_1(B_{m1}) + s_2(B_{m2})$ and

$$A_{m+1} := (I - s_1(B_{m1}))(I + A_m)(I - s_2(B_{m2})) - I.$$

Because $d^m \to 0$ as $m \to \infty$, the limit

$$Q_i := \lim_{m\to\infty} (1 - B_{mi}) \cdots (1 - B_{1i})$$

exists in $M_n(\mathcal{S}(X_i))$ and $Q_i \in \mathrm{GL}_n(\mathcal{S}(X_i))$ for $i = 1, 2$. By construction,

$$s_1(Q_1) \cdot M \cdot s_2(Q_2) = I$$

so $M = s_1(Q_1)^{-1} \cdot s_2(Q_2)^{-1}$ as claimed. □

5.4. **Theorem.** Suppose that $\mathcal{N}$ is an $\{X_1, X_2\}$-coherent sheaf of $\mathcal{S}$-modules. Then the canonical $\mathcal{S}(X_i)$-linear maps $\mathcal{S}(X_i) \otimes_{\mathcal{S}(X)} \mathcal{N}(X) \to \mathcal{N}(X_i)$ are surjective for $i = 1$ and $i = 2$.

Similarly, if $\mathcal{N}$ is an $\{X_1, X_2\}$-coherent sheaf of right $\mathcal{S}$-modules, then the canonical $\mathcal{S}(X_i)$-linear maps $\mathcal{N}(X) \otimes_{\mathcal{S}(X)} \mathcal{S}(X_i) \to \mathcal{N}(X_i)$ are surjective for $i = 1$ and $i = 2$.

Proof. We first deal with the case of left $\mathcal{S}$-modules. Let us identify $\mathcal{N}(X_3)$ with $\mathcal{S}(X_3) \otimes_{\mathcal{S}(X_1)} \mathcal{N}(X_1)$ and with $\mathcal{S}(X_3) \otimes_{\mathcal{S}(X_2)} \mathcal{N}(X_2)$. Suppose that $a_1, \ldots, a_n$ generate $\mathcal{N}(X_1)$ as a $\mathcal{S}(X_1)$-module and $b_1, \ldots, b_n$ generate $\mathcal{N}(X_2)$ as a $\mathcal{S}(X_2)$-module. Then the sets $\{1 \otimes a_1, \ldots, 1 \otimes a_n\}$ and $\{1 \otimes b_1, \ldots, 1 \otimes b_n\}$ each generate $\mathcal{N}(X_3)$ as a $\mathcal{S}(X_3)$-module.

Consider $\mathcal{N}(X_3)^n$ as a left module over the $n \times n$ matrix ring $M_n(\mathcal{S}(X_3))$ and let $\mathbf{a}, \mathbf{b} \in \mathcal{N}(X_3)^n$ be the column vectors whose jth entries are $1 \otimes a_j$ and $1 \otimes b_j$, respectively. Then we may find non-zero $U, V \in M_n(\mathcal{S}(X_3))$ such that

$$\mathbf{a} = U\mathbf{b} \quad \text{and} \quad \mathbf{b} = V\mathbf{a}.$$

Let c denote the constant from Lemma 5.3. Since the image of $s_1 \colon \mathcal{S}(X_1) \to \mathcal{S}(X_3)$ is dense by Remark 4.3, we can find $V' \in M_n(\mathcal{S}(X_1))$ such that

$$||s_1(V') - V|| < c/||U||.$$

Therefore $||(s_1(V') - V)U|| < c$, and by Lemma 5.3, we can find $Q_i \in \mathrm{GL}_n(\mathcal{S}(X_i))$ for $i = 1, 2$ such that

$$I + (s_1(V') - V)U = s_1(Q_1)^{-1} s_2(Q_2)^{-1}.$$

Applying this matrix identity to the vector $\mathbf{b} \in \mathcal{N}(X_3)^n$ we obtain

$$s_1(Q_1 V')\mathbf{a} = s_2(Q_2^{-1})\mathbf{b}.$$

Writing $a'_i = \sum_{j=1}^n (Q_1 V')_{ij} a_j \in \mathcal{N}(X_1)$ and $b'_i = \sum_{j=1}^n (Q_2^{-1})_{ij} b_j \in \mathcal{N}(X_2)$, we see that $1 \otimes a'_i = 1 \otimes b'_i$ in $\mathcal{N}(X_3)$ for each $i = 1, \ldots, n$. Since $\mathcal{N}$ is a sheaf, we can find elements $d_1, \ldots, d_n \in \mathcal{N}(X)$ such that the image of d_i in $\mathcal{N}(X_1)$ is a'_i and the image of d_i in $\mathcal{N}(X_2)$ is b'_i for each $i = 1, \ldots, n$. Since the matrix Q_2^{-1} is invertible, the elements $b'_1, \ldots, b'_n$ generate $\mathcal{N}(X_2)$ as an $\mathcal{S}(X_2)$-module. Therefore the map $\mathcal{S}(X_2) \otimes_{\mathcal{S}(X)} \mathcal{N}(X) \to \mathcal{N}(X_2)$ is surjective.

Now consider an arbitrary element $v \in \mathcal{N}(X_1)$. Since $1 \otimes b'_1, \ldots, 1 \otimes b'_n$ generate $\mathcal{N}(X_3)$ as a $\mathcal{S}(X_3)$-module we can write $1 \otimes v = \sum_{i=1}^n z_i \otimes b'_i$ for some $z_i \in \mathcal{S}(X_3)$.

The surjectivity of $\mathcal{S}(X_1) \oplus \mathcal{S}(X_2) \to \mathcal{S}(X_3)$ means that we can find $x_i \in \mathcal{S}(X_1)$ and $y_i \in \mathcal{S}(X_2)$ such that $z_i = s_1(x_i) + s_2(y_i)$ for each $i = 1, \ldots, n$. Therefore

$$1 \otimes (v - \sum_{i=1}^n x_i a_i') = 1 \otimes v - \sum_{i=1}^n s_1(x_i) \otimes a_i' = \sum_{i=1}^n s_2(y_i) \otimes b_i' = 1 \otimes \sum_{i=1}^n y_i b_i'$$

inside $\mathcal{N}(X_3)$, because $1 \otimes a_i' = 1 \otimes b_i'$ for all i. Since $\mathcal{N}$ is a sheaf, there is an element $w \in \mathcal{N}(X)$ whose image in $\mathcal{N}(X_1)$ is $v - \sum_{i=1}^n x_i a_i'$ and whose image in $\mathcal{N}(X_2)$ is $\sum_{i=1}^n y_i b_i'$. In particular, v is the image of $1 \otimes w + \sum_{i=1}^n x_i \otimes d_i$ under the map $\mathcal{S}(X_1) \otimes_{\mathcal{S}(X)} \mathcal{N}(X) \to \mathcal{N}(X_1)$. Therefore this map is also surjective.

In the case of right $\mathcal{S}$-modules, again we can find a generating set $\{a_1, \ldots, a_n\}$ for $\mathcal{N}(X_1)$ as a right $\mathcal{S}(X_1)$-module, and a generating set $\{b_1, \ldots, b_n\}$ for $\mathcal{N}(X_2)$ as a right $\mathcal{S}(X_2)$-module. Then $\{a_1 \otimes 1, \ldots, a_n \otimes 1\}$ and $\{b_1 \otimes 1, \ldots, b_n \otimes 1\}$ each generate $\mathcal{N}(X_3)$ as a right $\mathcal{S}(X_3)$-module. We consider $\mathcal{N}(X_3)^n$ as a right module over the $n \times n$ matrix ring $M_n(\mathcal{S}(X_3))$ and let $\mathbf{a}, \mathbf{b} \in \mathcal{N}(X_3)^n$ be the *row* vectors whose jth entries are $a_j \otimes 1$ and $b_j \otimes 1$, respectively. Then we may find non-zero $U, V \in M_n(\mathcal{S}(X_3))$ such that $\mathbf{a} = \mathbf{b}U$ and $\mathbf{b} = \mathbf{a}V$. Choose $V' \in M_n(\mathcal{S}(X_1))$ as above satisfying $||U(s_1(V') - V)|| < c$, and let $T := U(s_1(V') - V)$. Then $||(I + T)^{-1} - I|| < c$ also, so by Lemma 5.3, we can find $Q_i \in \mathrm{GL}_n(\mathcal{S}(X_i))$ for $i = 1, 2$ such that $(I+T)^{-1} = s_1(Q_1)^{-1} s_2(Q_2)^{-1}$. Hence $I+T = s_2(Q_2)s_1(Q_1)$, and applying this matrix identity to the vector $\mathbf{b} \in \mathcal{N}(X_3)^n$ we obtain $\mathbf{a}s_1(V'Q_1^{-1}) = \mathbf{b}s_2(Q_2)$. Therefore the elements $b_j' := \sum_{i=1}^n b_i (Q_2)_{ij} \in \mathcal{N}(X_2)$ extend to global sections of $\mathcal{N}$ and generate $\mathcal{N}(X_2)$ as a right $\mathcal{S}(X_2)$-module because the matrix Q_2 is invertible. Thus $\mathcal{N}(X) \otimes_{\mathcal{S}(X)} \mathcal{S}(X_2) \to \mathcal{S}(X_1)$ is surjective, and the same argument as in the case of left modules now shows that $\mathcal{N}(X) \otimes_{\mathcal{S}(X)} \mathcal{S}(X_1) \to \mathcal{N}(X_1)$ is also surjective. □

Corollary. *If $\mathcal{N}$ is an $\{X(f), X(1/f)\}$-coherent sheaf of $\mathcal{S}$-modules then there is a finitely generated $\mathcal{S}(X)$-module N such that* $\mathrm{Loc}(N) \cong \mathcal{N}$. *A similar statement holds for an $\{X(f), X(1/f)\}$-coherent sheaf of right $\mathcal{S}$-modules.*

Proof. By symmetry, it will suffice to treat the case of left $\mathcal{S}$-modules. As before write $X_1 = X(f)$ and $X_2 = X(1/f)$. By the Theorem, the natural maps $\mathcal{S}(X_i) \otimes_{\mathcal{S}(X)} \mathcal{N}(X) \to \mathcal{N}(X_i)$ are surjective for $i = 1, 2$. Since $\mathcal{N}(X_i)$ is a Noetherian $\mathcal{S}(X_i)$-module, we can find a finitely generated $\mathcal{S}(X)$-submodule M of $\mathcal{N}(X)$ such that $\mathcal{S}(X_i) \otimes_{\mathcal{S}(X)} M \to \mathcal{N}(X_i)$ is surjective for $i = 1, 2$. Thus the natural map $\alpha\colon \mathrm{Loc}(M) \to \mathcal{N}$ is surjective since its restrictions to X_1 and X_2 are both surjective. Since $\mathrm{Loc}(M)$ and $\mathcal{N}$ are both $\{X_1, X_2\}$-coherent, $\ker \alpha$ is also $\{X_1, X_2\}$-coherent by Proposition 5.2 so we may find a finitely generated $\mathcal{S}(X)$-submodule M' of $(\ker \alpha)(X)$ such that $\mathrm{Loc}(M') \to \ker \alpha$ is surjective. Thus $\mathcal{N}$ is isomorphic to the cokernel of $\mathrm{Loc}(M') \to \mathrm{Loc}(M)$. Since Loc is full, this cokernel is isomorphic to $\mathrm{Loc}(\mathrm{coker}(M' \to M))$ and we are done. □

Here is our non-commutative version of Kiehl's Theorem for sheaves of $\mathcal{S}$-modules and $\mathcal{L}$-accessible Laurent coverings.

5.5. **Theorem.** Suppose that $f_1, \ldots, f_n \in \mathcal{O}(X)$ are such that $\mathcal{L} \cdot f_i \subseteq \mathcal{A}$ for each $i = 1, \ldots, n$. Let $\mathcal{U}$ be the Laurent covering $\{X(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n}) \mid \alpha_i \in \{\pm 1\}\}$. Then $\mathcal{U}$ is $\mathcal{L}$-accessible and every $\mathcal{U}$-coherent sheaf $\mathcal{M}$ of left (respectively, right) $\mathcal{S}$-modules on $X_{ac}(\mathcal{L})$ is isomorphic to $\mathrm{Loc}(M)$ for some finitely generated left (respectively, right) $\mathcal{S}(X)$-module M.

Proof. The $\mathcal{L}$-accessibility of $\mathcal{U}$ follows from Corollary 4.7. By symmetry, it is sufficient to treat the case of left $\mathcal{S}$-modules. We proceed by induction on n, the case $n = 1$ being Corollary 5.4.

Suppose that $n > 1$, and that for every family $(X, \mathcal{A}, \mathcal{L})$ satisfying our standing hypotheses, the result is known for all smaller values of n. Suppose also that $f_1, \ldots, f_n \in \mathcal{O}(X)$ satisfy the hypotheses of the Proposition and that $\mathcal{M}$ is $\mathcal{U}$-coherent.

Consider the cover $\mathcal{V} := \{X(f_n)(f_1^{\alpha_1}, \ldots, f_{n-1}^{\alpha_n}) \mid \alpha_i \in \{\pm 1\}\}$ of $X(f_n)$. Let $\mathcal{B}$ be an $\mathcal{L}$-stable affine formal model for $X(f_n)$; then $\mathcal{L}' = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ is a smooth $(\mathcal{R}, \mathcal{B})$-Lie algebra. Now $\mathcal{L}' \cdot f_i \subseteq \mathcal{B}$ for all $i < n$, and since $\mathcal{M}|_{\mathcal{V}}$ is $\mathcal{V}$-coherent the induction hypothesis gives that $\mathcal{M}|_{X(f_n)}$ is isomorphic to $\operatorname{Loc}(M_1)$ for some finitely generated $\mathcal{S}(X(f_n))$-module M_1.

