REPRESENTATION THEORY

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Lecture 1

1. Introduction

Representation Theory is the study of how symmetries occur in nature; that is the study of how groups act by linear transformations on vector spaces.

One major goal of this course will be to understand how to go about classifying all representations of a given (finite) group. For this we will need to be precise about what it means for two representations to be the same as well as how representations may decompose into smaller pieces.

We'll also use Representation Theory to better understand groups themselves. An example of the latter that we'll see later in the course is the Burnside p^aq^b-theorem which tells us that the order of a finite simple group cannot have precisely two distinct prime factors.

1.1. **Linear algebra revision.**

By *vector space* we will always mean a finite dimensional vector space over a field k unless we say otherwise. This field k will usually be algebraically closed and of characteristic zero, for example $\mathbb{C}$, because this is typically the easiest case. However there are rich theories for more general fields and we will sometimes hint at them.

Given a vector space V, we define the *general linear group* of V

$$GL(V) = \mathrm{Aut}(V) = \{\alpha\colon V \to V \mid \alpha \text{ linear and invertible}\}.$$

This is a group under composition of maps.

Because all our vector spaces are finite dimensional, there is an isomorphism $k^d \xrightarrow{\sim} V$ for some $d \geqslant 0$.[1] Here d is the isomorphism invariant of V called its dimension. The choice of isomorphism determines a basis $e_1, \dots, e_d$ for V.[2] Then

$$GL(V) \cong \{A \in \mathrm{Mat}_d(k) \mid \det(A) \neq 0\}.$$

This isomorphism is given by the map that sends the linear map α to the matrix A such that $\alpha(e_i) = \sum A_{ji}e_j$.

Exercise. Check that this does indeed define an isomorphism of groups. ie check that α is an invertible if and only if $\det A \neq 0$; and that the given map is a bijective group homomorphism.

The choice of isomorphism $k^d \xrightarrow{\sim} V$ also induces a decomposition of V as a direct sum of one-dimensional subspaces

$$V = \bigoplus_{i=1}^{d} ke_i.$$

This decomposition is not unique is general[3] but the number of summands is always $\dim V$.

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[1] In fact the set of such isomorphisms is in bjiection with $GL(V)$ so typically there are very many such.

[2] Here e_i is the image of the ith standard basis vector for k^d under the isomorphism.

[3] that is it depends on the choice of basis up to rescaling the basis vectors so there is more than one such decomposition if $d > 1$

1.2. **Group representations — definitions and examples.** Recall that an action of a group G on a set X is a function $\cdot\colon G \times X \to X; (g,x) \mapsto g \cdot x$ such that

(i) $e \cdot x = x$ for all $x \in X$;
(ii) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

Recall also that to define such an action is equivalent to defining a group homomorphism $\rho\colon G \to S(X)$ where $S(X)$ denotes the symmetric group on the set X; that is the set of bijections from X to itself equipped with the binary operation of composition of functions. The notions are seen to be equivalent by the formula $\rho(g)(x) = g \cdot x$ for all $g \in G$ and $x \in X$.

Definition. A *representation* ρ of a group G on a vector space V is a group homomorphism $\rho\colon G \to GL(V)$, the group of invertible linear transformations of V.

By abuse of notation we will sometimes refer to the representation by ρ, sometimes by the pair (ρ, V) and sometimes just by V with the ρ implied. This can sometimes be confusing but we have to live with it.

Defining a representation of G on V corresponds to assigning a linear map $\rho(g)\colon V \to V$ to each $g \in G$ such that

(i) $\rho(e) = \mathrm{id}_V$;
(ii) $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$;
(iii) $\rho(g^{-1}) = \rho(g)^{-1}$ for all $g \in G$.

Exercise. Show that, given condition (ii) holds, conditions (i) and (iii) are equivalent to one another in the above. Show moreover that conditions (i) and (iii) can be replaced by the condition that $\rho(g) \in GL(V)$ for all $g \in G$.

Given a basis for V a representation ρ is an assignment of a matrix $\rho(g)$ to each $g \in G$ such that (i),(ii) and (iii) hold.

Definition. The *degree* of ρ or *dimension* of ρ is $\dim V$.

Definition. We say a representation ρ is *faithful* if $\ker \rho = \{e\}$.

Examples.

(1) Let G be any group and $V = k$. Then $\rho\colon G \to \mathrm{Aut}(V); g \mapsto \mathrm{id}$ is called the *trivial representation*.
(2) Let $G = C_2 = \{\pm 1\}$, $V = \mathbb{R}^2$, then
$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
is a group rep of G on V.
(3) Let $G = (\mathbb{Z}, +)$, V a vector space, and ρ a representation of G on V. Then necessarily $\rho(0) = \mathrm{id}_V$, and $\rho(1)$ is some invertible linear map α on V. Now $\rho(2) = \rho(1+1) = \rho(1)^2 = \alpha^2$. Inductively we see $\rho(n) = \alpha^n$ for all $n > 0$. Finally $\rho(-n) = (\alpha^n)^{-1} = (\alpha^{-1})^n$. So $\rho(n) = \alpha^n$ for all $n \in \mathbb{Z}$.

   Notice that conversely given any invertible linear map $\alpha\colon V \to V$ we may define a representation of G on V by $\rho(n) = \alpha^n$.

   Thus we see that there is a 1-1 correspondence between representations of $\mathbb{Z}$ and invertible linear transformations given by $\rho \mapsto \rho(1)$.

(4) Let $G = (\mathbb{Z}/N, +)$, and $\rho\colon G \to GL(V)$ a rep. As before we see $\rho(n + N\mathbb{Z}) = \rho(1 + N\mathbb{Z})^n$ for all $n \in \mathbb{Z}$ but now we have the additional constraint that $\rho(N + N\mathbb{Z}) = \rho(0 + N\mathbb{Z}) = \mathrm{id}_V$.

Thus representations of $\mathbb{Z}/N$ correspond to invertible linear maps α such that $\alpha^N = \mathrm{id}_V$. Of course any linear map such that $\alpha^N = \mathrm{id}_V$ is invertible so we may drop the word invertible from this correspondence.

(5) Let $G = S_3$, the symmetric group of $\{1, 2, 3\}$, and $V = \mathbb{R}^2$. Take an equilateral triangle in V centred on 0; then G acts on the triangle by permuting the vertices. Each such symmetry induces a linear transformation of V. For example $g = (12)$ induces the reflection through the vertex three and the midpoint of the opposite side, and $g = (123)$ corresponds to a rotation by $2\pi/3$.

Exercise. Choose a basis for $\mathbb{R}^2$. Write the coordinates of the vertices of the triangle in this basis. For each $g \in S_3$ write down the matrix of the corresponding linear map. Check that this does define a representation of S_3 on V. Would the calculations be easier in a different basis?

Lecture 2

(6) Given a finite set X we may form the vector space kX of functions X to k with basis $\langle \delta_x \mid x \in X\rangle$ where $\delta_x(y) = \delta_{xy}$.[4]

Then an action of G on X induces a representation $\rho\colon G \to \mathrm{Aut}(kX)$ by $(\rho(g)f)(x) = f(g^{-1} \cdot x)$ called the *permutation representation* of G on X.

It is straightforward to verify that $\rho(g)$ is linear and that $\rho(e) = \mathrm{id}_{kX}$. So to check that ρ is a representation we must show that $\rho(gh) = \rho(g)\rho(h)$ for each $g, h \in G$.

For this observe that for each $x \in X$,

$$\rho(g)(\rho(h)f)(x) = (\rho(h)f)(g^{-1}x) = f(h^{-1}g^{-1}x) = \rho(gh)f(x).$$

Notice that $\rho(g)\delta_x(y) = \delta_{x,g^{-1}\cdot y} = \delta_{g\cdot x,y}$ so $\rho(g)\delta_x = \delta_{g\cdot x}$. So by linearity $\rho(g)(\sum_{x\in X} \lambda_x\delta_x) = \sum \lambda_x\delta_{g\cdot x}$.

(7) In particular if G is finite then the action of G on itself by left multiplication induces the *regular representation* kG of G. The regular representation is always faithful because $\rho(g)\delta_e = \delta_e$ implies that $ge = e$ and so $g = e$.

(8) If $\rho\colon G \to GL(V)$ is a representation of G then we can use ρ to define a representation of G on V^*

$$\rho^*(g)(\theta)(v) = \theta(\rho(g^{-1})v); \quad \forall \theta \in V^*, v \in V.$$[5]

(9) More generally, if $(\rho, V), (\rho', W)$ are representations of G then $(\sigma, \mathrm{Hom}_k(V, W))$ defined by

$$\sigma(g)(\alpha) = \rho'(g) \circ \alpha \circ \rho(g^{-1}); \quad \forall g \in G \text{ and } \alpha \in \mathrm{Hom}_k(V, W)$$

is a rep of G.

Note that if $W = k$ is the trivial rep. this reduces to example 8.

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[4]Each $f \in kX$ can be written $f = \sum_{x\in X} f(x)\delta_x$.

[5]$\rho^*(g)$ can be viewed as the adjoint of $\rho(g)^{-1}$; recall that with respect to a pair of dual bases for V and V^* the matrix of adjoint of a linear map is the transpose of the matrix of the linear map itself. So this is saying $A \mapsto (A^{-1})^T$ is a homomorphism $GL_d(k) \to GL_d(k)$.

Exercise. Check the details.[6] Moreover show that if $V = k^n$ and $W = k^m$ with the standard bases, so that $\mathrm{Hom}_k(V, W) = \mathrm{Mat}_{m,n}(k)$, then

$$\sigma(g)(A) = \rho'(g)A\rho(g)^{-1} \text{ for all } A \in \mathrm{Mat}_{m,n}(k) \text{ and } g \in G.$$

(10) If $\rho\colon G \to GL(V)$ is a representation of G and $\theta\colon H \to G$ is a group homomorphism then $\rho\theta\colon H \to GL(V)$ is a representation of H. If H is a subgroup of G and θ is inclusion we call this the *restriction* of ρ to H.

1.3. **The category of representations.** We want to classify all representations of a group G but first we need a good notion of when two representations are the same.

Notice that if $\rho\colon G \to GL(V)$ is a representation and $\varphi\colon V \to W$ is a vector space isomorphism then we may define $\sigma\colon G \to GL(W)$ by $\sigma(g) = \varphi \circ \rho(g) \circ \varphi^{-1}$ and σ is also a representation.