Using an identical argument for $X(1/f_n)$, $\mathcal{M}|_{X(1/f_n)}$ is isomorphic to $\operatorname{Loc}(M_2)$ for some finitely generated $\mathcal{S}(X(1/f_n))$-module M_2. Applying Corollary 5.4 again completes the proof. $\square$

6. Fréchet–Stein enveloping algebras

We assume throughout Section 6 that A is a K-affinoid algebra and that L is a coherent (K, A)-Lie algebra.

6.1. Lie lattices.

Definition. Let $\mathcal{A}$ be an affine formal model in A and let $\mathcal{L} \subseteq L$ be an $\mathcal{A}$-submodule.

(a) $\mathcal{L}$ is an $\mathcal{A}$-*lattice* if it is finitely generated as an $\mathcal{A}$-module, and $K\mathcal{L} = L$.
(b) $\mathcal{L}$ is a $\mathcal{A}$-*Lie lattice* if in addition it is a sub $(\mathcal{R}, \mathcal{A})$-Lie algebra of L.

Lemma. *Let $\mathcal{L}$ be an $\mathcal{A}$-lattice in L.*

(a) If $\mathcal{L}$ is an $\mathcal{A}$-Lie lattice then $\pi^n\mathcal{L}$ is also an $\mathcal{A}$-Lie lattice for all $n \geqslant 0$.
(b) If $\mathcal{B}$ is another affine formal model in A then there is $n \geqslant 0$ such that

$$\pi^m \mathcal{L} \cdot \mathcal{B} \subseteq \mathcal{B} \quad \textit{for all} \quad m \geqslant n.$$

(c) There is $n \geqslant 0$ such that $\pi^m\mathcal{L}$ is an $\mathcal{A}$-Lie lattice in L for all $m \geqslant n$.

Proof. (a) This is clear.

(b) Let $x_1, \ldots, x_d$ generate $\mathcal{L}$ as an $\mathcal{A}$-module, and let $\rho : L \to \operatorname{Der}_K(A)$ be the anchor map. The derivation $\rho(x_i)\colon A \to A$ is bounded for each $i = 1, \ldots, d$ — see the discussion in Section 2.4. So there is $m_i \geqslant 0$ such that $\pi^{m_i}\rho(x_i)(\mathcal{B}) \subseteq \mathcal{B}$. By Lemma 3.1, we can find $t \geqslant 0$ such that $\pi^t\mathcal{A} \subseteq \mathcal{B}$. Let $n = t + \max m_i$ and suppose that $m \geqslant n$. Then

$$\pi^m \mathcal{L} \cdot \mathcal{B} \subseteq \sum_{i=1}^{d} \pi^t \mathcal{A}\, \pi^{m_i} \rho(x_i)(\mathcal{B}) \subseteq \mathcal{B}.$$

(c) Since L is a (K, A)-Lie algebra generated by $x_1, \ldots, x_d$ as an A-module, there are $a_{ij}^k \in A$ such that $[x_i, x_j] = \sum_{k=1}^d a_{ij}^k x_k$ for $1 \leqslant i, j \leqslant d$. Since $A = K \cdot \mathcal{A}$, there is $s \geqslant 0$ such that $\pi^s a_{ij}^k \in \mathcal{A}$ for all i, j and k. Then for $m \geqslant s$ we can compute

$$[\pi^m x_i, \pi^m x_j] \in \sum_{k=1}^{d} \pi^{2m} a_{ij}^k x_k \in \pi^m \mathcal{L}$$

and hence $\pi^m\mathcal{L}$ is an $\mathcal{R}$-Lie algebra for $m \geqslant s$. Using part (b), we can find $s' \geqslant 0$ such that $\pi^m\mathcal{L} \cdot \mathcal{A} \subseteq \mathcal{A}$ for all $m \geqslant s'$. Now take $n = \max\{s, s'\}$. □

6.2. **Fréchet completions of enveloping algebras.** Let $\mathcal{A}$ be an affine formal model in A, and let $\mathcal{L}$ be an $\mathcal{A}$-Lie lattice in L. We define

$$\widehat{U(L)}_{\mathcal{A},\mathcal{L}} := \varprojlim \widehat{U(\pi^n\mathcal{L})_K}.$$

Being a countable inverse limit of K-Banach algebras, $\widehat{U(L)}_{\mathcal{A},\mathcal{L}}$ is a Fréchet algebra.

Lemma. *Let $\mathcal{A}$ be an affine formal model in A and let $\mathcal{L}_1, \mathcal{L}_2$ be two $\mathcal{A}$-Lie lattices in L. Then there is a unique continuous K-algebra isomorphism*

$$\widehat{U(L)}_{\mathcal{A},\mathcal{L}_1} \xrightarrow{\cong} \widehat{U(L)}_{\mathcal{A},\mathcal{L}_2}$$

which restricts to the identity map on $U(L)$.

Proof. Since $\mathcal{L}_1 \cap \mathcal{L}_2$ is again an $\mathcal{A}$-Lie lattice in L, we may assume without loss of generality that $\mathcal{L}_1 \subseteq \mathcal{L}_2$. The universal property of $U(-)$ induces K-Banach algebra homomorphisms $\widehat{U(\pi^n\mathcal{L}_1)_K} \to \widehat{U(\pi^n\mathcal{L}_2)_K}$ for each $n \geqslant 0$ and hence a continuous K-algebra homomorphism

$$\alpha : \widehat{U(L)}_{\mathcal{A},\mathcal{L}_1} \to \widehat{U(L)}_{\mathcal{A},\mathcal{L}_2}.$$

Because $\mathcal{L}_1$ and $\mathcal{L}_2$ are $\mathcal{A}$-lattices in L, we can find an integer s such that $\pi^s\mathcal{L}_2 \subseteq \mathcal{L}_1$. This gives K-Banach algebra homomorphisms $\widehat{U(\pi^{n+s}\mathcal{L}_2)_K} \to \widehat{U(\pi^n\mathcal{L}_1)_K}$ for each $n \geqslant 0$ and hence a continuous K-algebra homomorphism

$$\beta : \widehat{U(L)}_{\mathcal{A},\mathcal{L}_2} \to \widehat{U(L)}_{\mathcal{A},\mathcal{L}_1}.$$

It is easy to see that α and β are mutually inverse. □

Thus $\widehat{U(L)}_{\mathcal{A},\mathcal{L}}$ is independent of the choice of $\mathcal{L}$ up to unique isomorphism, and we write $\widehat{U(L)}_{\mathcal{A}}$ to denote any of these Fréchet algebra completions of $U(L)$.

Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be two affine formal models in A. Then there is a unique continuous isomorphism*

$$\widehat{U(L)}_{\mathcal{A}} \xrightarrow{\cong} \widehat{U(L)}_{\mathcal{B}}$$

which restricts to the identity map on $U(L)$.

Proof. Choose an $\mathcal{A}$-Lie lattice $\mathcal{L}$ and a $\mathcal{B}$-Lie lattice $\mathcal{J}$ in L. By Lemma 3.1, we can find an integer r such that $\pi^r \cdot \mathcal{A} \subseteq \mathcal{B}$. Similarly we can find an integer s such that $\pi^s \cdot \mathcal{L} \subseteq \mathcal{J}$.

Let $x_1, \ldots, x_d$ generate $\mathcal{L}$ as an $\mathcal{A}$-module, and let T be the image of $\widehat{U(\mathcal{J})}$ inside $\widehat{U(\mathcal{J})_K}$. The universal property of $U(-)$ induces an $\mathcal{R}$-algebra homomorphism $\theta_0 : U(\pi^s\mathcal{L}) \to \widehat{U(\mathcal{J})_K}$. Now $U(\pi^s\mathcal{L})$ is generated as an $\mathcal{A}$-module by the set

$$\{(\pi^s x_1)^{\alpha_1} \cdots (\pi^s x_d)^{\alpha_d} : \alpha \in \mathbb{N}^d\}.$$

Since θ_0 sends all these elements to T and since $\mathcal{A} \subseteq \pi^{-r}\mathcal{B}$, we see that the image of θ_0 is contained in $\pi^{-r}T$. Hence θ_0 extends to a K-algebra homomorphism

$$\theta_0 : \widehat{U(\pi^s\mathcal{L})_K} \to \widehat{U(\mathcal{J})_K}.$$

Applying the same argument to $\pi^{s+n} \cdot \mathcal{L} \subseteq \pi^n \mathcal{J}$ for each $n \geqslant 0$, we obtain a compatible sequence of K-algebra homomorphisms

$$\theta_n : \widehat{U(\pi^{s+n}\mathcal{L})_K} \to \widehat{U(\pi^n \mathcal{J})_K}$$

and hence a continuous K-algebra homomorphism

$$\theta_{\mathcal{A},\mathcal{B}} := \varprojlim \theta_n : \widehat{U(L)}_{\mathcal{A}} \to \widehat{U(L)}_{\mathcal{B}}$$

which restricts to the identity map on $U(L)$. Since $\theta_{\mathcal{B},\mathcal{A}} \circ \theta_{\mathcal{A},\mathcal{B}}$ is the identity map on the dense image of $U(L)$ inside $\widehat{U(L)}_{\mathcal{A}}$, it must be equal to $\mathrm{id}_{\widehat{U(L)}_{\mathcal{A}}}$. Similarly $\theta_{\mathcal{A},\mathcal{B}} \circ \theta_{\mathcal{B},\mathcal{A}} = \mathrm{id}_{\widehat{U(L)}_{\mathcal{B}}}$. $\square$

Definition. Let A be a K-affinoid algebra and let L be a (K, A)-Lie algebra which is finitely generated as an A-module. The *Fréchet completion* of $U(L)$ is

$$\widehat{U(L)} := \widehat{U(L)}_{\mathcal{A}} = \varprojlim \widehat{U(\pi^n \mathcal{L})_K}$$

for any choice of affine formal model $\mathcal{A}$ in A and $\mathcal{A}$-Lie lattice $\mathcal{L}$ in L.

The above Lemma and Proposition ensure that this definition does not depend on the choice of $\mathcal{A}$ or $\mathcal{L}$, up to unique isomorphism.

6.3. **Functoriality.** Whenever $\sigma : A \to B$ is an étale morphism of affinoid algebras, there is a Lie homomorphism $\psi : \mathrm{Der}_K(A) \to \mathrm{Der}_K(B)$ by Lemma 2.4, and we may view $B \otimes_A L$ as a (K, B)-Lie algebra by Corollary 2.4.

Proposition. *Let $\sigma : A \to B$ be an étale morphism of K-affinoid algebras, and let $\varphi : L \to L'$ be a morphism of coherent (K, A)-Lie algebras. Then there are unique continuous K-algebra homomorphisms*

(a) $\widehat{U(L)} \to \widehat{U(B \otimes_A L)}$ extending the natural map $U(L) \to U(B \otimes_A L)$, and
(b) $\widehat{U(L)} \to \widehat{U(L')}$ extending the natural map $U(L) \to U(L')$.

Proof. Choose some affine formal model $\mathcal{A}$ in A and any affine formal model $\mathcal{B}$ in B containing $\sigma(\mathcal{A})$. We will construct an $\mathcal{A}$-Lie lattice $\mathcal{L}$ in L and a $\mathcal{B}$-Lie lattice $\mathcal{J}$ in $B \otimes_A L$ (respectively, an $\mathcal{A}$-Lie lattice $\mathcal{J}$ in L') such that $(\sigma \otimes 1)(\mathcal{L}) \subseteq \mathcal{J}$ (respectively, $\varphi(\mathcal{L}) \subseteq \mathcal{J}$). Then the universal property of $U(-)$ induces continuous K-algebra homomorphisms

$$\widehat{U(\pi^m \mathcal{L})_K} \to \widehat{U(\pi^m \mathcal{J})_K}$$

for all $m \geqslant 0$, and passing to the inverse limit gives the required map $\widehat{U(L)} \to \widehat{U(B \otimes_A L)}$ (respectively, $\widehat{U(L)} \to \widehat{U(L')}$). In each case uniqueness follows from the density of the image of $U(L)$ in $\widehat{U(L)}$.

(a) Choose an $\mathcal{A}$-Lie lattice $\mathcal{L}$ in L using Lemma 6.1, and let $\mathcal{J}$ be the image of $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ in $B \otimes_A L$. Then $\mathcal{J}$ is a $\mathcal{B}$-lattice in $B \otimes_A L$ so by Lemma 6.1(b), $\pi^n \mathcal{J}$ is a $\mathcal{B}$-Lie lattice in $B \otimes_A L$ for some $n \geqslant 0$. Now $(\sigma \otimes 1)(\pi^n \mathcal{L}) \subseteq \pi^n \mathcal{J}$.

(b) Let $\mathcal{J}$ be an $\mathcal{A}$-Lie lattice in L'. Then $\varphi^{-1}(\mathcal{J})$ generates L as a K-vector space and hence contains an $\mathcal{A}$-lattice in L. By Lemma 6.1(c), $\varphi^{-1}(\mathcal{J})$ contains an $\mathcal{A}$-Lie lattice $\mathcal{L}$ in L and $\varphi(\mathcal{L}) \subseteq \mathcal{J}$. $\square$

6.4. **Fréchet-Stein algebras.** Following [26, §3] we say that a K-algebra U is *Fréchet-Stein* if

- there is a tower $U_0 \leftarrow U_1 \leftarrow U_2 \leftarrow \cdots$ of Noetherian K-Banach algebras,
- the image of U_{n+1} is dense in U_n for all $n \geqslant 0$,
- U_n is a flat right U_{n+1}-module for all $n \geqslant 0$,
- $U = \varprojlim U_n$.

This definition is designed with a view towards categories of *left* modules. Because we will also need to work with *right* modules in the future, we make this definition more precise by saying that U is *left Fréchet-Stein*. If there is a tower $U_0 \leftarrow U_1 \leftarrow U_2 \leftarrow \cdots$ of Noetherian K-Banach algebras with dense images such that $U \cong \varprojlim U_n$ and each U_n is a flat *left* U_{n+1}-module for all $n \geqslant 0$, then we say that U is *right Fréchet-Stein*. If both conditions are satisfied, then we say that U is *two-sided Fréchet-Stein*.

Theorem. *Let A be a K-affinoid algebra and let L be a coherent (K, A)-Lie algebra. Suppose L has a smooth $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some affine formal model $\mathcal{A}$ in $\mathcal{O}(X)$. Then $\widehat{U(L)}$ is a two-sided Fréchet-Stein algebra.*

We start preparing for the proof of this Theorem, which is given below in Section 6.7: the main problem is to establish flatness.

Recall [1, §3.5] that a positively filtered $\mathcal{R}$-algebra U is said to be *deformable* if $\operatorname{gr} U$ is flat over $\mathcal{R}$. Its *n-th deformation* is by definition its subring

$$U_n := \sum_{i \geqslant 0} \pi^{in} F_i U.$$

It follows from [1, Lemma 3.5] that U_n is again a deformable $\mathcal{R}$-algebra, whose filtration is given by

$$F_j U_n = U_n \cap F_j U = \sum_{i=0}^{j} \pi^{in} F_i U.$$

We begin by recording some useful general facts on deformable algebras.

Lemma. *Let U be a deformable $\mathcal{R}$-algebra. Then*
(a) $U_1 \cap \pi^t U = \sum_{i \geqslant t} \pi^i F_i U$ for any $t \geqslant 0$.
(b) $(U_n)_m$ is equal to U_{m+n} for any $n, m \geqslant 0$.

Proof. (a) The $\mathcal{R}$-module $U/F_t U$ is a direct limit of iterated extensions of $\mathcal{R}$-modules of the form $\operatorname{gr}_j U$, each of which is flat by assumption. Hence $U/F_t U$ has no $\mathcal{R}$-torsion and consequently $F_t U \cap \pi^t U = \pi^t F_t U$. Since $\sum_{i>t} \pi^i F_i U \subseteq \pi^t U$,

$$U_1 \cap \pi^t U \subseteq \left(F_t U + \sum_{i>t} \pi^i F_i U \right) \cap \pi^t U \subseteq (F_t U \cap \pi^t U) + \sum_{i>t} \pi^i F_i U = \sum_{i \geqslant t} \pi^i F_i U$$

by the modular law, and the reverse inclusion is clear.

(b) $(U_n)_m = \sum_{j \geqslant 0} \pi^{jm} \sum_{i=0}^{j} \pi^{in} F_i U = \sum_{i \geqslant 0} (\sum_{j \geqslant i} \pi^{jm+in} \mathcal{R}) F_i U = U_{n+m}$. $\square$

6.5. **The subspace filtration on U_1.** We will need to study the subspace filtration on U_1 induced from the π-adic filtration on U in detail.

Lemma. *Let U be a deformable $\mathcal{R}$-algebra such that $\operatorname{gr} U$ is commutative. Suppose that $\operatorname{gr} U$ is generated by the symbols of the elements $x_1, \ldots, x_m \in U$ as an algebra over $\operatorname{gr}_0 U$. Let $r_j = \deg x_j$. Then*

$$F_i U = F_0 U \cdot \left\{ x_1^{\alpha_1} \cdots x_m^{\alpha_m} \mid \sum \alpha_j r_j \leqslant i \right\}$$

for each $i \geqslant 0$.

Proof. It is sufficient to prove that $F_i U$ is contained in the right hand side, the reverse inclusion being clear. We proceed by induction on i, the case $i = 0$ being trivial. For every $z \in F_i U$, the image of z in $\operatorname{gr}_i U$ is a $\operatorname{gr}_0 U$-linear combination of monomials in the symbols of the x_j's by our assumption. Hence for each $\alpha \in \mathbb{N}^d$ such that $\sum \alpha_j r_j = i$ we can find $\lambda_\alpha \in \operatorname{gr}_0 U = F_0 U$ such that

$$z - \sum \lambda_\alpha x^\alpha \in F_{i-1} U.$$

The result follows immediately by applying the inductive hypothesis. □

Proposition. *Let U be a deformable $\mathcal{R}$-algebra such that $\operatorname{gr} U$ is a commutative Noetherian graded ring, and let $I := U_1 \cap \pi U$. Then the subspace filtration on U_1 arising from the π-adic filtration on U and the I-adic filtration on U_1 are topologically equivalent.*

Proof. Because $\operatorname{gr} U$ is commutative and Noetherian, there are elements $x_1, \ldots, x_m$ in U whose symbols generate $\operatorname{gr} U$ as an algebra over $\operatorname{gr}_0 U$ by [4, Proposition 10.7]. We may assume that each $r_j := \deg x_j$ is positive; then

$$\pi \in I \quad \text{and} \quad \pi^{r_j} x_j \in I \quad \text{for all} \quad j \geqslant 1.$$

Let $r_0 := 1$; it follows from the Lemma that $\pi^i F_i U$ is generated as an $F_0 U$-module by all possible elements of the form

$$(\pi^{r_0})^{\alpha_0} (\pi^{r_1} x_1)^{\alpha_1} \cdots (\pi^{r_m} x_m)^{\alpha_m}$$

where $\alpha_j \in \mathbb{N}$ for all $j = 0, \ldots, m$ and $\sum_{j=0}^m \alpha_j r_j = i$. If the integer t is given and $i \geqslant t \max r_j$, then $\left(\sum_{j=0}^m \alpha_j\right) \max r_j \geqslant \sum_{j=0}^m \alpha_j r_j = i \geqslant t \max r_j$, so

$$(\pi^{r_0})^{\alpha_0} (\pi^{r_1} x_1)^{\alpha_1} \cdots (\pi^{r_m} x_m)^{\alpha_m} \in I^t$$

because $\pi \in I$ and $\pi^{r_j} x_j \in I$ for all $j \geqslant 1$. Therefore by Lemma 6.4(a) we have

$$U_1 \cap \pi^{t \max r_j} U = \sum_{i \geqslant t \max r_j} \pi^i F_i U \subseteq I^t \subseteq U_1 \cap \pi^t U \quad \text{for all} \quad t \geqslant 0$$

because I is an $F_0 U$-submodule of U. □

6.6. **π-adic completions.** Recall that if U is a deformable $\mathcal{R}$-algebra, then $\widehat{U_n} := \varprojlim U_n / \pi^a U_n$ denotes the π-adic completion of U_n and that

$$\widehat{U_{n,K}} := K \otimes_{\mathcal{R}} \widehat{U_n}$$

may be equipped with the structure of a K-Banach algebra, with unit ball $\widehat{U_n}$. Since $U_0 = U$, we will abbreviate $\widehat{U_{0,K}}$ to $\widehat{U_K}$.

Theorem. *Let U be a deformable $\mathcal{R}$-algebra such that $\operatorname{gr} U$ is a commutative Noetherian ring. Then $\widehat{U_K}$ is a flat $\widehat{U_{1,K}}$ module on both sides.*

Proof. In this proof, "flat module" will mean "flat module on both sides". Since $\widehat{U_{1,K}} = \widehat{U_1} \otimes_{\mathcal{R}} K$, it will be enough to prove that $\widehat{U_K}$ is a flat $\widehat{U_1}$-module. By Proposition 6.5, the I-adic completion V of U_1 is isomorphic to the closure of the image of U_1 in $\widehat{U}$. Thus we have natural maps $\widehat{U_1} \to V \to \widehat{U_K}$. We observe that V is π-adically complete by the proof of [32, Theorem VIII.5.14] noting that ideals in V are I-adically closed by [21, Theorem II.2.1.2, Proposition II.2.2.1]

We begin by filtering both $\widehat{U_1}$ and V π-adically. Notice that $V/\pi V$ is the $I/\pi U_1$-adic completion of $U_1/\pi U_1$ which is flat by [4, Proposition 10.14]. Since U_1 is π-torsion free, $\operatorname{gr} \widehat{U_1} \cong (U_1/\pi U_1)[t]$. Similarly, since V is isomorphic to a subring of $\widehat{U}$, it has no π-torsion, and so $\operatorname{gr} V \cong (V/\pi V)[t]$. Hence $\operatorname{gr} V$ is flat as a $\operatorname{gr}\widehat{U_1}$-module. Since both $\widehat{U_1}$ and V are π-adically complete, [26, Proposition 1.2] implies that V is a flat $\widehat{U_1}$-module.

Next, we again consider the subspace filtration on U_1 induced by the π-adic filtration on U. We have $\operatorname{gr} U \cong \overline{U}[t]$, where $t := \operatorname{gr} \pi$ and $\overline{U} := U/\pi U$ has degree zero. It follows from Lemma 6.4(a) that the image of $\operatorname{gr} U_1$ inside $\operatorname{gr} U$ is equal to $\oplus_{j \geqslant 0} t^j \cdot \overline{F_j U}$, where $\overline{F_j U}$ is the image of $F_j U$ in $\overline{U}$. Since the quotient filtration $\overline{F_j U}$ on $\overline{U}$ is exhaustive, the localisation of this image obtained by inverting t is equal to $\overline{U}[t, t^{-1}]$. Now V is the completion of U_1 so

$$(\operatorname{gr} V)_t = (\operatorname{gr} U_1)_t = \overline{U}[t, t^{-1}] = \operatorname{gr} \widehat{U_K}$$

and therefore $\operatorname{gr} \widehat{U_K}$ is a flat $\operatorname{gr} V$-module. Hence we can again invoke [26, Proposition 1.2] to deduce that $\widehat{U_K}$ is a flat V-module. $\square$

Remark. *Essentially all ideas involved in this proof can already be found in* [26].

6.7. **The functor $U \mapsto \widehat{U_K}$.** Let U be a deformable $\mathcal{R}$-algebra. By functoriality of π-adic completion, the descending chain

$$U = U_0 \supset U_1 \supset U_2 \supset \cdots$$

induces an inverse system of K-Banach algebras and bounded algebra maps

$$\widehat{U_K} = \widehat{U_{0,K}} \leftarrow \widehat{U_{1,K}} \leftarrow \widehat{U_{2,K}} \leftarrow \cdots$$

whose inverse limit we denote by

$$\widehat{U_K} := \varprojlim \widehat{U_{n,K}}.$$

The natural maps $\widehat{U_K} \to \widehat{U_{n,K}}$ may be used to construct semi-norms $|\cdot|_n$ on $\widehat{U_K}$ so that the completion of $\widehat{U_K}$ with respect to $|\cdot|_n$ is $\widehat{U_{n,K}}$. In this way $\widehat{U_K}$ becomes a Fréchet algebra.

Theorem. *Let U be a deformable $\mathcal{R}$-algebra such that $\operatorname{gr} U$ is commutative and Noetherian. Then $\widehat{U_K}$ is a two-sided Fréchet–Stein algebra.*

Proof. Each $\widehat{U_{n,K}}$ is Noetherian because $\operatorname{gr} U$ is Noetherian. The image of $\widehat{U_{n+1,K}}$ in $\widehat{U_{n,K}}$ is dense because it contains the image of U_K in $\widehat{U_{n,K}}$, which is dense because $\widehat{U_{n,K}}$ is the completion of U_K with respect the semi-norm $|\cdot|_n$.

Each U_n is a deformable $\mathcal{R}$-algebra with $\operatorname{gr} U_n \cong \operatorname{gr} U$ by [1, Lemma 3.5], and the first deformation $(U_n)_1$ of U_n is equal to U_{n+1} by Lemma 6.4(b). Hence $\widehat{U_{n,K}}$ is a flat $\widehat{U_{n+1,K}}$-module on both sides by Theorem 6.6. $\square$

Proof of Theorem 6.4. The algebra $U = U(\mathcal{L})$ is deformable because $\operatorname{gr} U = \operatorname{Sym}(\mathcal{L})$ by [23, Theorem 3.1], and U_n is naturally isomorphic to $U(\pi^n\mathcal{L})$ for all $n \geqslant 0$. Therefore

$$\widehat{U(L)} = \varprojlim \widehat{U(\pi^n\mathcal{L})_K} \cong \varprojlim \widehat{U_{n,K}} = \widehat{U_K}$$

is two-sided Fréchet-Stein by Theorem 6.7. □

7. The functor $\widehat{\otimes}$

From now on we will work with categories of *left* modules, however all our results will have analogues valid for categories of right modules. We omit giving the necessary repetitive details to save space.

7.1. **Co-admissible completion.** Suppose that U is a left Fréchet–Stein algebra. Recall, [26, §3], that if $U = \varprojlim U_n$ is a presentation of U as a left Fréchet–Stein algebra then a *coherent sheaf of $U_\bullet$-modules* is a family (M_n) of finitely generated U_n-modules M_n together with isomorphisms $U_n \otimes_{U_{n+1}} M_{n+1} \xrightarrow{\cong} M_n$ for each n. The coherent sheaves of $U_\bullet$-modules form an abelian category $\operatorname{Coh}(U_\bullet)$ with respect to the obvious notion of morphism. Then a U-module M is said to be *co-admissible* if it is isomorphic as a U-module to $\varprojlim M_n$ for some coherent sheaf of $U_\bullet$-modules (M_n) . By [26, Lemma 3.8] the question of whether a given U-module is co-admissible does not depend of the choice of $U_\bullet$ presenting U. The co-admissible U-modules form a full subcategory $\mathcal{C}_U$ of all U-modules. By [26, Corollary 3.3] the natural functors

$$\Gamma\colon \operatorname{Coh}(U_\bullet) \to \mathcal{C}_U \quad \text{and} \quad \operatorname{Loc}_{U_\bullet}\colon \mathcal{C}_U \to \operatorname{Coh}(U_\bullet)$$

that send a coherent sheaf (M_n) of $U_\bullet$-modules to the co-admissible U-module $\varprojlim M_n$, and a co-admissible U-module M to the coherent sheaf $(U_n \otimes_U M)$ of $U_\bullet$-modules, are mutually inverse equivalences of categories.