Definition. We say that $\rho\colon G \to GL(V)$ and $\sigma\colon G \to GL(W)$ are *isomorphic* representations if there is a linear isomorphism $\varphi\colon V \to W$ such that

$$\sigma(g) = \varphi \circ \rho(g) \circ \varphi^{-1} \text{ for all } g \in G$$

i.e. if $\sigma(g) \circ \varphi = \varphi \circ \rho(g)$. We say that φ *intertwines* ρ and σ.

Notice that id_V intertwines ρ and ρ; if φ intertwines ρ and σ then φ^{-1} intertwines σ and ρ; and if moreover φ' intertwines σ and τ then $\varphi'\varphi$ intertwines ρ and τ. Thus isomorphism is an equivalence relation.

Since every vector space is isomorphic to k^d for some $d \geqslant 0$, every representation is isomorphic to a matrix representation $G \to GL_d(k)$.

If $\rho, \sigma\colon G \to GL_d(k)$ are matrix representations of the same degree then an intertwining map $k^d \to k^d$ is an invertible matrix P and the matrices of the reps it intertwines are related by $\sigma(g) = P\rho(g)P^{-1}$. Thus matrix representations are isomorphic precisely if they represent the same family of linear maps with respect to different bases.

Examples.

(1) If $G = \{e\}$ then a representation of G is just a vector space and two vector spaces are isomorphic as representations precisely if they have the same dimension.
(2) If $G = \mathbb{Z}$ then $\rho\colon G \to GL(V)$ and $\sigma\colon G \to GL(W)$ are isomorphic reps if and only if there are bases of V and W such that $\rho(1)$ and $\sigma(1)$ are the same matrix. In other words isomorphism classes of representations of $\mathbb{Z}$ correspond to conjugacy classes of invertible matrices. Over $\mathbb{C}$ the latter is classified by Jordan Normal Form (more generally by rational canonical form).
(3) If $G = C_2 = \{\pm 1\}$ then isomorphism classes of representations of G correspond to conjugacy classes of matrices that square to the identity. Since the minimal polynomial of such a matrix divides $X^2 - 1 = (X - 1)(X + 1)$ provided the field does not have characteristic 2 every such matrix is conjugate to a diagonal matrix with diagonal entries all ± 1.

    Exercise. Show that there are precisely $n + 1$ isomorphism classes of representations of C_2 of dimension n.

[6] This will also appear on Examples Sheet 1.

(4) If X, Y are finite sets with a G-action and $f: X \to Y$ is a G-equivariant bijection i.e. f is a bijection such that $g \cdot f(x) = f(g \cdot x)$ for all $x \in X$ and $g \in G$, then $\varphi\colon kX \to kY$ defined by $\varphi(\theta)(y) = \theta(f^{-1}y)$ intertwines kX and kY. (Note that $\varphi(\delta_x) = \delta_{f(x)}$)

Lecture 3

Definition. Suppose that $\rho\colon G \to GL(V)$ is a rep. We say that a k-linear subspace W of V is *G-invariant* if $\rho(g)(W) \subseteq W$ for all $g \in G$ (ie $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$).

In that case we may define a representation $\rho_W : G \to GL(W)$ by

$$\rho_W(g)(w) = \rho(g)(w) \text{ for } w \in W.$$

We call (ρ_W, W) a *subrepresentation* of (ρ, V).

We call a subrepresentation W of V *proper* if $W \neq V$ and $W \neq 0$. We say that $V \neq 0$ is *irreducible* or *simple* if it has no proper subreps.

Examples.

(1) Any one-dimensional representation of a group is irreducible.

(2) Suppose that $\rho\colon C_2 \to GL_2(k)$ is given by $-1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (char $k \neq 2$). Then ρ has precisely two proper subrepresentations spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

Proof. It is easy to see that these two subspaces are G-invariant. Any proper subrepresentation must be one dimensional and so by spanned by an eigenvector of $\rho(-1)$. But the eigenspaces of $\rho(-1)$ are precisely those already described. $\square$

(3) If G is C_2 then the only irreducible representations are one-dimensional.

Proof. Suppose $\rho\colon G \to GL(V)$ is an irreducible rep. The minimal polynomial of $\rho(-1)$ divides $X^2 - 1 = (X-1)(X+1)$. Thus $\rho(-1)$ has an eigenvector v. Now $0 \neq \langle v \rangle$ is a subrepresentation of V. Thus $V = \langle v \rangle$. $\square$

Notice we've shown along the way that there are precisely two simple representations of G (up to isomorphism) if k doesn't have characteristic 2 and only one if it does.

(4) If $G = D_6$ then every irreducible complex representation has dimension at most 2.

Proof. Suppose $\rho\colon G \to GL(V)$ is an irreducible representation of G. Let r be a non-trivial rotation and s a reflection in G so that $r^3 = e = s^2$, $srs = r^{-1}$ and r and s generate G.

Since $\rho(r)^3 = \rho(r^3) = \mathrm{id}_V$, $\rho(r)$ has a eigenvector v, say with eigenvalue λ for some $\lambda \in \mathbb{C}$ such that $\lambda^3 = 1$.[7]

Consider $W := \langle v, \rho(s)v \rangle \leqslant V$ so $\dim W \leqslant 2$. Since

$$\rho(s)\rho(s)v = v$$

[7] This is the only point we use that $k = \mathbb{C}$. In fact suffices that $X^3 - 1$ completely factorises in k.

and

$$\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v,$$

W is G-invariant. Since V is irreducible, $W = V$. □

Exercise. Show that there are precisely three irreducible complex representations of D_6 up to isomorphism, one of dimension 2 and two of dimension 1. (Hint: We can split into cases depending on λ and whether $\rho(s)(v) \in \langle v \rangle$ or $\rho(s)(v) \notin \langle v \rangle$).

Definition. If W is a subrep of a rep (ρ, V) of G then we may define a *quotient representation* $\rho_{V/W} \colon G \to GL(V/W)$ by

$$\rho_{V/W}(g)(v + W) = \rho(g)(v) + W.$$

Since $\rho(g)W \subset W$ for all $g \in G$ this is well-defined.

We'll start dropping ρ now and write g for $\rho(g)$ where it won't cause confusion.

Definition. If (ρ, V) and (ρ', W) are reps of G we say a linear map $\varphi \colon V \to W$ is a *G-linear* map if $\varphi g = g\varphi$ (ie $\varphi \circ \rho(g) = \rho'(g) \circ \varphi$) for all $g \in G$. We write

$$\mathrm{Hom}_G(V, W) = \{\varphi \in \mathrm{Hom}_k(V, W) \mid \varphi \text{ is } G\text{-linear}\},$$

a k-vector subspace of $\mathrm{Hom}_k(V, W)$.

Remarks.

(1) $\varphi \in \mathrm{Hom}_k(V, W)$ is an intertwining map precisely if φ is a bijection and φ is in $\mathrm{Hom}_G(V, W)$.

(2) If $W \leqslant V$ is a subrepresentation then the natural inclusion map $\iota \colon W \to V$; $w \mapsto w$ is in $\mathrm{Hom}_G(W, V)$ and the natural projection map $\pi \colon V \to V/W$; $v \mapsto v + W$ is in $\mathrm{Hom}_G(V, V/W)$.

(3) Recall that $\mathrm{Hom}_k(V, W)$ is a G-rep via $(g\varphi)(v) = g(\varphi(g^{-1}v))$ for $\varphi \in \mathrm{Hom}_k(V, W)$, $g \in G$ and $v \in V$. Then $\varphi \in \mathrm{Hom}_G(V, W)$ precisely if $g\varphi = \varphi$ for all $g \in G$.

Lemma. *If U, V and W are representations of a group G with $\varphi_1 \in \mathrm{Hom}_k(V, W)$ and $\varphi_2 \in \mathrm{Hom}_k(U, V)$ then*

$$g \cdot (\varphi_1 \circ \varphi_2) = (g \cdot \varphi_1) \circ (g \cdot \varphi_2).$$

In particular

$$\varphi_1 \in \mathrm{Hom}_G(V, W) \implies g \cdot (\varphi_1 \circ \varphi_2) = \varphi_1 \circ (g \cdot \varphi_2),$$

$$\varphi_2 \in \mathrm{Hom}_G(U, V) \implies g \cdot (\varphi_1 \circ \varphi_2) = (g \circ \varphi_1) \circ \varphi_2 \textit{ and}$$

$$\varphi_1 \in \mathrm{Hom}_G(V, W) \textit{ and } \varphi_2 \in \mathrm{Hom}_G(U, V) \implies \varphi_1 \circ \varphi_2 \in \mathrm{Hom}_G(U, W).$$

Proof. With the notation in the statement we can compute

$$(g \cdot \varphi_1) \circ (g \cdot \varphi_2) = (g \circ \varphi_1 \circ g^{-1})(g \circ \varphi_2 \circ g^{-1}) = g \cdot (\varphi_1 \circ \varphi_2).$$

All the other statements follow immediately. □

Lemma (First isomorphism theorem for representations)**.** *Suppose (ρ, V) and (ρ', W) are representations of G and $\varphi \in \mathrm{Hom}_G(V, W)$ then*

(i) $\ker \varphi$ *is a subrepresentation of V;*

(ii) $\mathrm{Im}\, \varphi$ *is a subrepresentation of W;*

(iii) The linear isomorphism $\overline{\varphi} \colon V/\ker \varphi \to \mathrm{Im}\, \varphi$ given by the first isomorphism of vector spaces is an intertwining map. Thus $V/\ker \varphi \cong \mathrm{Im}\, \varphi$ as representations of G.

Proof.
(i) if $v \in \ker\varphi$ and $g \in G$ then $\varphi(gv) = g\varphi(v) = 0$
(ii) if $w = \varphi(v) \in \operatorname{Im}\varphi$ and $g \in G$ then $gw = \varphi(gv) \in \operatorname{Im}\varphi$.
(iii) We know that the linear map φ induces a linear isomorphism

$$\overline{\varphi}\colon V/\ker\varphi \to \operatorname{Im}\varphi; v + \ker\varphi \mapsto \varphi(v)$$

then $g\overline{\varphi}(v + \ker\varphi) = g(\varphi(v)) = \varphi(gv) = \overline{\varphi}(gv + \ker\varphi)$ □

Proposition. *Suppose $\rho\colon G \to GL(V)$ is a rep and $W \leqslant V$. Then the following are equivalent:*

(i) W is a subrepresentation;
(ii) there is a basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of W and the matrices $\rho(g)$ are all block upper triangular;
(iii) for every basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of W the matrices $\rho(g)$ are all block upper triangular.

Proof. Think about it (see also Linear Algebra Examples Sheet 1 Q11 from Michaelmas 2022). □

Lecture 4

2. Complete reducibility and Maschke's Theorem

Question. When can a representation V of a group G be decomposed as a direct sum of simple subrepresentations?

Examples.

(1) If $G = \{e\}$ the answer is always as noted in Lecture 1 since a simple subrep is the same as a 1-dimensional subspace.

(2) Suppose $G = C_2$, $V = \mathbb{R}^2$ and $\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We've seen that the only irreducible subrepresentations are $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ and $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$. So

$$\mathbb{R}^2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

is the only such decomposition in this case.

(3) Suppose $G = (\mathbb{Z}, +)$ and $\rho\colon G \to GL_2(k)$ is the representation determined by

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then $W = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ is the only proper G-invariant subspace so k^2 cannot be decomposed as a direct sum of irreducible subrepresentations — if it could then $\rho(1)$ would be diagonalisable.

Definition. We say a representation V is a *direct sum* of $(V_i)_{i=1}^k$ if each V_i is a subrepresentation of V and $V = \bigoplus_{i=1}^k V_i$ as vector spaces.[8]

Given a family of representations $(\rho_i, V_i)_{i=1}^k$ of G we may define a representation of G on the vector space

$$V := \bigoplus_{i=1}^{k} V_i := \{(v_i)_{i=1}^k \mid v_i \in V_i\} \text{ with pointwise operations}$$ [9]

by

$$\rho(g)((v_i)) = (\rho_i(g)v_i).$$

We write $(\rho, V) = \bigoplus_{i=1}^k (\rho_i, V_i) = \bigoplus \rho_i = \bigoplus V_i$.

Examples.

(1) Suppose G acts on a finite set X and X may be written as the disjoint union of two G-invariant subsets X_1 and X_2 (i.e. $g \cdot x \in X_i$ for all $x \in X_i$ and $g \in G$). Then $kX \cong kX_1 \oplus kX_2$ under $f \mapsto (f|_{X_1}, f|_{X_2})$.
Internally $kX = \{f \mid f(x) = 0\ \forall x \in X_2\} \oplus \{f \mid f(x) = 0\ \forall x \in X_1\}$.

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[8] i.e. $V = \sum_{i=1}^k V_i$ and for each $j = 1, \dots k$, $V_j \cap \sum_{i \neq j} V_i = 0$ as in Linear Algebra Examples Sheet 1 Q8 from Michaelmas 2022.

[9] the *external* direct sum of the V_i

More generally if the G-action on X decomposes into orbits as a disjoint union $X = \bigcup_{i=1}^r \mathcal{O}_i$ then

$$kX = \bigoplus_{i=1}^r \mathbf{1}_{\mathcal{O}_i}(kX) \cong \bigoplus k\mathcal{O}_i.$$

where $\mathbf{1}_{\mathcal{O}_i}\colon kX \to kX$ is given by $\mathbf{1}_{\mathcal{O}_i}(f)(x) = \begin{cases} f(x) & x \in \mathcal{O}_i \\ 0 & x \notin \mathcal{O}_i. \end{cases}$

These $k\mathcal{O}_i$ are almost never irreducible as explained in the following example.

(2) If G acts transitively on a finite set X then $U := \{f \in kX \mid \sum_{x\in X} f(x) = 0\}$ and $W := \{f \in kX \mid f \text{ is constant}\}$ are subreps of kX.

Proof. If $f \in U$ then for $g \in G$,

$$\sum_{x\in X} (g \cdot f)(x) = \sum_{x \in X} f(g^{-1}x) = 0$$

since $x \mapsto g^{-1}x$ is a bijection $X \to X$. Similarly if $f \in W$; $f(x) = \lambda$ for all $x \in X$ then for $g \in G$, $(g.f)(x) = f(g^{-1}x) = \lambda$ for all $x \in X$. □

If k is characteristic 0 then $kX = U \oplus W$. What happens if k has characteristic $p > 0$?

Proposition. *Suppose $\rho\colon G \to GL(V)$ is a rep. and $V = U \oplus W$ as vector spaces. Then the following are equivalent:*

(i) $V = U \oplus W$ as reps;
(ii) there is a basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of U and $v_{r+1}, \ldots v_d$ is a basis for W and the matrices $\rho(g)$ are all block diagonal;
(iii) for every basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of U and $v_{r+1}, \ldots, v_d$ is a basis for W the matrices $\rho(g)$ are all block diagonal.

Proof. Think about it! □

But the following example provides a warning.

Example. $\rho\colon C_2 \to GL_2(\mathbb{R})$; $-1 \mapsto \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ defines a representation (check). The representation $\mathbb{R}^2$ breaks up as $\langle e_1 \rangle \oplus \langle e_1 + e_2 \rangle$ as subreps even though the matrix is upper triangular but not diagonal.

Definition. We say that a representation V of a group G is *completely reducible* if $V \cong \oplus_{i=1}^r V_i$ for some irreducible representations $V_1, \ldots, V_r$ of G.

We've seen by considering $G = \mathbb{Z}$ that it is not true that every representation of every group G is completely reducible. However we're going to prove the remarkable fact that if G is a finite group and k has characteristic 0 then every representation of G defined over k is completely reducible.

Lemma. *Suppose that (ρ, V) is a representation of a group G such that for every pair W_1, W_2 of G-invariant subspaces of V such that $W_1 \leqslant W_2 \leqslant V$ there is a G-invariant complement to W_1 in W_2. Then V is completely reducible.*

Proof. By induction on $\dim V$. If $\dim V = 0$ or V is irreducible then the result is clear. Otherwise V has a proper G-invariant subspace W. By the assumption there is a G-invariant complement U of W in V and $V \cong U \oplus W$ as G-reps. Moreover $\dim U, \dim W < \dim V$ and U and W inherit the assumption on V. Thus by induction there are simple representations $U_1, \ldots, U_r$ such that $U \cong \oplus_{i=1}^r U_i$ and $W_1, \ldots, W_s$ such that $W \cong \oplus_{j=1}^s W_j$. Thus

$$V \cong \bigoplus_{i=1}^r U_i \oplus \bigoplus_{j=1}^s W_j$$

is complete reducible. □

Recall, if V is a complex vector space then a *Hermitian inner product* is a positive definite Hermitian sesquilinear form; i.e. $(-,-)\colon V \times V \to \mathbb{C}$ satisfying

(i) (a) $(ax+by, z) = \overline{a}(x,z) + \overline{b}(y,z)$ and
   (b) $(x, ay+bz) = a(x,y) + b(x,z)$ for $a, b \in \mathbb{C}$, $x, y, z \in V$ (sesquilinear);

(ii) $(x,y) = \overline{(y,x)}$ (Hermitian);[10]

(iii) $(x,x) > 0$ for all $x \in V \backslash \{0\}$ (positive definite).[11]

The standard inner product on $\mathbb{C}^n$ is given by

$$\langle x, y \rangle = \sum_{i=1}^n \overline{x_i} y_i.$$

Recall also that the *unitary group* $U(n)$ is the subgroup of $GL_n(\mathbb{C})$

$$\begin{aligned} U(n) &= \{A \in GL_n(\mathbb{C}) : \overline{A^T} A = I\} \\ &= \{A \in GL_n(\mathbb{C} : \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{C}^n\}. \end{aligned}$$

Definition. We say that a representation (ρ, V) of a group G is *unitary* if there is a basis for V so the corresponding map $G \to GL_n(\mathbb{C})$ has image inside $U(n)$.

Lecture 5

Definition. A Hermitian inner product $(-,-)$ on a representation V of G is G-invariant if

$$(gx, gy) = (x, y) \text{ for all } g \in G \text{ and } x, y \in V;$$

or, equivalently, if

$$(gx, gx) = (x, x) \text{ for all } g \in G \text{ and } x \in V.$$

Proposition. *A representation (ρ, V) of G is unitary if and only if V has a G-invariant inner product.*

Proof. If (ρ, V) is unitary then let $e_1, \ldots, e_n$ be a basis for V with respect to which $\rho(g) \in U(n)$ for all $g \in G$. Now

$$\left(\sum_{i=1}^n \lambda_i e_i, \sum_{j=1}^n \mu_j e_j \right) = \sum_{i=1}^n \overline{\lambda_i} \mu_i$$

defines a G-invariant inner product on V.

[10] if (ii) holds then (i)(a) is equivalent to (i)(b).

[11] (ii) gives that $(x,x) \in \mathbb{R}$.

Conversely, if V has a G-invariant inner product $(-,-)$ we can find an orthonormal basis $v_1,\dots,v_n$ for V with respect to $(-,-)$.[12] Then $(-,-)$ corresponds to the standard inner product with respect to this basis and so each $\rho(g)$ is unitary with respect to the basis. $\square$

We note that it follows easily that subrepresentations of unitary representations are unitary since a G-invariant inner product on the representation will restrict to the subrepresentation.

Lemma. *If (ρ, V) is a unitary representation of a group G then every subrepresentation W of V has a G-invariant complement. In particular V is completely reducible.*

Proof. Let $(-,-)$ be a G-invariant inner product on V. Then

$$W^\perp := \{v \in V : (v,w) = 0 \text{ for all} w \in W\}$$

is a vector-space complement to W in V by standard linear algebra. Moreover if $g \in G$, $v \in W^\perp$ and $w \in W$. Then $\langle gv, w\rangle = \langle v, g^{-1}w\rangle = 0$ since $g^{-1}w \in W$. Thus $gv \in W^\perp$ and $W^\perp$ is a G-invariant complement. Complete reducibility follows from a lemma from the last lecture. $\square$

Theorem (Maschke's Theorem)**.** *Let G be a finite group and (ρ, V) a representation of G over a field k of characteristic zero. Suppose $W \leqslant V$ is a G-invariant subspace. Then there is a G-invariant complement to W ie a G-invariant subspace U of V such that $V = U \oplus W$. In particular V is completely reducible.*

Key idea. *If (ρ, V) is a representation of a finite group G then for all $v \in V$*

$$\sum_{g\in G} g\cdot v \in V^G := \{v \in V : g \cdot v = v \textit{ for all } g \in G\} \leqslant V.$$

Proof. If $h \in G$,

$$h \cdot \left(\sum_{g\in G} g\cdot v\right) = \sum_{g\in G}(hg)\cdot v = \sum_{g'\in G} g'\cdot v$$

since $h\colon G \to G; g \mapsto hg$ is a permutation of G. $\square$

Proposition (Weyl's unitary trick)**.** *If V is a complex representation of a finite group G, then there is a G-invariant Hermitian inner product on V. In particular V is unitary and every G-invariant subspace has a G-invariant complement.*

Proof. Pick any Hermitian inner product $\langle -,-\rangle$ on V (e.g. choose a basis $e_1,\dots,e_n$ and take the standard inner product $\langle \sum \lambda_i e_i, \sum \mu_i e_i\rangle = \sum \overline{\lambda_i}\mu_i$). Then define a new inner product $(-,-)$ on V via:

$$(x,y) := \sum_{g\in G}\langle gx, gy\rangle.$$

[12] Choose any basis and then apply Gram-Schmidt.