Definition. We say that a co-admissible U-module $\widehat{M}$ is a *co-admissible completion* of a U-module M if there is a U-linear map $\iota_M\colon M \to \widehat{M}$ such that for every co-admissible U-module N and every U-linear map $f\colon M \to N$ there is a unique U-linear map $g\colon \widehat{M} \to N$ such that $g \circ \iota_M = f$.

By usual arguments with universal properties, if a U-module M has a co-admissible completion, it (together with the map ι) is uniquely determined up to unique isomorphism.

Proposition. *Suppose that $U = \varprojlim U_n$ is a presentation of U as a left Fréchet–Stein algebra. If M is a U-module such that each $U_n \otimes_U M$ is finitely generated as a U_n-module then $\varprojlim U_n \otimes_U M$ (together with the natural map $\iota_M\colon M \to \varprojlim U_n \otimes_U M$) is a co-admissible completion of M.*

Proof. Certainly $\varprojlim U_n \otimes_U M$ is a co-admissible U-module, so suppose that N is also a co-admissible U-module and $f\colon M \to N$ is a U-linear map. By functoriality, there is a natural commutative diagram

$$\begin{tikzcd} M \arrow[r, "\iota_M"] \arrow[d, "f"'] & \varprojlim U_n \otimes_U M \arrow[d, "\widehat{f}"] \\ N \arrow[r, "\iota_N"'] & \varprojlim U_n \otimes_U N \end{tikzcd}$$

where $\widehat{f} = \varprojlim 1 \otimes f$. Since N is co-admissible, ι_N is an isomorphism so we may define $g := \iota_N^{-1} \circ \widehat{f}$. Then $g \circ \iota_M = f$.

Suppose that $h : \varprojlim U_n \otimes_U M \to N$ is another U-linear map such that $h \circ \iota_M = f$. Then $\iota_N \circ h \circ \iota_M = \iota_N \circ f = \widehat{f} \circ \iota_M$, so the U-linear map

$$q := \iota_N \circ h - \widehat{f} : \varprojlim U_n \otimes_U M \to \varprojlim U_n \otimes_U N$$

is zero on the image of ι_M. By [26, Corollary 3.3], q is the inverse limit of U_n-linear maps $q_n : U_n \otimes_U M \to U_n \otimes_U N$ where $q_n(x_n) = q(x)_n$ for any $x = (x_n) \in \varprojlim U_n \otimes_U M$. Now for any $m \in M$, $\iota_M(m) = (1 \otimes m) \in \varprojlim U_n \otimes_U M$, so $q_n(1 \otimes m) = q(\iota_M(m))_n = 0$. Since q_n is U_n-linear, we see that $q_n = 0$ for all n and hence $q = 0$. So $\iota_N \circ h = \widehat{f}$ and $h = \iota_N^{-1} \circ \widehat{f} = g$. $\square$

7.2. **A Fréchet structure on** Hom **sets for co-admissible modules.** Suppose that U is a left Fréchet–Stein algebra. Let $\mathcal{Q}_U$ be the partially ordered set of continuous seminorms q on U such that the corresponding Banach completion U_q is a left Noetherian K-algebra.

Lemma. *Let $U_\bullet$ be a Fréchet–Stein structure on U.*

(a) For each $q \in \mathcal{Q}_U$ and $M, N \in \mathcal{C}_U$, $\operatorname{Hom}_{U_q}(U_q \otimes_U M, U_q \otimes_U N)$ is naturally a K-Banach space.

(b) There is a natural bifunctor from $\mathcal{Q}_U \times \mathcal{C}_U$ to the category of K-Banach spaces and continuous maps sending the pair (q, M) to $U_q \otimes_U M$.

(c) For each M and N in $\mathcal{C}_U$,

$$\begin{aligned} \operatorname{Hom}_U(M,N) &\cong \varprojlim_{q \in \mathcal{Q}_U} \operatorname{Hom}_{U_q}(U_q \otimes_U M, U_q \otimes_U N) \\ &\cong \varprojlim_n \operatorname{Hom}_{U_n}(U_n \otimes_U M, U_n \otimes_U N). \end{aligned}$$

Proof. Write q_n for the semi-norm on U such that $U_n = U_{q_n}$. For each $q \in \mathcal{Q}_U$, there is some n such that $q \leqslant q_n$: the set $\{q_n\}$ is cofinal in $\mathcal{Q}_U$.

(a) Suppose M, N are co-admissible U-modules and $q \in \mathcal{Q}_U$. We can find n such that there is a continuous homomorphism of Noetherian K-Banach algebras $U_n \to U_q$. Since M and N are co-admissible $U_n \otimes_U M$ and $U_n \otimes_U N$ are finitely generated U_n-modules. Thus $U_q \otimes_U M \cong U_q \otimes_{U_n} U_n \otimes_U M$ and $U_q \otimes_U N$ are finitely generated U_q-modules. Thus by [26, Proposition 2.1], $U_q \otimes_U M$ and $U_q \otimes_U N$ have canonical Banach topologies and $\operatorname{Hom}_{U_q}(U_q \otimes_U M, U_q \otimes_U N)$ consists of continuous K-linear maps. In particular $\operatorname{Hom}_{U_q}(U_q \otimes_U M, U_q \otimes_U N)$ is a closed subspace of the Banach space consisting of all continuous K-linear maps from $U_q \otimes_U M$ to $U_q \otimes_U N$.

(b) Suppose now that $q' \leqslant q \in \mathcal{Q}_U$ and $M \in \mathcal{C}_U$. Then we can define

$$\psi_{M,q,q'} : U_q \otimes_U M \to U_{q'} \otimes_U M$$

by identifying $U_{q'} \otimes_U M$ with $U_{q'} \otimes_{U_q} U_q \otimes_U M$ and setting $\psi_{M,q,q'}(u_q \otimes m) = 1 \otimes u_q \otimes m$. Now if $q, q' \in \mathcal{Q}_U$, $M, N \in \mathcal{C}_U$ and $f \in \operatorname{Hom}_U(M,N)$, then

$$\begin{tikzcd} U_q \otimes_U M \arrow[r, "\mathrm{id} \otimes f"] \arrow[d, "\psi_{M,q,q'}"'] & U_q \otimes_U N \arrow[d, "\psi_{N,q,q'}"] \\ U_{q'} \otimes_U M \arrow[r, "\mathrm{id} \otimes f"'] & U_{q'} \otimes_U N \end{tikzcd}$$

is a commutative diagram. Hence $(q, M) \mapsto U_q \otimes_U M$ is a bifunctor.

(c) Write $M_n = U_n \otimes_U M$ and $N_n = U_n \otimes_U N$. Since the set q_n is cofinal in $\mathcal{Q}_U$, it suffices to show that $\operatorname{Hom}_U(M, N) \cong \varprojlim_n \operatorname{Hom}_{U_n}(M_n, N_n)$. Now by the equivalence of categories between coherent $U_\bullet$-modules and coadmissible U-modules there is a K-linear isomorphism $\operatorname{Hom}_U(M, N) \cong \operatorname{Hom}_{\operatorname{Coh} U_\bullet}(M_\bullet, N_\bullet)$. Thus it remains to observe that if $(f_n) \in \prod_{n \geqslant 0} \operatorname{Hom}_{U_n}(M_n, N_n)$ then $f_\bullet$ is a morphism of coherent $U_\bullet$-modules if and only if $\psi_{N,q_{n+1},q_n} \circ f_{n+1} = f_n \circ \psi_{M,q_{n+1},q_n}$ for each $n \geqslant 0$. $\square$

Definition. Suppose that M and N are co-admissible U-modules. Using the Lemma we can make

$$\operatorname{Hom}_U(M, N) \cong \varprojlim_{q \in \mathcal{Q}_U} \operatorname{Hom}_{U_q}(U_q \otimes_U M, U_q \otimes_U N)$$

into a K-Fréchet space by giving it the inverse limit topology in the category of locally convex vector spaces.

7.3. **The functor $M \mapsto P \widehat{\otimes}_V M$.** Let U and V be left Fréchet–Stein algebras.

Definition. We say that a Fréchet space P is a *U-co-admissible (U, V)-bimodule* if P is a co-admissible left U-module equipped with a continuous homomorphism $V^{\mathrm{op}} \to \operatorname{End}_U(P)$ with respect to the topology on $\operatorname{End}_U(P)$ defined in §7.2.

For any Fréchet-Stein structures $U_\bullet$ and $V_\bullet$ on U and V respectively, the definition of the Fréchet topology on $\operatorname{End}_U(P)$ implies that $V^{\mathrm{op}} \to \operatorname{End}_U(P)$ is continuous if and only if for every $n \geqslant 0$, there is some $m \geqslant 0$ and a continuous algebra homomorphism $V_m^{\mathrm{op}} \to \operatorname{End}_{U_n}(U_n \otimes_U P)$ such that the diagram

$$\begin{tikzcd} V^{\mathrm{op}} \arrow[r] \arrow[d] & \operatorname{End}_U(P) \arrow[d] \\ V_m^{\mathrm{op}} \arrow[r] & \operatorname{End}_{U_n}(U_n \otimes_U P) \end{tikzcd}$$

commutes. Thus for example U is a U-co-admissible (U, V)-bimodule whenever $V \to U$ is a continuous homomorphism of left Fréchet–Stein algebras.

Lemma. *Suppose that P is a U-co-admissible (U, V)-bimodule. Then for every co-admissible V-module M, there is a co-admissible U-module*

$$P \widehat{\otimes}_V M$$

and a V-balanced U-linear map

$$\iota \colon P \times M \to P \widehat{\otimes}_V M$$

satisfying the following universal property: if $f \colon P \times M \to N$ is a V-balanced U-linear map with $N \in \mathcal{C}_U$, then there is a unique U-linear map $g \colon P \widehat{\otimes}_V M \to N$ such that $g \circ \iota = f$. Moreover, $P \widehat{\otimes}_V M$ is determined by its universal property up to canonical isomorphism.

Proof. Let $U = \varprojlim U_n$ and $V = \varprojlim V_n$ be presentations of U and V as left Fréchet–Stein algebras and let $n \geqslant 0$ be fixed. Then $P_n := U_n \otimes_U P$ is a (U_n, V)-bimodule that is finitely generated as a U_n-module. Because $V^{\mathrm{op}} \to \operatorname{End}_U(P)$ is continuous, the map $V^{\mathrm{op}} \to \operatorname{End}_{U_n}(P_n)$ factors through V_m for some m. Thus

$$P_n \otimes_V M \cong P_n \otimes_{V_m} (V_m \otimes_V M)$$

is a finitely generated U_n-module because M is co-admissible. Therefore $P \otimes_V M$ has a co-admissible completion by Proposition 7.1, and we define

$$P\widehat{\otimes}_V M := \widehat{P \otimes_V M} = \varprojlim P_n \otimes_V M.$$

The universal properties of $\otimes_V$ and of co-admissible completion ensure that $P\widehat{\otimes}_V M$ satisfies the required universal property. $\square$

We note that if U, V and P are as in the Lemma then for any choice of $U_\bullet$ presenting U as a left Fréchet–Stein algebra, and any $M \in \mathcal{C}_U$, we have isomorphisms

$$U_n \otimes_U (P\widehat{\otimes}_V M) \cong U_n \otimes_U P \otimes_V M.$$

For any $f \in \operatorname{Hom}_{\mathcal{C}_V}(M, M')$, the universal property for $\widehat{\otimes}_V$ uniquely determines an element $1\widehat{\otimes} f \in \operatorname{Hom}_{\mathcal{C}_U}(P\widehat{\otimes}_V M, P\widehat{\otimes}_V M')$ since the composite $P \times M \xrightarrow{1\times f} P \times M' \to P\widehat{\otimes}_V M'$ is V-balanced and U-linear. Thus we have defined the *co-admissible base change* functor

$$P\widehat{\otimes}_V - : \mathcal{C}_V \longrightarrow \mathcal{C}_U.$$

7.4. Associativity of $\widehat{\otimes}$.

Lemma. *Suppose that U, V and W are left Noetherian K-Banach algebras, P is a (U,V)-bimodule, and Q is a (V,W)-bimodule. Suppose further that P and Q are finitely generated over U and V respectively, and that $V^{\mathrm{op}} \to \operatorname{End}_U(P)$ and $W^{\mathrm{op}} \to \operatorname{End}_V(Q)$ are both continuous. Then $P \otimes_V Q$ is a finitely generated left U-module and the natural map $W^{\mathrm{op}} \to \operatorname{End}_U(P \otimes_V Q)$ is continuous.*

Proof. Suppose that $X := \{x_1, \ldots, x_n\}$ generates P as a left U-module and $Y := \{y_1, \ldots, y_m\}$ generates Q as a left V-module. Then if $p \otimes q \in P \otimes Q$, we can write

$$p \otimes q = \sum_{i=1}^m p \otimes v_i y_i = \sum_{i=1}^m p v_i \otimes y_i$$

for some $v_1, \ldots, v_m \in V$. Now for each i, $pv_i = \sum_{j=1}^n u_{ij} x_j$ for some $u_{ij} \in U$. Thus $p \otimes q = \sum_{i,j} u_{ij} x_j \otimes y_i$. Since $P \otimes_V Q$ is generated by elementary tensors as an abelian group, it follows that it is generated as a U-module by the set $X \otimes Y := \{x \otimes y \mid x \in X, y \in Y\}$.

Choose sub-multiplicative norms on U, V and W that define their Banach topologies and let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be the corresponding unit balls. By a non-commutative version of [11, §3.7], $\mathcal{U}X$, $\mathcal{V}Y$ and $\mathcal{U}(X \otimes Y)$ are unit balls with respect to some norms on P, Q and $P \otimes_V Q$ that define their respective Banach topologies.