It is easy to see that $(-,-)$ is a Hermitian inner product because $\langle -,-\rangle$ is. For example if $a, b \in \mathbb{C}$ and $x, y, z \in V$, then

$$\begin{aligned}(x, ay+bz) &= \sum_{g\in G}\langle gx, g(ay+bz)\rangle \\ &= \sum_{g\in G}\langle gx, ag(y)+bg(z)\rangle \\ &= \sum_{g\in G}(a\langle gx, gy\rangle + b\langle gx, gz\rangle) \\ &= a(x,y)+b(z,y)\end{aligned}$$

as required.

But now if $h \in G$ and $x, y \in V$ then

$$(hx, hy) = \sum_{g\in G}\langle ghx, ghy\rangle = \sum_{g'\in G}\langle g'x, g'y\rangle$$

and so $(-,-)$ is G-invariant. Complete reducibility now follows by a lemma proven in the last lecture. □

Remark. The proof can be phrased as follows

(i) $\mathrm{Herm}(V) := \{$Hermitian sesquilinear forms on $V\}$ is naturally an $\mathbb{R}$-vector space.

(ii) $G \to \mathrm{Aut}(\mathrm{Herm}(V))$; $g \cdot (-,-)(x,y) := (g^{-1}x, g^{-1}y)$ defines an $\mathbb{R}$-linear representation of G.[13]

(iii) An $\mathbb{R}^{>0}$-linear combination of positive definite elements of $\mathrm{Herm}(V)$ is positive definite.

(iv) Given (i)-(iii) the key idea transforms any inner product into a G-invariant inner product.

It follows that studying complex representations of a finite group is equivalent to studying unitary, i.e. distance preserving, representations.

Corollary. *Every finite subgroup G of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.*

Proof. If $G \leqslant GL_n(\mathbb{C})$ the inclusion map $\rho\colon G \to GL_n(\mathbb{C})$ is a representation. By the unitary trick, ρ is a unitary representation i.e. there is $P \in GL_n(\mathbb{C})$ such that $PgP^{-1} \in U(n)$ for all $g \in G$. □

We now generalise our idea to general k of characteristic zero — one way to explain our argument when the representation is unitary is that the orthogonal projection map $V \to W$ is G-linear with kernel $W^\perp$ a G-invariant complement.

Proof of Maschke's Theorem. Idea: if $\pi\colon V \to V$ is a projection i.e. $\pi^2 = \pi$ then $V = \mathrm{Im}\,\pi \oplus \ker \pi$ as vector spaces. If π is G-linear then $\ker\pi$ and $\mathrm{Im}\,\pi$ are both G-invariant. So we pick a projection $V \to V$ with image W and average it.

[13] Added after lecture: a choice of basis $v_1, \dots, v_n$ for V gives an identification of $\mathrm{Herm}(V)$ with Hermitian matrices in $\mathrm{Mat}_{\dim V, \dim V}(\mathbb{C})$, i.e. A such that $A = \overline{A^T}$ where $A_{ij} = (v_i, v_j)$ and the corresponding representation of G on these Hermitian matrices is given by $g \cdot A = \overline{\rho(g^{-1})^T} A \rho(g^{-1})$ where $\rho\colon G \to GL_{\dim V}(\mathbb{C})$ is the matrix representation corresponding to V equipped with the same basis. Note that $g \mapsto \overline{\rho(g^{-1})^T}$ is a homomorphism since $g \mapsto g^{-1}$ and $X \mapsto \overline{X^T}$ are both antihomomorphisms; that is the second two reverse the order of operations so their composite, the first, preserves it.

Let $\pi\colon V \to V$ be any k-linear projection with $\pi(w) = w$ for all $w \in W$ and $\operatorname{Im}\pi = W$.

Recall that $\operatorname{Hom}_k(V,V)$ is a rep of G via $(g\varphi)(v) = g\varphi g^{-1}v$. Let $\pi^G\colon V \to V$ be defined by

$$\pi^G := \frac{1}{|G|}\sum_{g\in G}(g\pi) \in \operatorname{Hom}_G(V,V)$$

by the key idea. Moreover $\operatorname{Im}\pi^G \leqslant W$ since

$$(g\cdot\pi)(v) = (g\circ\pi\circ g^{-1})\cdot v \in W$$

for all $g \in G$ and $v \in V$. Also, for $w \in W$,

$$\pi^G(w) = \frac{1}{|G|}\sum_{g\in G} g(\pi(g^{-1}w)) = \frac{1}{|G|}\sum_{g\in G} gg^{-1}(w) = w$$

since $g^{-1}w \in W$ for all $g \in G$ and $w \in W$.

Thus π^G is a G-invariant projection $V \to V$ with image W. So $\ker\pi^G$ is the required G-invariant complement to W. $\square$

Remarks (on the Proof of Maschke's Theorem).

(1) We can explicitly compute π^G and $\ker\pi^G$ given (ρ, V) and (π, W) via the formula

$$\pi^G = \frac{1}{|G|}\sum_{g\in G} g\cdot\pi.$$

(2) Notice that we only used that char $k = 0$ when we inverted $|G|$. So in fact we only need that the characteristic of k does not divide $|G|$.

(3) As an extension of our key idea: for any G-rep V (with char k not dividing $|G|$), the map

$$\pi\colon v \mapsto \frac{1}{|G|}\sum_{g\in G} g\cdot v$$

is a projection in $\operatorname{Hom}_G(V,V)$ with image $V^G := \{v \in V \mid g\cdot v = v\}$. As a foreshadowing of what is coming soon, notice that

$$\dim V^G = \operatorname{tr}\pi = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(g)$$

since tr is linear and for $\pi\colon V \to V$ any projection $\operatorname{tr}\pi = \operatorname{Im}\pi$.

Lecture 6

3. Schur's Lemma

Recall that if V is a vector space of dimension d then $\mathrm{Aut}(V) \cong GL_d(k)$.

Theorem (Schur's Lemma). *Suppose that V and W are irreducible representations of G over k. Then*

(i) every element of $\mathrm{Hom}_G(V, W)$ is either 0 or an isomorphism;
(ii) if k is algebraically closed then $\dim_k \mathrm{Hom}_G(V, W)$ is either 0 or 1.

In other words, when k is algebraically closed, irreducible representations are rigid in the same sense that one-dimensional vector spaces are rigid since they have the same automorphism group.

Proof. (i) Let φ be a non-zero G-linear map from V to W. Then $\ker\varphi \lneq V$ is a G-invariant subspace of V. So as V is simple, $\ker\varphi = 0$. Similarly $0 \neq \mathrm{Im}\,\varphi \leqslant W$ so $\mathrm{Im}\,\varphi = W$ since W is simple. Thus φ is an isomorphism by the first isomorphism theorem.

(ii) Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$ are non-zero. Then by (i) they are both isomorphisms. Consider $\varphi = \varphi_1^{-1}\varphi_2 \in \mathrm{Hom}_G(V, V)$. Since k is algebraically closed we may find λ an eigenvalue of φ then $\varphi - \lambda\,\mathrm{id}_V$ has non-zero G-invariant kernel and so the map is zero. Thus $\varphi_1^{-1}\varphi_2 = \lambda\,\mathrm{id}_V$ and $\varphi_2 = \lambda\varphi_1$ as required. □

Proposition. *If V, V_1 and V_2 are k-representations of G then*

$$\mathrm{Hom}_G(V, V_1 \oplus V_2) \cong \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$$

and

$$\mathrm{Hom}_G(V_1, \oplus V_2, V) \cong \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V).$$

Proof. There are natural G-linear inclusion maps

$$\iota_i \colon V_i \to V_1 \oplus V_2 \text{ for } i = 1, 2$$

that induce (by post-composition)

$$\mathrm{Hom}_k(V, V_i) \to \mathrm{Hom}_k(V, V_1 \oplus V_2).$$

These together induce a linear isomorphism

$$\mathrm{Hom}_k(V, V_1) \oplus \mathrm{Hom}_k(V, V_2) \to \mathrm{Hom}_k(V, V_1 \oplus V_2)$$

given by

$$(f_1, f_2) \mapsto \iota_1 f_1 + \iota_2 f_2.$$

Since ι_1, ι_2 are G-linear this is an intertwining map:

$$g \cdot (\iota_1 f_1 + \iota_2 f_2) = \iota_1(g \cdot f_1) + \iota_2(g \cdot f_2).$$

Since in general an intertwining map $\varphi \colon U \to W$ between representations of G induces an isomorphism of G-fixed points — $g \cdot \varphi(u) = \varphi(u)$ if and only if $g \cdot u = u$ for all $g \in G$ — and $\mathrm{Hom}_G(U, W)$ consists of the G-fixed points of $\mathrm{Hom}_k(U, W)$, it follows that there is an induced isomorphism

$$\mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2) \to \mathrm{Hom}_G(V, V_1 \oplus V_2)$$

as claimed.

Similarly the natural projection maps

$$\pi_i \colon V_1 \oplus V_2 \to V_i \text{ for } i = 1, 2$$

induce a G-linear isomorphism

$$\mathrm{Hom}_k(V_1, V) \oplus \mathrm{Hom}_k(V_2, V) \to \mathrm{Hom}_k(V_1 \oplus V_2, V)$$

given by

$$(f_1, f_2) \mapsto f_1\pi_1 + f_2\pi_2$$

and again it follows that there is an induced isomorphism

$$\mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V) \to \mathrm{Hom}_G(V_1 \oplus V_2, V)$$

as claimed. □

Corollary. *If* $V \cong \bigoplus_{i=1}^r V_i$ *and* $W \cong \bigoplus_{j=1}^s W_j$ *then*

$$\mathrm{Hom}_G(V, W) \cong \bigoplus_{i=1}^r \bigoplus_{j=1}^s \mathrm{Hom}_G(V_i, W_j).$$

Proof. This follows from the Proposition by a straightforward induction argument. □

Corollary. *Suppose k is algebraically closed and*

$$V \cong \bigoplus_{i=1}^r V_i$$

is a decomposition of a representation of G over k into irreducible components.

Then for each irreducible representation W of G,

$$|\{i \mid V_i \cong W\}| = \dim \mathrm{Hom}_G(W, V) = \dim \mathrm{Hom}_G(V, W).$$

Proof. By the last result

$$\mathrm{Hom}_G(W, V) = \bigoplus_{i=1}^r \mathrm{Hom}_G(W, V_i)$$

and so

$$\dim \mathrm{Hom}_G(W, V) = \sum_{i=1}^r \dim \mathrm{Hom}_G(W, V_i).$$

and similarly

$$\mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^r \mathrm{Hom}_G(V_i, W)$$

and so

$$\dim \mathrm{Hom}_G(V, W) = \sum_{i=1}^r \dim \mathrm{Hom}_G(V_i, W).$$

Thus it suffices to show that

$$\dim \mathrm{Hom}_G(W, V_i) = \dim \mathrm{Hom}_G(V_i, W) = \begin{cases} 1 & \text{if } W \cong V_i \\ 0 & \text{if } W \not\cong V_i \end{cases}$$

and this is precisely the statement of Schur's Lemma when k is algebraically closed.[14] □

[14] A question to ponder for those who like to think about such things: what can be said if k is not algebraically closed?

It follows that the number of times each simple representation occurs in a decomposition of a representation as a direct sum of simple subrepresentations is independent of the choice of decomposition. Important question: How can we compute these numbers $\dim \operatorname{Hom}_G(V, W)$?[15]

Corollary. *(of Schur's Lemma) Every irreducible complex representation of an abelian group G is one-dimensional.*

Proof. Let (ρ, V) be a complex irreducible representation of G. Since G is abelian,

$$\rho(g)\rho(h) = \rho(h)\rho(g) \text{ for all } g, h \in G$$

and so

$$\rho(g) \in \operatorname{Hom}_G(V, V) \text{ for each } g \in G.$$

Thus, since V is irreducible and $\mathbb{C}$ is algebraically closed, by Schur, each $\rho(g)$ is a scalar multiple of id_V. It follows that for $v \in V$ non-zero, $\langle v \rangle$ is a subrep of V and so $V = \langle v \rangle$ by irreducibility of V again. In particular $\dim V = 1$. □

Corollary. *(of Schur's Lemma) If a finite group G has a faithful irreducible representation over an algebraically closed field k then the centre of G, $Z(G)$ is cyclic.*

Proof. Let (ρ, V) be a faithful irreducible representation of G, and let $z \in Z(G)$. Since $gz = zg$ for all $g \in G$, $\rho(z) \in \operatorname{Hom}_G(V, V)$. Thus, since V is irreducible and k is algebraically closed, by Schur, $\rho(z) = \lambda_z \mathrm{id}_V$, say, with $\lambda_z \in k$.

Moreover for z_1, z_2 in $Z(G)$,

$$\rho(z_1 z_2) = \rho(z_1)\rho(z_2) \text{ and so } \lambda_{z_1 z_2} = \lambda_{z_1}\lambda_{z_2}.$$

Since also, $\lambda_e = 1$,

$$Z(G) \to k^\times; z \mapsto \lambda_z$$

is a representation of $Z(G)$ that is faithful since V is faithful. In particular $Z(G)$ is isomorphic to a finite subgroup of $k^\times$. But every such subgroup is cyclic. □

Examples. We can list all the irreducible complex representations of C_4 and $C_2 \times C_2$

$G = C_4 = \langle x \rangle$.

	1	x	x^2	x^3
ρ_1	1	1	1	1
ρ_2	1	i	-1	$-i$
ρ_3	1	-1	1	1
ρ_4	1	$-i$	-1	i

$G = C_2 \times C_2 = \langle x, y \rangle$.

	1	x	y	xy
ρ_1	1	1	1	1
ρ_2	1	-1	1	-1
ρ_3	1	1	-1	-1
ρ_4	1	-1	-1	1

Lecture 7

Proposition. *Every finite abelian group G has precisely $|G|$ complex irreducible representations.*

[15] We saw in our remarks on the proof of Maschke's Theorem that if k denotes the trivial representation then $\dim \operatorname{Hom}_G(k, V) = \dim V^G = \frac{1}{|G|}\sum_{g \in G} \operatorname{tr} \rho(g)$ when k has characteristic zero.

Proof. Let ρ be an irreducible complex rep of G. By the last corollary, $\dim \rho = 1$. So $\rho\colon G \to \mathbb{C}^\times$ is a group homomorphism.

If $G = H \times K$ decomposes as a direct product of its subgroups H and K then there is a 1-1 correspondence

$$\mathrm{Hom}(G, \mathbb{C}^\times) \xleftrightarrow{\sim} \mathrm{Hom}(H, \mathbb{C}^\times) \times \mathrm{Hom}(K, \mathbb{C}^\times)$$

given by restriction $\varphi \mapsto (\varphi|_H, \varphi|_K)$.[16]

Since G is a finite abelian group $G \cong C_{n_1} \times \cdots \times C_{n_r}$ some $n_1, \ldots, n_r$. Thus by an induction argument on r we may reduce to the case $G = C_n = \langle x \rangle$ is cyclic.

Now ρ is determined by $\rho(x)$ and $\rho(x)^n = 1$ so $\rho(x)$ must be an nth root of unity. Moreover for each $0 \leqslant j < n$ we can define the representation

$$\rho_j(x^m) = e^{\frac{2\pi i j m}{n}} \text{ for each } m \in \mathbb{Z}$$

giving the required set of n representations. □

Lemma. *If (ρ_1, V_1) and (ρ_2, V_2) are non-isomorphic one-dimensional representations of a finite group G then $\sum_{g \in G} \rho_1(g^{-1})\rho_2(g) = 0$.* [17]

Proof. We've seen that $\mathrm{Hom}_k(V_1, V_2)$ is a representation of G under

$$g \cdot \varphi = \rho_2(g)\varphi\rho_1(g^{-1}).$$

Moreover $\sum_{g \in G} g \cdot \varphi \in \mathrm{Hom}_G(V_1, V_2) = 0$ by Schur. Pick an isomorphism $\varphi \in \mathrm{Hom}_k(V_1, V_2)$. Then

$$0 = \sum_{g \in G} \rho_2(g)\varphi\rho_1(g^{-1}) = \left(\sum_{g \in G} \rho_1(g^{-1})\rho_2(g)\right)\varphi.$$

Since φ is injective this suffices. □

If V is a representation of a group G that is completely reducible and W is any irreducible representation of G then the *W-isotypic component of V* is the smallest subrepresentation of V containing all simple subrepresentations isomorphic to W. This exists since if $(V_i)_{i \in I}$ are subrepresentations of V containing all simple subrepresentations isomorphic to W then so is $\bigcap_{i \in I} V_i$.[18]

We say that V has a *unique isotypical decomposition* if V is the direct sum of its W-isotypic components as W varies over all simple representations of V (up to isomorphism).

Corollary. *Suppose G is a finite abelian group then every complex representation V of G has a unique isotypical decomposition.*

Proof. For each homomorphism $\theta_i\colon G \to \mathbb{C}^\times$ $(i = 1, \ldots, |G|)$ we can define W_i to be the subspace of V defined by

$$W_i = \{v \in V \mid \rho(g)v = \theta_i(g)v \text{ for all } g \in G\},$$

the θ_i-isotypic component of V.

Since V is completely reducible and every irreducible rep of G is one dimensional $V = \sum W_i$. We need to show that $\sum w_i = 0$ with each $w_i \in W_i$ implies $w_i = 0$ for all i.

---

[16] This crucially uses that $\mathbb{C}^\times$ is abelian.

[17] We note in passing that if $k = \mathbb{C}$ then $\rho(g^{-1}) = \overline{\rho(g)}$ since $\rho(g)^{o(g)} = 1$.

[18] It can also be realised as the vector space sum of all subrepresentations isomorphic to W.

But $\sum w_i = 0$ with w_i in W_i certainly implies $0 = \rho(g)\sum w_i = \sum \theta_i(g)w_i$. By the last Lemma it follows that for each j,

$$0 = \sum_i \left(\sum_{g\in G} \theta_j(g^{-1})\theta_i(g) \right) w_i = \sum_{g\in G} \theta_j(g^{-1})\theta_j(g)w_j = |G|w_j.$$

Thus $w_j = 0$.[19] □

You will extend this result to all finite groups on Example Sheet 2.

[19]If you inspect the proof you'll see we only really use k is algebraically closed and $|G| \neq 0 \in k$.

4. Characters

Summary so far. We want to classify all representations of groups G. We've seen that if G is finite and k has characteristic zero then every representation V decomposes as $V \cong \bigoplus n_i V_i$ with V_i irreducible and pairwise non-isomorphic and $n_i \geqslant 0$. Moreover if k is also algebraically closed, we've seen that $n_i = \dim \mathrm{Hom}_G(V_i, V)$.

Our next goals are to classify all irreducible representations of a finite group and understand how to compute the n_i given V. We're going to do this using character theory.

4.1. Definitions.

Definition. Given a representation $\rho\colon G \to GL(V)$, the *character* of ρ is the function $\chi = \chi_\rho = \chi_V : G \to k$ given by $g \mapsto \operatorname{tr} \rho(g)$.

Since for matrices $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, the character does not depend on the choice of basis for V [$\mathrm{tr}(X^{-1}AX) = \mathrm{tr}(AXX^{-1}) = \mathrm{tr}(A)$]. By the same argument we also see that isomorphic representations have the same character.

Example. Let $G = D_6 = \langle s, t \mid s^2 = 1, t^3 = 1, sts^{-1} = t^{-1}\rangle$, the dihedral group of order 6. This acts on $\mathbb{R}^2$ by symmetries of the triangle; with t acting by rotation by $2\pi/3$ and s acting by a reflection. To compute the character of this rep we just need to know the eigenvalues of the action of each element. Each reflection (element of the form st^i) will act by a matrix with eigenvalues ± 1. Thus $\chi(st^i) = 0$ for all i. The eigenvalues of each non-trivial rotation must be non-real cube roots of unity and sum to a real number. Thus $\rho(t) = \rho(t^2) = e^{\frac{2\pi i}{3}} + e^{-\frac{2\pi i}{3}} = -1$ and $\rho(1) = 1 + 1 = 2$.