Since $V^{\mathrm{op}} \to \operatorname{End}_U(P)$ and $W^{\mathrm{op}} \to \operatorname{End}_V(Q)$ are continuous, there are natural numbers a and b such that $\mathcal{U}X\mathcal{V} \subseteq \pi^{-a}\mathcal{U}X$ and $\mathcal{V}Y\mathcal{W} \subseteq \pi^{-b}\mathcal{V}Y$. Thus

$$\mathcal{U}(X \otimes Y)\mathcal{W} \subseteq \mathcal{U}(X \otimes \pi^{-b}\mathcal{V}\mathcal{Y}) = \pi^{-b}\mathcal{U}X\mathcal{V} \otimes Y \subseteq \pi^{-(a+b)}\mathcal{U}(X \otimes Y)$$

and so $W^{\mathrm{op}} \to \operatorname{End}_U(P \otimes_V Q)$ is continuous as claimed. $\square$

Proposition. *Suppose that U, V and W are left Fréchet–Stein algebras, that P is a U-co-admissible (U,V)-bimodule and that Q is a V-co-admissible (V,W)-bimodule. Then $P\widehat{\otimes}_V Q$ is a U-co-admissible (U,W)-bimodule, and for every co-admissible W-module M there is a canonical isomorphism*

$$P\widehat{\otimes}_V (Q\widehat{\otimes}_W M) \xrightarrow{\cong} (P\widehat{\otimes}_V Q)\widehat{\otimes}_W M$$

of co-admissible U-modules.

Proof. Let $U_\bullet$, $V_\bullet$ and $W_\bullet$ be Fréchet–Stein structures on U, V and W respectively. $P\widehat{\otimes}_V Q$ is a co-admissible U-module by Lemma 7.3, and to see that $W^{\mathrm{op}} \to \mathrm{End}_U(P\widehat{\otimes}_V Q)$ is continuous, it suffices to show that for each $n \geqslant 0$, $W^{\mathrm{op}} \to \mathrm{End}_{U_n}\left(U_n \otimes_U (P\widehat{\otimes}_V Q)\right)$ factors continuously through some W_l^{op}.

Fix $n \geqslant 0$ and write $P_n := U_n \otimes_U P$. Because $V^{\mathrm{op}} \to \mathrm{End}_U(P)$ is continuous, there is some $m \geqslant 0$ such that $V^{\mathrm{op}} \to \mathrm{End}_{U_n}(P_n)$ factors through a continuous map $V_m^{\mathrm{op}} \to \mathrm{End}_{U_n}(P_n)$. Let $Q_m := V_m \otimes_V Q$ so that there is a canonical isomorphism of (U_n, W)-bimodules $P_n \otimes_{V_m} Q_m \cong P_n \otimes_V Q$. Because $W^{\mathrm{op}} \to \mathrm{End}_V(Q)$ is continuous, there is some $l \geqslant 0$ such that $W^{\mathrm{op}} \to \mathrm{End}_{V_m}(Q_m)$ factors through a continuous map $W_l^{\mathrm{op}} \to \mathrm{End}_{V_m}(Q_m)$. Hence $W_l^{\mathrm{op}} \to \mathrm{End}_{U_n}(P_n \otimes_{V_m} Q_m)$ is continuous by the Lemma, so $W^{\mathrm{op}} \to \mathrm{End}_U(P\widehat{\otimes}_V Q)$ is continuous.

Now, for the choice of m above, there are canonical isomorphisms

$$\begin{aligned} U_n \otimes_U \left(P\widehat{\otimes}_V (Q\widehat{\otimes}_W M)\right) &\cong P_n \otimes_V (Q\widehat{\otimes}_W M) \\ &\cong P_n \otimes_{V_m} \left(V_m \otimes_V \left(Q\widehat{\otimes}_W M\right)\right) \\ &\cong P_n \otimes_{V_m} Q_m \otimes_W M \\ &\cong P_n \otimes_V Q \otimes_W M \\ &\cong U_n \otimes_U (P\widehat{\otimes}_V Q) \otimes_W M \\ &\cong U_n \otimes_U \left((P\widehat{\otimes}_V Q)\widehat{\otimes}_W M\right). \end{aligned}$$

We note that the composite isomorphism

$$U_n \otimes_U \left(P\widehat{\otimes}_V (Q\widehat{\otimes}_W M)\right) \xrightarrow{\cong} U_n \otimes_U \left((P\widehat{\otimes}_V Q)\widehat{\otimes}_W M\right)$$

does not depend on m provided that it is sufficiently large with respect to n. Since n is arbitrary, the result follows. □

Corollary. *Let $W \to V \to U$ be a sequence of continuous morphisms of left Fréchet–Stein algebras. Then there is a canonical isomorphism*

$$U\widehat{\otimes}_V (V\widehat{\otimes}_W M) \xrightarrow{\cong} U\widehat{\otimes}_W M$$

of U-modules, for every co-admissible W-module M.

7.5. **Co-admissible flatness.** Let U and V be left Fréchet–Stein algebras.

Definition. Let P be a U-co-admissible (U, V)-bimodule.

(a) P is a *c-flat* right V-module if $P\widehat{\otimes}_V -$ is exact.

(b) P is a *faithfully c-flat* right V-module if in addition $P\widehat{\otimes}_V M = 0$ only if $M = 0$.

Proposition. *Let P be a U-co-admissible (U, V)-bimodule.*

(a) The functor $P\widehat{\otimes}_V -$ is right exact.

(b) If $U = \varprojlim U_n$ is a presentation of U as a left Fréchet–Stein algebra such that $U_n \otimes_U P$ is a flat right V-module for all n, then P is c-flat over V.

(c) If additionally, for all non-zero $M \in \mathcal{C}_V$ there exists n such that $U_n \otimes_U P \otimes_V M$ is non-zero, then P is a faithfully c-flat right V-module.

Proof. Let $P_n = U_n \otimes_U P$ and consider the functor $\mathrm{Loc}_{U_\bullet} \circ (P\widehat{\otimes}_V -) \colon \mathcal{C}_V \to \mathrm{Coh}(U_\bullet)$. This is equivalent to the functor $(P_n \otimes_V -)$. Since $\mathrm{Loc}_{U_\bullet}$ is an equivalence of categories, it suffices to show that a) $P_n \otimes_V -$ is always right exact, b) it is exact if each P_n is flat over V, and c) if for all non-zero $M \in \mathcal{C}_V$ there exists n such that $P_n \otimes_V M$ is non-zero then $(P_n \otimes_V M)$ is non-zero. All these statements are either well-known or clear. □

7.6. **Rescaling the Lie lattice.** We will now apply the theory developed in Section 7 to the Fréchet-Stein enveloping algebras $\widehat{U(L)}$ introduced in Section 6.

Let Y be an affinoid subdomain of the K-affinoid variety X and suppose that $\mathcal{L}$ is a coherent $(\mathcal{R}, \mathcal{A})$-Lie algebra for some formal model $\mathcal{A}$ in $\mathcal{O}(X)$.

Lemma. *(a) For each $g \in \mathcal{O}(X)$, there is an $n \geqslant 0$ such that $\pi^n \mathcal{L} \cdot g \subseteq \mathcal{A}$.*
(b) There exists $l \geqslant 0$ such that Y is $\pi^n\mathcal{L}$-admissible for all $n \geqslant l$.

Proof. Choose a set of generators $\{x_1, \ldots, x_d\}$ for $\mathcal{L}$ as an $\mathcal{A}$-module.

(a) Since $x_i \cdot g \in \mathcal{O}(X)$ for each $1 \leqslant i \leqslant d$ and $\mathcal{O}(X) = K \cdot \mathcal{A}$, there are $n_i \geqslant 0$ such that for each such i, $\pi^{n_i} x_i \cdot g \in \mathcal{A}$. Taking $n = \sup\{n_i\}$ we see that $\pi^n\mathcal{L} \cdot g \subseteq \mathcal{A}$.

(b) Choose an affine formal model $\mathcal{B}$ in $\mathcal{O}(Y)$ which contains the image of $\mathcal{A}$ in $\mathcal{O}(Y)$. Since $\mathcal{B}$ is topologically finitely generated and the action of each x_i on $\mathcal{O}(Y)$ is bounded we can find $m_i \geqslant 0$ such that for each $1 \leqslant i \leqslant d$, $\pi^{m_i} x_i \cdot \mathcal{B} \subseteq \mathcal{B}$. Taking $l = \sup\{m_i\}$ we see that $\mathcal{B}$ is $\pi^n\mathcal{L}$-stable for all $n \geqslant l$. □

Proposition. *There is an $m \geqslant 0$ such that Y is $\pi^n\mathcal{L}$-accessible for all $n \geqslant m$.*

Proof. First suppose that Y is a rational subdomain of X. By [11, Proposition 7.2.4/1], there is a chain $Y = Z_r \subseteq Z_{r-1} \subseteq \cdots \subseteq Z_1 = X$ such that $Z_{k+1} = Z_k(g_k)$ or $Z_{k+1} = Z_k(1/g_k)$ for some $g_k \in \mathcal{O}(Z_k)$. By part (a) of the Lemma, we may then inductively find $m_k \geqslant m_{k-1}$ (with $m_0 = 0$) and $\pi^{m_k}\mathcal{L}$-stable affine formal models $\mathcal{B}_k$ in $\mathcal{O}(Z_k)$ such that $\pi^{m_k}\mathcal{L} \cdot g_k \subseteq \mathcal{B}_k$. Then $Y \subseteq X$ is $\pi^n\mathcal{L}$-accessible for all $n \geqslant m_r$.

Returning to the general case, let l be given by part (b) of the Lemma. By [15, Theorem 4.10.4], we can find rational subdomains $X_1, \ldots, X_r$ of X such that $Y = \bigcup_{j=1}^r X_j$. By part (a) of the Lemma, we can find integers $m_1, \ldots, m_r \geqslant l$ such that each rational subdomain $X_j \subseteq X$ is a $\pi^n\mathcal{L}$-accessible for $n \geqslant m_j$. We may then take $m = \sup\{m_j\}$. □

Here is the main result of Section 7.

7.7. **Theorem.** Let $X = \operatorname{Sp} A$ be a K-affinoid variety, let $\mathcal{A}$ be an affine formal model in A, and let $\mathcal{L}$ be a smooth $\mathcal{A}$-Lie lattice in the (K, A)-Lie algebra L.

(a) If Y is an affinoid subdomain of X, then $\widehat{U(\mathcal{O}(Y) \otimes_A L)}$ is a c-flat $\widehat{U(L)}$-module on both sides.
(b) If $\{Y_1, \ldots, Y_n\}$ is an affinoid covering of X, then $\oplus_{i=1}^n \widehat{U(\mathcal{O}(Y_i) \otimes_A L)}$ is a faithfully c-flat $\widehat{U(L)}$-module on both sides.

Proof. Replacing $\mathcal{L}$ by a π-power multiple if necessary, by Proposition 7.6 we may assume that Y and each Y_i are $\pi^n\mathcal{L}$-accessible affinoid subdomains of X for all $n \geqslant 0$. Choose an $\mathcal{L}$-stable affine formal model $\mathcal{B}$ in $B := \mathcal{O}(Y)$; then $\mathcal{L}' := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ is a smooth $\mathcal{B}$-Lie lattice in $B \otimes_A L$ so using Definition 6.2, we may write

$$\widehat{U(L)} = \varprojlim \widehat{U(\pi^n\mathcal{L})_K} \quad \text{and} \quad \widehat{U(B \otimes_A L)} = \varprojlim \widehat{U(\pi^n\mathcal{L}')_K}.$$

Now $\widehat{U(L)}$ is a two-sided Fréchet-Stein algebra by Theorem 6.4, so $\widehat{U(\pi^n\mathcal{L})_K}$ is a flat $\widehat{U(L)}$-module on both sides by the two-sided version of [26, Remark 3.2]. Also $\widehat{U(\pi^n\mathcal{L}')_K}$ is a flat $\widehat{U(\pi^n\mathcal{L})_K}$-module on both sides by Theorem 4.9(a). Therefore $\widehat{U(\pi^n\mathcal{L}')_K}$ is a flat $\widehat{U(L)}$-module on both sides, and hence $\widehat{U(B \otimes_A L)}$ is a c-flat

$\widehat{U(L)}$-module on both sides by the two-sided version of Proposition 7.5(b). This establishes part (a), and part (b) follows from the two-sided version of Proposition 7.5(c) and Theorem 4.9(b). □

8. Co-admissible $\widehat{\mathscr{U}(L)}$-modules on affinoid varieties

In this Section we suppose that X is a K-affinoid variety, $\mathcal{A}$ is an affine formal model in $\mathcal{O}(X)$, $\mathcal{L}$ is a smooth $(\mathcal{R}, \mathcal{A})$-Lie algebra, and $L = \mathcal{L} \otimes_{\mathcal{R}} K$.

8.1. **Sheaves of Fréchet–Stein enveloping algebras.**

Definition. For each affinoid subdomain Y of X, write

$$\widehat{\mathscr{U}(L)}(Y) := U(\widehat{\mathcal{O}(Y) \otimes_{\mathcal{O}(X)}} L)$$

for the Fréchet completion of the enveloping algebra $U(\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} L)$.

Theorem. $\widehat{\mathscr{U}(L)}$ *is a sheaf of two-sided Fréchet–Stein algebras on* X_w.

Proof. Let Y be an affinoid subdomain of X. By replacing $\mathcal{L}$ by a π-power multiple if necessary and applying Lemma 7.6(b), we may assume that Y is $\mathcal{L}$-admissible. Let $\mathcal{B}$ be an $\mathcal{L}$-stable affine formal model in Y. Then $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ is a smooth $\mathcal{B}$-Lie lattice in $\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} L$, so $\widehat{\mathscr{U}(L)}(Y)$ is a two-sided Fréchet-Stein algebra by Theorem 6.4.