Proposition. *Let (ρ, V) be a rep of G with character χ*

(i) $\chi(e) = \dim V$;
(ii) $\chi(g) = \chi(hgh^{-1})$ for all $g, h \in G$;
(iii) If χ' is the character of (ρ', V') then $\chi + \chi'$ is the character of $V \oplus V'$.
(iv) If V is unitary[20] then $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$;

Proof.
(i) $\chi(e) = \operatorname{tr} \mathrm{id}_V = \dim V$.
(ii) $\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1}$. Thus $\rho(hgh^{-1})$ and $\rho(g)$ are conjugate and so have the same trace.
(iii) is clear.
(iv) By choosing a basis we may view ρ as a homomorphism $G \to U(n)$. Then

$$\rho(g^{-1}) = \rho(g)^{-1} = \overline{\rho(g)^T}$$

and so $\operatorname{tr} \rho(g^{-1}) = \overline{\operatorname{tr} \rho(g)}$ for all $g \in G$ since trace is invariant under taking transposes. □

The proposition tells us that the character of ρ contains very little data; an element of k for each conjugacy class in G. The extraordinary thing that we will see is that, at least when G is finite and $k = \mathbb{C}$, it contains all we need to know to reconstruct ρ up to isomorphism.

Definition. We say a function $f\colon G \to k$ is a *class function* if $f(hgh^{-1}) = f(g)$ for all $g, h \in G$. We'll write $\mathcal{C}_G$ for the k-vector space of class functions on G.

[20] For example whenever G is finite and $k = \mathbb{C}$ by Weyl's unitary trick.

Notice that if $\mathcal{O}_1, \dots, \mathcal{O}_r$ is a list of the conjugacy classes of G then the indicator functions $\mathbf{1}_{\mathcal{O}_i} : G \to \mathbb{C}$ given by

$$\mathbf{1}_{\mathcal{O}_i}(g) = \begin{cases} 1 & \text{if } g \in \mathcal{O}_i \\ 0 & \text{if } g \notin \mathcal{O}_i \end{cases}$$

form a basis for $\mathcal{C}_G$. In particular $\dim \mathcal{C}_G$ is the number of conjugacy classes in G.

Lecture 8

4.2. **Orthogonality of characters.** We'll now assume that G is a finite group and $k = \mathbb{C}$ unless we say otherwise. [21]

Recall that $\mathcal{C}_G = \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g) \text{ for all } g, h \in G\} \leqslant \mathbb{C}G$ and $\mathcal{C}_G$ has a basis consisting of indicator functions $\mathbf{1}_{\mathcal{O}_i}$ where $\mathcal{O}_1, \dots, \mathcal{O}_r$ are the conjugacy classes in G.

We can make $\mathcal{C}_G$ into a Hermitian inner product space by defining

$$\langle f_1, f_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g).$$

This even defines an Hermitian inner product on $\mathbb{C}G$ which then restricts to $\mathcal{C}_G$. The functions $\mathbf{1}_{\mathcal{O}_i}$ are pairwise orthogonal and

$$\langle \mathbf{1}_{\mathcal{O}_i}, \mathbf{1}_{\mathcal{O}_i} \rangle_G = \frac{|\mathcal{O}_i|}{|G|} = \frac{1}{|\mathcal{C}_G(x_i)|} \text{ for any } x_i \in \mathcal{O}_i.$$

Thus if $x_1, \dots, x_r$ are representatives of $\mathcal{O}_1, \dots, \mathcal{O}_r$ respectively, then we can write for $f_1, f_2 \in \mathcal{C}_G$

$$\langle f_1, f_2 \rangle_G = \sum_{i=1}^{r} \frac{1}{|C_G(x_i)|} \overline{f_1(x_i)} f_2(x_i).$$

Example. $G = D_6 = \langle s, t \mid s^2 = t^3 = e, sts = t^{-1} \rangle$ has conjugacy classes $\{e\}, \{t, t^{-1}\}, \{s, st, st^2\}$ and

$$\langle f_1, f_2 \rangle_G = \frac{1}{6}\overline{f_1(e)} f_2(e) + \frac{1}{2}\overline{f_1(s)} f_2(s) + \frac{1}{3}\overline{f_1(t)} f_2(t).$$

Morever if $\mathbb{C}$ is the trivial representation of D_6 and V is the natural representation of degree 2 then

$$\chi_{\mathbb{C}} = \mathbf{1}_G \text{ and } \chi_V(e) = 2, \chi_V(s) = 0 \text{ and } \chi_V(t) = -1$$

so

$$\begin{aligned} \langle \chi_{\mathbb{C}}, \chi_{\mathbb{C}} \rangle_G = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} &\quad = \quad 1 \\ \langle \chi_V, \chi_V \rangle_G = \frac{4}{6} + \frac{0}{2} + \frac{1}{3} &\quad = \quad 1 \\ \langle \chi_{\mathbb{C}}, \chi_V \rangle_G = \frac{2}{6} + \frac{0}{2} + \frac{-1}{3} &\quad = \quad 0 \end{aligned}$$

[21] If $k = \overline{k}$ has characteristic zero the main results are all essentially true but the story needs to be told slightly differently.

Theorem (Orthogonality of characters)**.** *If V and W are complex irreducible representations of a finite group G then*

$$\langle \chi_V, \chi_W \rangle_G = \begin{cases} 1 & \textit{if } V \cong W \\ 0 & \textit{otherwise.} \end{cases}$$

This should remind you of Schur's Lemma and in fact the similarity is no coincidence. It is a corollary of Schur. Before we prove it we need a couple of lemmas.

Lemma. *If V and W are (unitary) representations of G then*

$$\chi_{\mathrm{Hom}_k(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g)$$

for each $g \in G$.

Proof. Given $g \in G$ we may choose bases $v_1, \dots, v_n$ for V and $w_1, \dots, w_m$ for W such that $gv_i = \lambda_i v_i$ and $gw_j = \mu_j w_j$ for some $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m \in \mathbb{C}$. Then the functions $\alpha_{ij}(v_k) = \delta_{jk} w_i$ extend to linear maps that form a basis for $\mathrm{Hom}_k(V, W)$[22] and

$$(g \cdot \alpha_{ij})(v_k) = g \cdot (\alpha_{ij}(g^{-1} \cdot v_k)) = \delta_{jk}\lambda_k^{-1}\mu_i w_i$$

thus $g \cdot \alpha_{ij} = \lambda_j^{-1}\mu_i\alpha_{ij}$ and

$$\chi_{\mathrm{Hom}(V,W)}(g) = \sum_{i,j} \lambda_j^{-1}\mu_i = \chi_V(g^{-1})\chi_W(g) = \overline{\chi_V(g)}\chi_W(g)$$

as claimed. □

Lemma. *If U is a representation of G then*

$$\dim U^G = \frac{1}{|G|}\sum_{g \in G} \chi_U(g) = \langle 1, \chi_U \rangle.$$

Proof. We've seen previously that $\pi \colon U \to U$; $\pi(u) = \frac{1}{|G|}\sum_{g \in G} gu$ defines a projection from U onto U^G. Thus

$$\dim U^G = \operatorname{tr}\pi = \frac{1}{|G|}\sum_{g \in G} \operatorname{tr} g = \frac{1}{|G|}\sum_{g \in G} \chi_U(g) = \langle \mathbf{1}_G, \chi_U \rangle$$

as required. □

We can use these two lemmas to prove the following.

Proposition. *If V and W are representations of G then*

$$\dim \mathrm{Hom}_G(V, W) = \langle \chi_V, \chi_W \rangle.$$

Proof. By the lemmas

$$\begin{aligned} \dim \mathrm{Hom}_G(V, W) &= \langle \mathbf{1}, \overline{\chi_V}\chi_W \rangle \\ &= \frac{1}{|G|}\sum_{g \in G} \overline{\chi_V}(g)\chi_W(g) \\ &= \langle \chi_V, \chi_W \rangle_G \end{aligned}$$

as required. □

[22] α_{ij} is represented by the matrix with a 1 in entry ij and 0s elsewhere with respect to the given bases since v_j maps to w_i and all other v_k map to 0

Corollary (Orthogonality of characters)**.** *If V and W are irreducible representations of G then*

$$\langle \chi_V, \chi_W \rangle_G = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

In particular if $\chi_V = \chi_W$ then $V \cong W$.

Proof. Apply the Proposition and Schur's Lemma, noting that if $\chi_V = \chi_W$, with V and W irreducible, then $\dim \mathrm{Hom}_G(V, W) = \langle \chi_V, \chi_V \rangle_G > 0$ and so $V \cong W$. $\square$

Corollary. *If (ρ, V) is a representation of G then*

$$V \cong \bigoplus_{\substack{W \text{ irred} \\ \text{rep of } G/\sim}} \langle \chi_W, \chi_\rho \rangle_G W.$$

In particular if σ is another representation with $\chi_\rho = \chi_\sigma$ then $\sigma \cong \rho$.

Proof. By Machke's Theorem there are non-negative integers n_W such that

$$V \cong \bigoplus_{\substack{W \text{ irred} \\ \text{rep of } G/\sim}} n_W W.$$

Moreover we've seen that $n_W = \dim \mathrm{Hom}_G(W, V)$ and $\dim \mathrm{Hom}_G(W, V) = \langle \chi_W, \chi_\rho \rangle_G$ by the Proposition so the first part follows.

Since

$$\bigoplus_{\substack{W \text{ irred} \\ \text{rep of } G/\sim}} \langle \chi_W, \chi_\rho \rangle_G W$$

only depend on χ_ρ the second part follows. $\square$

Notice that complete irreducibility was a key part of the proof of this corollary, as well as orthogonality of characters. For example the two reps of $\mathbb{Z}$ given by $1 \mapsto id_{\mathbb{C}^2}$ and $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are not isomorphic but have the same trace. Indeed they both have trivial subrepresentations with trivial quotient. The slogan might be 'Characters can't see gluing data.'

Corollary. *If ρ is a complex representation of G with character χ then ρ is irreducible if and only if $\langle \chi, \chi \rangle_G = 1$.*

Proof. One direction follows immediately from the theorem on orthogonality of characters. For the other direction, assume that $\langle \chi, \chi \rangle_G = 1$. Then we may write $\chi = \sum n_W \chi_W$ for some non-negative integers n_W. By orthogonality of characters $1 = \langle \chi, \chi \rangle = \sum n_W^2$. Thus $\chi = \chi_W$ for some W and ρ is irreducible. $\square$

This is a good way of calcuating whether a representation is irreducible.

Example.

Consider the action of D_6 on $\mathbb{C}^2$ by extending the symmetries of a triangle. $\chi(1) = 2$, $\chi(s) = \chi(st) = \chi(st^2) = 0$, and $\chi(t) = \chi(t^2) = -1$. Now

$$\langle \chi, \chi \rangle = \frac{1}{6}(2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1$$

so this rep is irreducible. Of course we had already established this by hand in (an exercise in) Lecture 3.

Theorem (The character table is square)**.** *The irreducible characters of a finite group G form a orthonormal basis for the space of class functions $\mathcal{C}_G$ with respect to $\langle f_1, f_2\rangle_G = \frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)}f_2(g)$.*

Proof. We already know that the irreducible characters form an orthonormal set. So it remains to show that they span $\mathcal{C}_G$.

Let $I = \langle \chi_1, \ldots, \chi_r\rangle$ be the $\mathbb{C}$-linear span of the irreducible characters. We need to show that

$$I^\perp := \{f \in \mathcal{C}_G : \langle f, \chi_i\rangle_G = 0 \text{ for } i = 1, \ldots r\} = 0.$$

Lecture 9

Suppose $f \in \mathcal{C}_G$. For each representation (ρ, V) of G we may define a linear map $\varphi = \varphi_{f,V} \in \mathrm{Hom}_k(V, V)$ by $\varphi = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\rho(g)$.

Now,

$$\rho(h)^{-1}\varphi\rho(h) = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\rho(h^{-1}gh) = \frac{1}{|G|}\sum_{g'\in G}\overline{f(g')}\rho(g') = \varphi$$

since f is a class function and $G \to G; g \mapsto hgh^{-1}$ is a bijection, and we see that in fact $\varphi_{f,V} \in \mathrm{Hom}_G(V, V)$.

Moreover, if V is an irreducible representation then $\varphi_{f,V} = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}$ by Schur's Lemma. If additionally $f \in I^\perp$ then

$$\lambda \dim V = \operatorname{tr}\varphi_{f,V} = \langle f, \chi_V\rangle = 0$$

so $\varphi_{f,V} = 0$.

But every representation breaks up as a direct sum of irreducible representations $V = \bigoplus V_i$ and $\varphi_{f,V}$ breaks up as $\bigoplus \varphi_{f,V_i}$. So $\varphi_{f,V} = 0$ whenever $f \in I^\perp$ and V is a representation of G.

But now if we take V to be the regular representation $\mathbb{C}G$ then

$$0 = \varphi_{f,\mathbb{C}G}\delta_e = |G|^{-1}\sum_{g\in G}\overline{f(g)}\delta_g = |G|^{-1}\overline{f}.$$

Thus $f = 0$. $\square$

Corollary. *The number of irreducible representations is the number of conjugacy classes in the group.*

Notation. *For $g \in G$ we'll write*

$$[g]_G := \{xgx^{-1} : x \in G\}$$

for the conjugacy class containing g.

Corollary. *For each $g \in G$, $\chi(g)$ is real for every character χ if and only if $[g]_G = [g^{-1}]_G$.*

Proof. Since $\chi(g^{-1}) = \overline{\chi(g)}$, $\chi(g) \in \mathbb{R}$ if and only if $\chi(g) = \chi(g^{-1})$.

Since the irreducible characters span the space of class functions, $\chi(g) = \chi(g^{-1})$ for every character χ if and only if $f(g) = f(g^{-1})$ for every $f \in \mathcal{C}_G$.

Since $\mathbf{1}_{[g]_G}$ is a class function, this last is equivalent to $[g] = [g^{-1}]_G$ as required. $\square$

4.3. **Character tables.** We now want to classify all the irreducible representations of a given finite group and we know that it suffices to write down the characters of each one.

The character table of a group is defined as follows: we list the conjugacy classes of G, $[g_1]_G, \dots, [g_r]_G$ (by convention always $g_1 = e$); we then list the irreducible characters $\chi_1, \dots, \chi_r$ (by convention $\chi_1 = \chi_{\mathbb{C}}$ the character of the trivial rep; then we write the matrix

$$\begin{array}{c|cccccc} & e & g_2 & \cdots & g_i & \cdots & g_r \\ \hline \chi_1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & & & & \vdots & & \\ \chi_j & \cdots & \cdots & \cdots & \chi_j(g_i) & \cdots & \cdots \\ \vdots & & & & \vdots & & \\ \chi_r & & & & \vdots & & \end{array}$$

We sometimes write the size of the conjugacy class $[g_i]_G$ above g_i and sometimes $|C_G(g_i)|$.[23]

Examples.

(1) $C_3 = \langle x \rangle$ and let $\omega = e^{\frac{2\pi i}{3}}$ so $\omega^2 = \overline{\omega}$.

$$\begin{array}{c|ccc} & e & x & x^2 \\ \hline \chi_1 & 1 & 1 & 1 \\ \chi_2 & 1 & \omega & \overline{\omega} \\ \chi_3 & 1 & \overline{\omega} & \omega \end{array}$$

Notice that the rows are indeed pairwise orthogonal with respect to $\langle -, - \rangle_G$. The columns are too with respect to the standard inner product in this case.

(2) S_3

There are three conjugacy classes $\{e\}$, $\{(12), (23), (13)\}$ and $\{(123), (132)\}$. Thus there are also three irreducible representations. We know that the trivial representation has character $\mathbf{1}_G$ for all $g \in G$. We also know another 1-dimensional representation $\epsilon\colon S_3 \to \{\pm 1\}$ given by $g \mapsto 1$ if g is even and $g \mapsto -1$ if g is odd.

To compute the character χ of the last representation we may use orthogonality of characters. Let $\chi(e) = a$, $\chi((12)) = b$ and $\chi((123)) = c$ (a, b and c are each real since each g in S_3 is conjugate to its inverse). We know that

$$\begin{aligned} 0 = \langle \mathbf{1}, \chi \rangle &= \frac{1}{6}(a + 3b + 2c), \\ 0 = \langle \epsilon, \chi \rangle &= \frac{1}{6}(a - 3b + 2c) \text{ and} \\ 1 = \langle \chi, \chi \rangle &= \frac{1}{6}(a^2 + 3b^2 + 2c^2). \end{aligned}$$

Thus we see quickly that $b = 0$, $a + 2c = 0$ and $a^2 + 2c^2 = 6$. We also know that a is a positive integer. Thus $a = 2$ and $c = -1$.

[23] Recall that $|[g]_G||C_G(g)| = |G|$ by the orbit-stabiliser theorem.

$\lvert C_{S_3}(g)\rvert$	1	3	2
g	e	(12)	(123)
$\mathbf{1}$	1	1	1
ϵ	1	-1	1
χ	2	0	-1

In fact we already knew about this 2-dimensional representation; it is the one coming from the symmetries of a triangle inside $\mathbb{R}^2$.

Once again the columns are orthogonal with respect to the standard inner product. If we compute their length we get:

$$\begin{aligned} 1^2+1^2+2^2 &= 6 = |C_{S_3}(e)| \\ 1^2+(-1)^2+0^2 &= 2 = |C_{S_3}((12))| \\ 1^2+1^2+(-1)^2 &= 3 = |C_{S_3}((123))|. \end{aligned}$$

This is an instance of a more general phenomenon.

Proposition (Column Orthogonality)**.** *If G is a finite group and $\chi_1,\dots,\chi_r$ is a complete list of the irreducible characters of G then for each $g,h\in G$,*

$$\sum_{i=1}^{r}\overline{\chi_i(g)}\chi_i(h)=\begin{cases}0 & \text{if } [g]_G\neq[h]_G\\ |C_G(g)| & \text{if } [g]_G=[h]_G.\end{cases}$$

In particular

$$\sum_{i=1}^{r}(\dim V_i)^2=\sum_{i=1}^{r}\chi_i(e)^2=|G|.$$

Proof. Let X be the character table thought of as a matrix; $X_{ij}=\chi_i(g_j)$ and let D be the diagonal matrix with entries $D_{ii}=|C_G(g_i)|$.

Orthogonality of characters tell us that

$$\langle\chi_i,\chi_j\rangle_G=\sum_k |C_G(g_k)|^{-1}\overline{X_{ik}}X_{jk}=\delta_{ij}$$

ie $\overline{X}D^{-1}X^T=I$.

Since X is square and invertible and D is real we may rewrite this as

$$D^{-1}\overline{X}^T=X^{-1}.$$

Thus $\overline{X}^TX=D$. That is

$$\sum_k\overline{\chi_k(g_i)}\chi_k(g_j)=\delta_{ij}|C_G(g_i)|$$

as required. □

4.4. **Permuation representations.** Recall that if X is a finite set with G-action then $\mathbb{C}X=\{f\colon X\to\mathbb{C}\}$ is a representation of G via $gf(x)=f(g^{-1}x)$ for all $f\in\mathbb{C}X$, $g\in G$ and $x\in X$ or equivalently $g\cdot\delta_x=\delta_{g\cdot x}$ for all $g\in G$ and $x\in X$.