By Proposition 6.3(a), $\widehat{\mathscr{U}(L)}$ is a presheaf on X_w. Let $\mathcal{U}$ be an X_w-covering of X. By replacing $\mathcal{L}$ by a π-power multiple again if necessary and applying Lemma 7.6(b), we may assume that $\mathcal{U}$ is $\pi^n\mathcal{L}$-admissible for all $n \geqslant 0$. Now

$$\widehat{\mathscr{U}(L)}(Y) \cong \varprojlim_{n \geqslant 0} \widehat{\mathscr{U}(\pi^n \mathcal{L})_K}(Y)$$

whenever Y is an intersection of members of $\mathcal{U}$, and the complex $C^\bullet_{\mathrm{aug}}(\mathcal{U}, \widehat{\mathscr{U}(\pi^n\mathcal{L})_K})$ is exact for each $n \geqslant 0$ by Theorem 3.5. Therefore

$$C^\bullet_{\mathrm{aug}}(\mathcal{U}, \widehat{\mathscr{U}(L)}) \cong \varprojlim C^\bullet_{\mathrm{aug}}(\mathcal{U}, \widehat{\mathscr{U}(\pi^n\mathcal{L})_K})$$

is also exact, and hence $\widehat{\mathscr{U}(L)}$ is a sheaf. □

8.2. **Localisation.** For every co-admissible $\widehat{U(L)}$-module M, we can define a presheaf $\mathrm{Loc}(M)$ of $\widehat{\mathscr{U}(L)}$-modules on X_w by setting

$$\mathrm{Loc}(M)(Y) := \widehat{\mathscr{U}(L)}(Y) \underset{\widehat{U(L)}}{\widehat{\otimes}} M$$

for each affinoid subdomain Y of X. The restriction maps in $\mathrm{Loc}(M)$ are obtained from the associativity isomorphism

$$\widehat{\mathscr{U}(L)}(Z) \underset{\widehat{\mathscr{U}(L)}(Y)}{\widehat{\otimes}} \left(\widehat{\mathscr{U}(L)}(Y) \underset{\widehat{U(L)}}{\widehat{\otimes}} M \right) \quad \cong \quad \widehat{\mathscr{U}(L)}(Z) \underset{\widehat{U(L)}}{\widehat{\otimes}} M$$

given by Corollary 7.4, which is applicable by Proposition 6.3(a).

Theorem. Loc *defines a full exact embedding of abelian categories from the category of co-admissible* $\widehat{U(L)}$*-modules to the category of sheaves of* $\widehat{\mathscr{U}(L)}$*-modules on* X_w *with vanishing higher Čech cohomology groups.*

Proof. Suppose that $f : M \to N$ is a morphism of co-admissible $\widehat{U(L)}$-modules. By the universal property of $\widehat{\otimes}$, for each Y in X_w there is a unique morphism of $\widehat{\mathscr{U}(L)}(Y)$-modules $\mathrm{id}\widehat{\otimes}f : \mathrm{Loc}(M)(Y) \to \mathrm{Loc}(N)(Y)$ making the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ \mathrm{Loc}(M)(Y) & \xrightarrow[\mathrm{id}\widehat{\otimes}f]{} & \mathrm{Loc}(M)(X) \end{array}$$

commute. It is now easy to see that Loc is full and faithful functor as claimed.

Suppose now that $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of co-admissible $\widehat{U(L)}$-modules. Since $\widehat{\mathscr{U}(L)}(Y)$ is a c-flat $\widehat{\mathscr{U}(L)}(X)$-module on both sides for each $Y \in X_w$ by Theorem 7.7, each sequence

$$0 \to \mathrm{Loc}(M_1)(Y) \to \mathrm{Loc}(M_2)(Y) \to \mathrm{Loc}(M_3)(Y) \to 0$$

is exact. This suffices to see that Loc is exact.

Finally, we prove that if M is a co-admissible $\widehat{U(L)}$-module, then any X_w-covering $\mathcal{U}$ of an affinoid subdomain Y of X is $\mathrm{Loc}(M)$-acyclic. This will imply that $\mathrm{Loc}(M)$ is a sheaf on X_w with vanishing higher Čech cohomology groups.

Using Proposition 7.6, we may assume that $\mathcal{U}$ is a $\pi^n\mathcal{L}$-accessible covering of Y for each $n \geqslant 0$. Write $M = \varprojlim M_n$ where $M_n := \widehat{U(\pi^n\mathcal{L})_K} \otimes_{\widehat{U(L)}} M$, and consider the sheaves $\mathcal{M}_n := \mathrm{Loc}(M_n)$ of $\widehat{\mathscr{U}(\pi^n\mathcal{L})_K}$-modules on $X_{ac}(\pi^n\mathcal{L})$. By Proposition 5.1, the augmented Čech complexes $C^\bullet_{\mathrm{aug}}(\mathcal{U}, \mathcal{M}_n)$ are exact for each $n \geqslant 0$.

Now $\mathrm{Loc}(M)(Y) = \varprojlim \mathcal{M}_n(Y)$ and $\mathrm{Loc}(M)(U) = \varprojlim \mathcal{M}_n(U)$ for each $U \in \mathcal{U}$. Moreover, by [26, Theorem B], $\varprojlim{}^{(j)}\mathcal{M}_n(Y) = 0$ and $\varprojlim{}^{(j)}\mathcal{M}_n(U) = 0$ for each $j > 0$ and each $U \in \mathcal{U}$. Consider the exact complex of towers of $\widehat{\mathscr{U}(L)}(Y)$-modules

$$C^\bullet_{\mathrm{aug}}(\mathcal{U}, (\mathcal{M}_n)).$$

An induction starting with the left-most term shows that $\varprojlim{}^{(j)}$ is zero on the kernel of every differential in this complex, for all $j > 0$. Therefore $\varprojlim C^\bullet_{\mathrm{aug}}(\mathcal{U}, \mathcal{M}_n)$ is exact. But this complex is isomorphic to $C^\bullet_{\mathrm{aug}}(\mathcal{U}, \mathrm{Loc}(M))$. $\square$

8.3. **Co-admissible $\widehat{\mathscr{U}(L)}$-modules.** We will now start to study the essential image of Loc.

Definition. Let $\mathscr{M}$ be a $\widehat{\mathscr{U}(L)}$-module. Given an X_w-covering $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X, we say that $\mathscr{M}$ is *$\mathcal{U}$-co-admissible* if for each $1 \leqslant i \leqslant n$ there is a co-admissible $\widehat{\mathscr{U}(L)}(U_i)$-module M_i such that $\mathscr{M}|_{U_i}$ is isomorphic to $\mathrm{Loc}(M_i)$ as sheaves of $\widehat{\mathscr{U}(L)}|_{U_i}$-modules. We say that $\mathscr{M}$ is *co-admissible* if there is some X_w-covering $\mathcal{U}$ of X such that $\mathscr{M}$ is $\mathcal{U}$-co-admissible.

Proposition. *Suppose that $\alpha\colon \mathscr{M} \to \mathscr{N}$ is a morphism of $\mathcal{U}$-co-admissible $\widehat{\mathscr{U}(L)}$-modules for some admissible covering $\mathcal{U}$. Then* $\ker\alpha, \mathrm{coker}\,\alpha$ *and* $\mathrm{Im}\,\alpha$ *are each $\mathcal{U}$-co-admissible.*

Proof. We can compute using Theorem 8.2 that $(\ker\alpha)|_{U_i} \cong \mathrm{Loc}(\ker\alpha(U_i))$, that $(\mathrm{coker}\,\alpha)|_{U_i} \cong \mathrm{Loc}(\mathrm{coker}\,\alpha(U_i))$ and that $\mathrm{Im}\,\alpha|_{U_i} = \mathrm{Loc}(\mathrm{Im}\,\alpha(U_i))$. $\square$

Lemma. *Suppose that $\mathscr{M}$ is a sheaf of $\widehat{\mathscr{U}(L)}$-modules isomorphic to* $\operatorname{Loc}(M)$ *for some co-admissible $\widehat{U(L)}$-module M. Then the sheaf* $\operatorname{Loc}\left(\widehat{U(\mathcal{L})_K} \otimes_{\widehat{U(L)}} M\right)$ *on $X_{ac}(\mathcal{L})$ has sections given by $Z \mapsto \widehat{\mathscr{U}(\mathcal{L})_K}(Z) \otimes_{\widehat{\mathscr{U}(L)}(Z)} \mathscr{M}(Z)$.*

Proof. The commutative diagram

$$\begin{tikzcd}
\widehat{U(L)} \arrow[r, equal] \arrow[d] & \widehat{\mathscr{U}(L)}(X) \arrow[r] \arrow[d] & \widehat{\mathscr{U}(L)}(Z) \arrow[d] \\
\widehat{U(\mathcal{L})_K} \arrow[r, equal] & \widehat{\mathscr{U}(\mathcal{L})_K}(X) \arrow[r] & \widehat{\mathscr{U}(\mathcal{L})_K}(Z)
\end{tikzcd}$$

induces an isomorphism

$$\widehat{\mathscr{U}(\mathcal{L})_K}(Z) \underset{\widehat{U(\mathcal{L})_K}}{\otimes} \widehat{U(\mathcal{L})_K} \underset{\widehat{U(L)}}{\otimes} M \quad \cong \quad \widehat{\mathscr{U}(\mathcal{L})_K}(Z) \underset{\widehat{\mathscr{U}(L)}(Z)}{\otimes} \widehat{\mathscr{U}(L)}(Z) \underset{\widehat{U(L)}}{\widehat{\otimes}} M$$

and the result follows. $\square$

8.4. **Kiehl's Theorem.** Here is the main result of Section 8, which shows that co-admissible $\widehat{U(L)}$-modules may be obtained by patching together appropriate local information. It can be viewed as an analogue of the classical Theorem of Kiehl [15, Theorem 4.5.2] on coherent $\mathcal{O}$-modules on rigid analytic spaces.

Theorem. *Let $\mathscr{M}$ be a sheaf of $\widehat{\mathscr{U}(L)}$-modules on X_w. Then the following are equivalent.*

(a) $\mathscr{M}$ is co-admissible.

(b) $\mathscr{M}$ is $\mathcal{U}$-co-admissible for all X_w-coverings $\mathcal{U}$ of X.

(c) $\mathscr{M}$ is isomorphic to $\operatorname{Loc}(M)$ *for some co-admissible $\widehat{U(L)}$-module M.*

Proof. Note that (c) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (a) are trivial. We will prove (a) $\Longrightarrow$ (c).

Suppose that $\mathcal{U}$ is a covering of X by affinoid subdomains such that $\mathscr{M}$ is $\mathcal{U}$-co-admissible. By [11, Lemmas 8.2.2/2-4], $\mathcal{U}$ may be refined to a Laurent covering $\mathcal{V} = \{X(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}) \mid \alpha_i \in \{\pm 1\}\}$ for some $f_1, \ldots, f_m \in \mathcal{O}(X)$. Certainly $\mathscr{M}$ is $\mathcal{V}$-co-admissible so we may, without loss of generality, assume that $\mathcal{U} = \mathcal{V}$. Using Proposition 7.6, we may also assume that $\mathcal{U}$ is $\pi^n\mathcal{L}$-accessible for all $n \geqslant 0$.

In an attempt to improve readability, we write $\mathcal{S}_n$ for the sheaf $\widehat{\mathscr{U}(\pi^n\mathcal{L})_K}$ on $X_{ac}(\pi^n\mathcal{L})$ and $\mathcal{S}_\infty$ for the sheaf $\widehat{\mathscr{U}(L)}$ on X_w, so that $\mathcal{S}_\infty(X) \cong \varprojlim \mathcal{S}_n(X)$ and

$$\mathcal{S}_\infty(Y) \cong \varprojlim \mathcal{S}_n(Y) \quad \text{for all} \quad Y \in \mathcal{U}.$$

Fix $n \geqslant 0$. Consider the sheafification $\mathcal{M}_n$ of the presheaf $Z \mapsto \mathcal{S}_n(Z) \otimes_{\mathcal{S}_\infty(Z)} \mathscr{M}(Z)$ on $X_{ac}(\pi^n\mathcal{L})$. Let $Y \in \mathcal{U}$, so that $\mathscr{M}(Y)$ is a co-admissible $\mathcal{S}_\infty(Y)$-module, and $\mathscr{M}_{|Y_w}$ is isomorphic to $\operatorname{Loc}(\mathscr{M}(Y))$ by assumption. By Lemma 8.3 applied to $\mathscr{M}_{|Y_w}$ there are isomorphisms

$$\mathcal{M}_n|_{Y_w} \cong \operatorname{Loc}\left(\mathcal{S}_n(Y) \otimes_{\mathcal{S}_\infty(Y)} \mathscr{M}(Y)\right)$$

for each $Y \in \mathcal{U}$. Thus $\mathcal{M}_n$ is a $\mathcal{U}$-coherent $\mathcal{S}_n$-module, so there is a finitely generated $\mathcal{S}_n(X)$-module M_n and an isomorphism $\operatorname{Loc}(M_n) \xrightarrow{\cong} \mathcal{M}_n$ by Theorem 5.5.

Now $\operatorname{Loc}(M_n) \cong \operatorname{Loc}(\mathcal{S}_n(X) \otimes_{\mathcal{S}_{n+1}(X)} M_{n+1})$ as sheaves on $X_{ac}(\pi^n\mathcal{L})$, because they have the same local sections on $\mathcal{U}$. Thus $M_\infty := \varprojlim M_n$ is a co-admissible $\mathcal{S}_\infty(X)$-module. We will show that $\operatorname{Loc}(M_\infty)$ is isomorphic to our sheaf $\mathscr{M}$.

Let θ_n denote the $\mathcal{S}_\infty(X)$-linear map $M_\infty \to \mathcal{M}_n(X)$ defined by the composite of the natural map $M_\infty \to M_n$ and the global sections of the isomorphism $\operatorname{Loc}(M_n) \to \mathcal{M}_n$. Let $Y \in \mathcal{U}$. Combining the isomorphism

$$\operatorname{Loc}(M_n)(Y) = \mathcal{S}_n(Y) \otimes_{\mathcal{S}_n(X)} M_n \xrightarrow{\cong} \mathcal{M}_n(Y)$$

together with the canonical isomorphism $M_n \cong \mathcal{S}_n(X) \otimes_{\mathcal{S}_\infty(X)} M_\infty$ given by [26, Corollary 3.1] produces a compatible family of isomorphisms

$$\alpha_n(Y) : \mathcal{S}_n(Y) \otimes_{\mathcal{S}_\infty(X)} M_\infty \xrightarrow{\cong} \mathcal{M}_n(Y)$$

given by the $\mathcal{S}_\infty(X)$-balanced map $(s, m) \mapsto s \cdot \theta_n(m)|_Y$.

Passing to the limit as $n \to \infty$ gives an isomorphism of $\mathcal{S}_\infty(Y)$-modules

$$\alpha(Y)\colon \operatorname{Loc}(M_\infty)(Y) = \mathcal{S}_\infty(Y)\widehat{\otimes}_{\mathcal{S}_\infty(X)} M_\infty \xrightarrow{\cong} \mathscr{M}(Y)$$

given by the $\mathcal{S}_\infty(X)$-balanced map $(s, m) \mapsto s \cdot \lim(\theta_n(m)|_Y)$. Since $\mathscr{M}_{|Y} \cong \operatorname{Loc}(\mathscr{M}(Y))$ by assumption, Theorem 8.2 gives an isomorphism

$$\alpha_Y : \operatorname{Loc}(M_\infty)_{|Y} \xrightarrow{\cong} \mathscr{M}_{|Y}$$

of sheaves of $\mathcal{S}_\infty|_Y$-modules whose local sections

$$\alpha_Y(Z)\colon \mathcal{S}_\infty(Z) \underset{\mathcal{S}_\infty(X)}{\widehat{\otimes}} M_\infty \to \mathscr{M}(Z)$$

are given by $\alpha_Y(Z)(s\widehat{\otimes} m) = s \cdot \lim(\theta_n(m)|_Y)|_Z$, whenever Z is an affinoid subdomain of Y. Because $\lim(\theta_n|_Y)|_Z = \lim \theta_n|_Z$, it follows that

$$\alpha_Y(Y \cap Y') = \alpha_{Y'}(Y \cap Y') \quad \text{for every} \quad Y, Y' \in \mathcal{U}.$$

Hence the α_Y patch together to an isomorphism of sheaves $\alpha\colon \operatorname{Loc}(M_\infty) \to \mathscr{M}$. $\square$

9. Sheaves on rigid analytic spaces

In this Section X is a rigid K-analytic space.

9.1. **Lie algebroids.** Let X_w denote the subset of X_{rig} consisting of the affinoid subdomains of X. Since we do not assume that X is separated, X_w is not closed under intersections in X_{rig} and thus is not a G-topology on X in general. However, every admissible open subset in X_{rig} has an admissible cover by affinoid subdomains of X.

Definition. [11, §9.2.1] A subset $\mathcal{B}$ of objects of X_{rig} is a *basis* for the topology if every admissible open has an admissible cover by objects in $\mathcal{B}$.

In particular, X_w is a basis of X.

Definition. If $\mathcal{B}$ is a basis of X, a presheaf $\mathcal{F}$ on $\mathcal{B}$ is a *sheaf* if for every admissible cover $\{U_i\}$ of U by objects in $\mathcal{B}$ and any choice of admissible covers $\{W_{ijk}\}$ of $U_i \cap U_j$,

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(W_{ijk})$$

is exact.

Theorem. *Suppose that $\mathcal{B} \subseteq X_{\mathrm{rig}}$ is a basis for the topology X. The restriction functor induces an equivalence of categories between sheaves on X_{rig} and sheaves on $\mathcal{B}$.*

This is a consequence of the Comparison Lemma [18, Theorem C.2.2.3], but we give a proof in Appendix A for the convenience of the reader.

Proposition. *There is a coherent sheaf $\mathcal{T}_X$ of K-Lie algebras on X_{rig} with*

$$\mathcal{T}_X(U) := \mathrm{Der}_K\, \mathcal{O}(U)$$

for every affinoid subdomain U of X. Moreover, for all admissible open subsets Y of X, $\mathcal{T}_X(Y)$ acts by derivations on $\mathcal{O}(Y)$.

Proof. We define the restriction maps $\mathcal{T}_X(U) \to \mathcal{T}_X(V)$ for $V \subseteq U$ affinoid subdomains in X using Lemma 2.4. By the uniqueness part of that Lemma this defines a presheaf of K-Lie algebras on X_w. Let $\{U_i\}$ be an admissible affinoid cover of an affinoid subdomain U of X. Then it is routine to check that the sequence

$$0 \to \mathcal{T}_X(U) \to \prod \mathcal{T}_X(U_i) \to \prod \mathcal{T}_X(U_i \cap U_j)$$

is exact, so $\mathcal{T}_X$ defines a sheaf of K-Lie algebras on X_w. By the Theorem, this extends to a sheaf of K-Lie algebras on X_{rig}. A similarly routine verification shows that $\mathcal{T}_X(Y)$ acts by derivations on $\mathcal{O}(Y)$ whenever Y is an admissible open subset of X. □

We call the sheaf $\mathcal{T}_X$ constructed in the Proposition the *tangent sheaf* of X.

Definition. A *Lie algebroid* on X is a pair $(\rho, \mathscr{L})$ such that

- $\mathscr{L}$ is a locally free sheaf of $\mathcal{O}$-modules of finite rank on X_{rig},
- $\mathscr{L}$ has the structure of a sheaf of K-Lie algebras, and
- $\rho\colon \mathscr{L} \to \mathcal{T}$ is an $\mathcal{O}$-linear map of sheaves of Lie algebras such that

$$[x, ay] = a[x, y] + \rho(x)(a)y$$

whenever U is an admissible open subset of X, $x, y \in \mathscr{L}(U)$ and $a \in \mathcal{O}(U)$.

For example, if X is smooth, then the tangent sheaf $\mathcal{T}_X$ is locally free of finite rank by definition, and thus $(\mathrm{id}_{\mathcal{T}_X}, \mathcal{T}_X)$ is a Lie algebroid on X by the Proposition.

9.2. **Lie-Rinehart algebras and Lie algebroids.** If $(\rho, \mathscr{L})$ is a Lie algebroid on X, then $(\rho(U), \mathscr{L}(U))$ is a $(K, \mathcal{O}(U))$-Lie algebra for every admissible open subset U of X. Moreover every affinoid subdomain U of X, $\mathscr{L}(U)$ is smooth by [15, Proposition 4.7.2].

Definition. A *morphism* $(\rho, \mathscr{L}) \to (\rho', \mathscr{L}')$ *of Lie algebroids* on X is a morphism of sheaves $\theta\colon \mathscr{L} \to \mathscr{L}'$ such that $\theta(U)$ is a morphism of $(K, \mathcal{O}(U))$-Lie algebras for every $U \subseteq X$ in X_{rig}.

Lemma. *Let $Y = \mathrm{Sp}(A)$ be a K-affinoid variety. The global sections functor $\Gamma(Y, -)$ defines an equivalence of categories between the category of Lie algebroids on Y_{rig} and the category of smooth (K, A)-Lie algebras.*

Proof. First, suppose that (L, ρ) is a smooth (K, A)-Lie algebra and define $\mathrm{Loc}(L)$ to be the locally free sheaf on Y_w given by $\mathrm{Loc}(L)(U) = \mathcal{O}(U) \otimes_A L$ for $U \subseteq Y$ affinoid and natural restriction maps. By Corollary 2.4 there is a unique structure of a $(K, \mathcal{O}(U))$-Lie algebra on $\mathrm{Loc}(L)(U)$ with anchor map $\rho(U)$ so that

$$\begin{tikzcd} L \arrow[r, "\rho"] \arrow[d] & \mathrm{Der}_K(A) \arrow[d] \\ \mathrm{Loc}(L)(U) \arrow[r, "\rho(U)"] & \mathcal{T}_Y(U) \end{tikzcd}$$

commutes. Suppose that $V \subseteq U$ are affinoid subdomains of Y, and consider the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\rho"] & \mathrm{Loc}(L)(U) \arrow[r] \arrow[d, "\rho(U)"] & \mathrm{Loc}(L)(V) \arrow[d, "\rho'(V)"] \\ \mathrm{Der}_K(A) \arrow[r] & \mathcal{T}_Y(U) \arrow[r] & \mathcal{T}_Y(V) \end{tikzcd}$$

where $\rho'(V)$ is the anchor map for the unique $(K, \mathcal{O}(V))$-Lie algebra structure on $\mathrm{Loc}(L)(V)$ making the right-hand square commute. Since the left-hand square also commutes, the outer square must commute and $\rho'(V) = \rho(V)$ by the uniqueness of $\rho(V)$. Thus $\rho\colon \mathrm{Loc}(L) \to \mathcal{T}_Y|_{Y_w}$ is a morphism of sheaves of Lie algebras on Y_w. By [11, Proposition 9.2.3/1], $\mathrm{Loc}(L)$ extends to a Lie algebroid $\mathrm{Loc}(L)$ on Y_{rig}.

Now, suppose that $f\colon L \to L'$ is a morphism of (K, A)-Lie algebras. By Corollary 2.4 there is a unique morphism of sheaves on Y_w

$$\mathrm{Loc}(f)\colon \mathrm{Loc}(L)|_{Y_w} \to \mathrm{Loc}(L')|_{Y_w}$$

such that $\mathrm{Loc}(f)(Y) = f$, given by $\mathrm{Loc}(f)(U) = \mathcal{O}(U) \otimes_A f$ for affinoid subdomains $U \subseteq Y$. By [11, Proposition 9.2.3/1] again, $\mathrm{Loc}(f)$ extends to a morphism of Lie algebroids. Thus Loc defines a functor inverse to $\Gamma(Y, -)$. □

Corollary. *If $(\rho, \mathscr{L})$ is a Lie algebroid on a rigid K-analytic space X, then for every affinoid subdomain U of X, $\mathscr{L}|_U \cong \mathrm{Loc}(\mathscr{L}(U))$.*

9.3. **The Fréchet completion of $\mathscr{U}(\mathscr{L})$.** We will need to work with a slightly coarser basis for X_{rig} than X_w.

Definition. Let $\mathscr{L}$ be a Lie algebroid on the rigid K-analytic space X. We say that $\mathscr{L}(X)$ *admits a smooth Lie lattice* if there is an affine formal model $\mathcal{A}$ in $\mathcal{O}(Y)$ and a smooth $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\mathscr{L}(Y)$. We let $X_w(\mathscr{L})$ denote the set of affinoid subdomains Y of X such that $\mathscr{L}(Y)$ admits a smooth Lie lattice.

Lemma. *$X_w(\mathscr{L})$ is a basis for X.*

Proof. Suppose that Y is an affinoid subdomain of X such that $\mathscr{L}(Y)$ is a free $\mathcal{O}(Y)$-module. Choose an affine formal model $\mathcal{A}$ in $\mathcal{O}(Y)$; then $\mathscr{L}(Y)$ has a free $\mathcal{A}$-lattice spanned by a generating set for $\mathscr{L}(Y)$ as an $\mathcal{O}(Y)$-module, and some π-power multiple of this lattice will be a free $\mathcal{A}$-Lie lattice by Lemma 6.1(c). Thus $\mathscr{L}(Y)$ has a smooth $\mathcal{A}$-Lie lattice whenever $\mathscr{L}(Y)$ is a free $\mathcal{O}(Y)$-module, so $X_w(\mathscr{L})$ is a basis for X since $\mathscr{L}$ is a locally free $\mathcal{O}$-module. □

Theorem. *Let X be a rigid K-analytic space. There is a natural functor $\widehat{\mathscr{U}(-)}$ from Lie algebroids on X to sheaves of K-algebras on X_{rig} such that there is a canonical isomorphism*

$$\widehat{\mathscr{U}(\mathscr{L})}|_{Y_w} \cong \widehat{\mathscr{U}(\mathscr{L}(Y))}$$

for every $Y \in X_w(\mathscr{L})$.

Proof. Given a Lie algebroid $\mathscr{L}$ on X, let $\widehat{\mathscr{U}(\mathscr{L})}$ be the presheaf of K-algebras on X_w given by

$$\widehat{\mathscr{U}(\mathscr{L})}(Y) := \widehat{U(\mathscr{L}(Y))}$$

on affinoid subdomains Y of X, with restriction maps given by Proposition 6.3(a).

By Theorem 8.1, $\widehat{\mathscr{U}(\mathscr{L})}$ is a sheaf of K-algebras on $X_w(\mathscr{L})$. Because $X_w(\mathscr{L})$ is a basis for X by the Lemma, $\widehat{\mathscr{U}(\mathscr{L})}$ extends uniquely to a sheaf of K-algebras on X_{rig} by Theorem 9.1.

Moreover if $\mathscr{L} \to \mathscr{L}'$ is a morphism of Lie algebroids on X then Proposition 6.3(b), together with Lemma A.1, gives a morphism of sheaves of K-algebras $\widehat{\mathscr{U}(\mathscr{L})} \to \widehat{\mathscr{U}(\mathscr{L}')}$ on X_{rig} in a functorial way. □

Definition. We call the sheaf $\widehat{\mathscr{U}(\mathscr{L})}$ constructed in the Theorem the *Fréchet completion of* $\mathscr{U}(\mathscr{L})$. If X is smooth, $\mathscr{L} = \mathcal{T}$ and $\rho = 1_{\mathcal{T}}$, we call

$$\widehat{\mathcal{D}} := \widehat{\mathscr{U}(\mathcal{T})}$$

the *Fréchet completion of* $\mathcal{D}$.

9.4. **Co-admissible sheaves of modules.** Let $\mathscr{L}$ be a Lie algebroid on X. By analogy with the definition of coherent sheaves given in [17, §II.5], we make the following

Definition. A sheaf of $\widehat{\mathscr{U}(\mathscr{L})}$-modules $\mathscr{M}$ on X_{rig} is *co-admissible* if there is an admissible covering $\{U_i\}$ of X by affinoids in $X_w(\mathscr{L})$ such that $\mathscr{M}|_{U_{i,w}}$ is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}|_{U_{i,w}}$-module for all i in the sense of Definition 8.3.

We record three equivalent ways of thinking about co-admissible modules.

Theorem. *The following are equivalent for a sheaf $\mathscr{M}$ of $\widehat{\mathscr{U}(\mathscr{L})}$-modules on X_{rig}:*
(a) $\mathscr{M}$ is co-admissible,
(b) $\mathscr{M}|_{U_w}$ is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}|_{U_w}$-module for every $U \in X_w(\mathscr{L})$,
(c) $\mathscr{M}(U)$ is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}(U)$-module, and the natural map

$$\widehat{\mathscr{U}(\mathscr{L})}(V) \underset{\widehat{\mathscr{U}(\mathscr{L})}(U)}{\widehat{\otimes}} \mathscr{M}(U) \longrightarrow \mathscr{M}(V)$$

is an isomorphism whenever $V, U \in X_w(\mathscr{L})$ and $V \subseteq U$.

Proof. (a) $\Rightarrow$ (b). Let $\{U_i\}$ be an admissible affinoid covering of X such that $\mathscr{M}|_{U_{i,w}}$ is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}|_{U_{i,w}}$-module and $U_i \in X_w(\mathscr{L})$ for all i. Let U be another object of $X_w(\mathscr{L})$; then $\{U \cap U_i\}$ is an admissible cover of U. Choose an admissible affinoid covering $\{V_{ij}\}_j$ of $U \cap U_i$ for each i; then $\{V_{ij}\}_{i,j}$ is an admissible affinoid covering of U and therefore admits a finite subcovering $\mathcal{W}$, say. Now $\mathscr{M}|_{U_w}$ is $\mathcal{W}$-co-admissible in the sense of Definition 8.3 since each $W \in \mathcal{W}$ is an affinoid subdomain of some U_i.

(b) $\Rightarrow$ (c). Let $U \in X_w(\mathscr{L})$. By Theorem 8.4, $\mathscr{M}|_{U_w}$ is isomorphic to $\mathrm{Loc}(M_U)$ for some co-admissible $\widehat{\mathscr{U}(\mathscr{L})}(U)$-module M_U. Applying $\Gamma(U, -)$ shows that $M_U = \mathrm{Loc}(M_U)(U) \cong \mathscr{M}(U)$, so $\mathscr{M}(U)$ is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}(U)$-module. Hence

$$\mathscr{M}(V) \cong \mathrm{Loc}\left(\mathscr{M}(U)\right)(V) = \widehat{\mathscr{U}(\mathscr{L})}(V) \underset{\widehat{\mathscr{U}(\mathscr{L})}(U)}{\widehat{\otimes}} \mathscr{M}(U)$$

for every affinoid subdomain V of U.

(c) $\Rightarrow$ (a). Using Lemma 9.3, choose an admissible covering $\{U_i\}$ of X by affinoids in $X_w(\mathscr{L})$, and let $M_i := \mathscr{M}(U_i)$ for each i. Then M_i is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}(U_i)$-module, and there is a natural isomorphism of sheaves of $\widehat{\mathscr{U}(\mathscr{L})}|_{U_{i,w}}$-modules

$$\operatorname{Loc}(M_i) \xrightarrow{\cong} \mathscr{M}|_{U_{i,w}}$$

for each i, by assumption. Hence $\mathscr{M}$ is co-admissible. $\square$

It follows readily from Proposition 8.3 that the full subcategory of sheaves of $\widehat{\mathscr{U}(\mathscr{L})}$-modules on X_{rig} whose objects are the co-admissible $\widehat{\mathscr{U}(\mathscr{L})}$-modules is abelian.

9.5. **Two corollaries.** We begin with a more general version of Corollary 1.4, which follows immediately from Theorems 8.2 and 8.4.

Theorem. *Suppose that $\mathscr{L}$ is a Lie algebroid on a K-affinoid variety X such that $\mathscr{L}(X)$ admits a smooth Lie lattice. Then* Loc *is an equivalence of abelian categories*

$$\left\{ \begin{array}{c} \text{co–admissible} \\ \widehat{\mathscr{U}(\mathscr{L})}(X)\text{ –modules} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{co–admissible sheaves of} \\ \widehat{\mathscr{U}(\mathscr{L})}\text{–modules on } X \end{array} \right\}.$$

Given an abelian sheaf $\mathcal{F}$ on X, write $H^\bullet(X, \mathcal{F})$ to denote the sheaf cohomology of $\mathcal{F}$, and let $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$ denote the Čech cohomology of $\mathcal{F}$ with respect to the covering $\mathcal{U}$.

Proposition. *Suppose that X is separated, $\mathscr{M}$ is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}$-module and $\mathcal{U}$ is any cover of X by affinoids in $X_w(\mathscr{L})$. Then*

$$H^i(X, \mathscr{M}) = \check{H}^i(\mathcal{U}, \mathscr{M})$$

for all $i \geqslant 0$. In particular, $H^i(X, \mathscr{M}) = 0$ for $i \geqslant |\mathcal{U}|$.

Proof. Since X is separated, every finite intersection V of elements of $\mathcal{U}$ is affinoid. Thus by Theorem 9.4, $\mathscr{M}|_V$ is a co-admissible $\widehat{\mathscr{U}(\mathscr{L})}|_V$-module and so has vanishing Čech cohomology groups by Theorem 8.4 and Theorem 8.2. Hence the spectral sequence (1) for the covering $\mathcal{U}$ and the sheaf $\mathscr{M}$ from Subsection 3.4 collapses on page 2, and the result follows. $\square$

It follows from the Proposition that in the setting of the Theorem, the global sections functor $\Gamma(X, -)$ is an exact quasi-inverse to the localisation functor Loc.

Appendix A.

When X is affinoid, [11, Proposition 9.2.3/1] gives that the restriction functor from sheaves on X_{rig} to sheaves on X_w is an equivalence of categories. In this appendix we extend this result to bases for general rigid K-analytic spaces X.

A.1. **Lemma.** Suppose that $\mathcal{B} \subseteq X_{\mathrm{rig}}$ is a basis. The restriction functor r from sheaves on X_{rig} to the category of sheaves on $\mathcal{B}$ is full and faithful.

Proof. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X_{rig} and θ is a natural transformation from $r(\mathcal{F})$ to $r(\mathcal{G})$. We must show that θ extends uniquely to a morphism of sheaves $t\colon \mathcal{F} \to \mathcal{G}$.

Suppose that U is an admissible open subset of X and $\mathcal{U} = \{U_i\}$ is a admissible cover of U by U_i in $\mathcal{B}$. Since $\mathcal{G}$ is a sheaf on X_{rig},

$$\mathcal{G}(U) \to \prod \mathcal{G}(U_i) \rightrightarrows \prod_{i,j} \mathcal{G}(U_i \cap U_j)$$

is exact. For each pair i, j, choose an admissible cover $\{W_{ijk}\}$ of $U_i \cap U_j$ by objects in $\mathcal{B}$. Since $\mathcal{G}$ is a sheaf, $\mathcal{G}(U_i \cap U_j) \to \prod_k \mathcal{G}(W_{ijk})$ is a monomorphism for each pair i, j and so $\mathcal{G}(U)$ is also the equaliser of $\prod \mathcal{G}(U_i) \rightrightarrows \prod_{ijk} \mathcal{G}(W_{ijk})$. Thus $\mathcal{G}|_{\mathcal{B}}$ is a sheaf on $\mathcal{B}$.

Since θ is a natural transformation, the two composites

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{G}(U_i) \rightrightarrows \prod \mathcal{G}(W_{ijk})$$

agree. Thus there is a unique $t(\mathcal{U}) \in \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ such that

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "t(\mathcal{U})"] & \prod \mathcal{F}(U_i) \arrow[d, "\prod \theta(U_i)"] \\
\mathcal{G}(U) \arrow[r] & \prod \mathcal{G}(U_i)
\end{tikzcd}$$

commutes. Next, suppose that $\mathcal{V} = \{V_j\}$ is a refinement of $\mathcal{U}$ with each V_j in $\mathcal{B}$. Then

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "t(\mathcal{U})"] & \prod \mathcal{F}(U_i) \arrow[r] \arrow[d, "\prod \theta(U_i)"] & \prod \mathcal{F}(V_j) \arrow[d, "\prod \theta(V_j)"] \\
\mathcal{G}(U) \arrow[r] & \prod \mathcal{G}(U_i) \arrow[r] & \prod \mathcal{G}(V_j)
\end{tikzcd}$$

also commutes, so $t(\mathcal{U}) = t(\mathcal{V})$. Since any two such covers of U have a common refinement, we see that $t(U) := t(\mathcal{U})$ does not depend on the choice of cover of U. In particular if U is in $\mathcal{B}$, $t(U) = \theta(U)$.

Now suppose that $V \subseteq U$ are admissible opens in X_{rig} with U in $\mathcal{B}$. Let $\{V_i\}$ be an admissible cover of V by objects in $\mathcal{B}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "\theta(U)"] & \mathcal{F}(V) \arrow[r] \arrow[d, "t(V)"] & \prod \mathcal{F}(V_i) \arrow[d, "\theta(V)"] \\
\mathcal{G}(U) \arrow[r] & \mathcal{G}(V) \arrow[r] & \prod \mathcal{G}(V_i).
\end{tikzcd}$$

The outer square commutes because θ is a natural transformation. The right-hand square commutes by the construction of $t(V)$. Since $\mathcal{G}$ is a sheaf, the bottom rightmost horizontal morphism is a monomorphism so it follows that the left-hand square commutes.

Finally, consider $V \subseteq U$ for general admissible opens in X_{rig}. Let $\{U_i\}$ be an admissible cover of U by objects in $\mathcal{B}$ and define $V_i := V \cap U_i$ so that $\{V_i\}$ is an admissible cover of V. Then consider the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "t(U)"] & \prod \mathcal{F}(U_i) \arrow[r] \arrow[d, "\prod \theta(U_i)"] & \prod \mathcal{F}(V_i) \arrow[d, "\prod t(V_i)"] \\
\mathcal{G}(U) \arrow[r] & \prod \mathcal{G}(U_i) \arrow[r] & \prod \mathcal{G}(V_i).
\end{tikzcd}$$

The left-hand square commutes by construction of $t(U)$. The right-hand square commutes by the previous paragraph since each U_i is in $\mathcal{B}$. Thus the outer square commutes. By repeating the argument used in the case U is in $\mathcal{B}$ we see that t is the unique morphism of sheaves extending θ as required. $\square$

A.2. **Proposition.** Suppose $\mathcal{B} \subseteq X_{\mathrm{rig}}$ is a basis of X. The essential image of the restriction functor from sheaves on X_{rig} to presheaves on $\mathcal{B}$ consists of the sheaves on $\mathcal{B}$.

Proof. Suppose that F is a sheaf on $\mathcal{B}$. We will construct a sheaf $\mathcal{F}$ on X_{rig} whose restriction is naturally isomorphic to F. Suppose U is an admissible open subvariety of X and $\mathcal{U} = \{U_i \mid i \in I\}$ is an admissible cover of U by objects in $\mathcal{B}$. For each $i, j \in I$ let $\mathcal{V}_{ij} = \{V_{ijk}\}$ denote an admissible cover of $U_i \cap U_j$ by objects in $\mathcal{B}$. Then define $H^0(\mathcal{U}, F)$ to be the equaliser of $\prod_{i\in I} F(U_i) \rightrightarrows \prod F(V_{ijk})$. Since F is a sheaf on $\mathcal{B}$, if $\mathcal{W}_{ij} = \{W_{ijl}\}$ is a refinement of $\mathcal{V}_{ij}$ then $\prod F(V_{ijk}) \to \prod F(W_{ijl})$ is a monomorphism. Thus $H^0(\mathcal{U}, F)$ only depends on the choice of cover $\mathcal{U}$ not on the choice of $\mathcal{V}_{ij}$. Note also that, by the definition of a sheaf on $\mathcal{B}$, $H^0(\mathcal{U}, F) = F(U)$ whenever $\mathcal{U}$ is a admissible cover of $U \in \mathcal{B}$.

Now, we can define for any admissible open subset U of X

$$\mathcal{F}(U) := \varinjlim H^0(\mathcal{U}, F)$$

where the direct limit is over all covers of U by objects in $\mathcal{B}$. In particular $\mathcal{F}(U) \cong F(U)$ for $U \in \mathcal{B}$. Suppose that $V \subseteq U$ are admissible open subsets of X. If $\mathcal{U} = \{U_i \mid i \in I\}$ is an admissible cover of U by objects in $\mathcal{B}$ then $\mathcal{V} = \{U_i \cap V \mid i \in I\}$ is an admissible cover of V. For each i we can find an admissible cover $\mathcal{V}_i$ of $U_i \cap V$ by objects in $\mathcal{B}$. Then $\bigcup \mathcal{V}_i$ is an admissible cover of V by objects in $\mathcal{B}$. Moreover, the universal property of equalisers defines a map $H^0(\mathcal{U}, F) \to H^0(\bigcup \mathcal{V}_i, F) \to \mathcal{F}(V)$. These patch together using the universal property of direct products to give a morphism $\mathcal{F}(V) \to \mathcal{F}(U)$. It is routine to check that in this way $\mathcal{F}$ defines a presheaf on X_{rig} whose restriction to $\mathcal{B}$ is naturally isomorphic to F. The proof of [11, Lemma 9.2.2/3] shows that $\mathcal{F}$ is in fact a sheaf. $\square$

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Mathematical Institute, University of Oxford, Oxford OX2 6GG
E-mail address: `ardakov@maths.ox.ac.uk`

Homerton College, Cambridge, CB2 8PQ
E-mail address: `S.J.Wadsley@dpmms.cam.ac.uk`