Lemma. *If χ is the character of $\mathbb{C}X$ then $\chi(g)=|\{x\in X\mid gx=x\}|$*

Proof. If $X=\{x_1,\dots,x_d\}$ and $gx_i=x_j$ then $g\delta_{x_i}=\delta_{x_j}$ so the ith column of g has a 1 in the jth entry and zeros elsewhere. So it contributes 1 to the trace precisely if $x_i=x_j$. □

Lecture 10

Corollary. *If $V_1, \ldots, V_r$ is a complete list of irreducible reps of a finite group G then the regular representation decomposes as*

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} (\dim V_i) V_i.$$

In particular every irreducible representation is isomorphic to a subrepresentation of the regular representation and

$$|G| = \sum (\dim V_i)^2.$$

Proof. We need to prove $\dim \operatorname{Hom}_G(V_i, \mathbb{C}G) = \dim V_i$ for $i = 1, \ldots, r$. But

$$\begin{aligned} \dim \operatorname{Hom}_G(V_i, \mathbb{C}G) &= \langle \chi_{V_i}, \chi_{\mathbb{C}G} \rangle_G \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V_i}}(g) \chi_{\mathbb{C}G}(g) \\ &= \dim V_i \end{aligned}$$

since $\chi_{\mathbb{C}G}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$ and $\overline{\chi_{V_i}} = \dim V_i$. □

Proposition (Burnside's Lemma)**.** *Let G be a finite group and X a finite set with a G-action. Then $\langle \mathbf{1}, \chi_{\mathbb{C}X} \rangle_G$ is the number of orbits of G on X.*

Proof.

$$\begin{aligned} |G| \langle \mathbf{1}, \chi_{\mathbb{C}X} \rangle_G &= \sum_{g \in G} \chi_{\mathbb{C}X}(g) \\ &= \sum_{g \in G} |\{x \in X \mid gx = x\} \\ &= |\{(g, x) \in G \times X \mid gx = x\}| \\ &= \sum_{x \in X} |\{g \in G \mid gx = x\} \\ &= \sum_{x \in X} |\operatorname{Stab}_G(x)| \end{aligned}$$

So

$$\begin{aligned} \langle \mathbf{1}, \chi_{\mathbb{C}G} \rangle_G &= \sum_{x \in X} \frac{1}{|\operatorname{Orb}_G(x)|} \text{ (by the Orbit-Stabiliser Theorem)} \\ &= \sum_{\substack{\text{orbits} \\ \mathcal{O}_i}} \left(\sum_{x \in \mathcal{O}_i} \frac{1}{|\mathcal{O}_i|} \right) \\ &= \text{number of orbits} \end{aligned}$$

as required. □

Note that if $X = \bigcup_{i=1}^{t} \mathcal{O}_i$ is the orbit decomposition of X then we saw before that $\mathbb{C}X = \bigoplus_{i=1}^{t} \mathbb{C}\mathcal{O}_i$ so Burnside's Lemma says that each $\mathbb{C}\mathcal{O}_i$ contains precisely one copy of the trivial representation $\mathbb{C}$ when it is decomposed as a direct sum of irreducible representations — the span of the constant function.

If X and Y are two sets with a G-action we may view $X \times Y$ as a set with a G-action via $(g,(x,y)) \mapsto (gx, gy)$ for all $g \in G$, $x \in X$ and $y \in Y$.

Lemma. *If X and Y are both finite sets with G-action then*

$$\chi_{\mathbb{C}X\times Y} = \chi_{\mathbb{C}X} \cdot \chi_{\mathbb{C}Y}.$$

Proof. Since

$$\{(x,y) \in X \times Y : g \cdot (x,y) = (x,y)\} = \{x \in X : g \cdot x = x\} \times \{y \in Y : g \cdot y = y\}$$

this follows from the lemma that computes characters of permutation representations in terms of fixed points. □

Corollary. *If G is a finite group and X and Y are finite sets with a G-action then $\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}Y}\rangle_G$ is the number of G-orbits on $X \times Y$.*

Proof. $\langle \chi_X, \chi_Y \rangle_G = \frac{1}{|G|}\sum_{g\in G} \chi_X(g)\chi_Y(g) = \langle \mathbf{1}, \chi_{X\times Y}\rangle_G$ and the result follows from Burnside's Lemma. □

Remark. If X is any set with a G-action with $|X| > 1$ then $\{(x,x)|x \in X\} \subset X \times X$ is G-stable and so is the complement $\{(x,y) \in X \times X \mid x \neq y\}$. Moreover both are non-empty.

Definition. We say that G acts 2-*transitively* on X if X has at least 2 elements and for all $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq y_1$ and $x_2 \neq y_2$ there is $g \in G$ such that $g \cdot x_1 = x_2$ and $g \cdot y_1 = y_2$.[24] Equivalently G has only two orbits on $X \times X$.

Example. The natural action of S_n on $\{1, \dots, n\}$ is 2-transitive whenever $n \geqslant 2$.

By the Corollary if G acts 2-transitively on X then $\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}X}\rangle = 2$. Thus if $\mathbb{C}X \cong \sum n_i V_i$ with V_i irreducible and pairwise non-isomorphic then $\sum n_i^2 = 2$ and so $\mathbb{C}X$ has two non-isomorphic irreducible summands — explicitly these are the set of constant functions and the set $V = \left\{f \in \mathbb{CX} : \sum_{x\in X} f(x) = 0\right\}$. Then χ_V is an irreducible character with

$$\chi_V(g) = (\text{number of fixed points of } g \text{ on } X) - 1.$$

Exercise. If $G = GL_2(\mathbb{F}_p)$ then decompose the permutation rep of G coming from the action of G on $\mathbb{F}_p \cup \{\infty\}$ by Möbius transformations.

Examples.

(1) $G = S_4$: the character table is as follows

$\lvert C_G(x_i)\rvert$	24	8	3	4	4
$\lvert\mathcal{O}_i\rvert$	1	3	8	6	6
x_i	e	(12)(34)	(123)	(12)	(1234)
$\mathbf{1}$	1	1	1	1	1
ϵ	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
χ_4	3	-1	0	-1	1
χ_5	2	2	-1	0	0

[24] i.e. $g \cdot (x_1, y_1) = (x_2, y_2)$.

Proof. The trivial **1** and sign ϵ characters may be constructed in the same way as for S_3.

By our discussion above

$$\chi_{\mathbb{C}\{1,2,3,4\}} = \mathbf{1} + \chi_V$$

for some irreducible representation V of dimension 3 and we may define χ_3 to be χ_V. Its values $\chi_3(g)$ are (number of fixed points of g) $- 1$ and can be computed directly to be the claimed values.

We saw on Example Sheet 1 (Q2) that given a 1-dimensional represntation θ and an irreducible representation ρ we may form another irreducible representation $\theta\otimes\rho$ by $\theta\otimes\rho(g) = \theta(g)\rho(g)$. It is not hard to see that $\chi_{\theta\otimes\rho}(g) = \theta(g)\chi_\rho(g)$. Thus we get another irreducible character $\epsilon\chi_3$ that we compute by multiplying characters and may set this to be χ_4.

We can then complete the character table using column orthogonality: We note that $24 = 1^2 + 1^2 + 3^2 + 3^2 + \chi_5(e)^2$ thus $\chi_5(e) = 2$. Then using $\sum_1^5 \chi_i(1)\chi_i(g) = 0$ we can construct the remaining values in the table. □

(2) $G = A_4$. Each irreducible representation of S_4 may be restricted to A_4 and its character values on elements of A_4 will be unchanged. In this way we get three characters of A_4: **1**, $\psi_2 = \chi_3|_{A_4}$ and $\psi_3 = \chi_5|_{A_4}$. Of course **1** is irreducible since it has dimension 1. Computing

$$\langle\psi_2, \psi_2\rangle_{A_4} = \frac{1}{12}(3^2 + 3(-1)^2 + 8(0^2) = 1$$

we see ψ_2 also remains irreducible.[25] However

$$\langle\psi_3, \psi_3\rangle = \frac{1}{12}(2^2 + 3(2^2) + 8(-1)^2) = 2$$

so ψ_3 breaks up into two non-isomorphic irreducible reps of A_4.

Exercise. Use this information to construct the whole character table of A_4.

[25] Note that the conjugacy class of (123) in S_4 breaks into two classes of size 4 in A_4 but that doesn't matter for this calculation since ψ_2 takes the same value on these two classes.

5. The character ring

We've seen already that algebraic structure on $\mathcal{C}_G$ for a finite group G is a shadow of representation theoretic information: if V_1 and V_2 are representations that $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$, $\chi_0 = 0$, $\dim \operatorname{Hom}_G(V_1, V_2) = \langle \chi_1, \chi_2 \rangle$. An alternative way of viewing this is that the category of representations is a model for the algebraic structure on $\mathcal{C}_G$.

We've seen that $\chi_{\mathbb{C}X \times Y} = \chi_{\mathbb{C}X} \cdot \chi_{\mathbb{C}Y}$. We've also seen that when θ and ρ are representations with $\dim \theta = 1$ there is a representation $\theta \otimes \rho$ such that $\chi_{\theta \otimes \rho} = \chi_\theta \cdot \chi_\rho$. We want to generalise these i.e. given any representations ρ_1, ρ_2 build a representation $\rho_1 \otimes \rho_2$ such that $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}$.

Lecture 11

5.1. **Tensor products.** Suppose that V and W are vector spaces over a field k, with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ respectively. We may view $V \oplus W$ either as the vector space with basis $v_1, \ldots, v_m, w_1, \ldots, w_n$ (so $\dim V \oplus W = \dim V + \dim W$) or more abstractly as the vector space of pairs (v, w) with $v \in V$ and $w \in W$ and pointwise operations.

Definition. The *tensor product* $V \otimes W$ of V and W is the k-vector space with basis given by symbols $v_i \otimes w_j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ and so

$$\dim V \otimes W = \dim V \cdot \dim W.$$

Example. If X and Y are sets then $kX \otimes kY$ has basis $\delta_x \otimes \delta_y$ for $x \in X$ and $y \in Y$. Notice that

$$\alpha_{X \times Y} : kX \otimes kY \to kX \times Y; \;\; \delta_x \otimes \delta_y \to \delta_{(x,y)}$$

defines an isomorphism.

Notation. *If* $v = \sum \lambda_i v_i \in V$ *and* $w = \sum \mu_j w_j \in W$,

$$v \otimes w \colon = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j) \in V \otimes W.$$

For example $\alpha_{X \times Y}(f \otimes g)(x, y) = f(x)g(y)$.

Note that, in general, not every element of $V \otimes W$ may be written in the form $v \otimes w$ (eg $v_1 \otimes w_1 + v_2 \otimes w_2$). The smallest number of summands that are required is known as the *rank* of the tensor.

Lemma. *The map* $V \times W \to V \otimes W$ *given by* $(v, w) \mapsto v \otimes w$ *is bilinear.*

Proof. We should prove that if $x, x_1, x_2 \in V$ and $y, y_1, y_2 \in W$ and $\nu_1, \nu_2 \in k$ then

$$x \otimes (\nu_1 y_1 + \nu_2 y_2) = \nu_1 (x \otimes y_1) + \nu_2 (x \otimes y_2)$$

and

$$(\nu_1 x_1 + \nu_2 x_2) \otimes y = \nu_1 (x_1 \otimes y) + \nu_2 (x_2 \otimes y).$$

We'll just do the first; the second follows by symmetry.

Write $x = \sum_i \lambda_i v_i$, $y_k = \sum_j \mu_j^k w_j$ for $k = 1, 2$. Then

$$x \otimes (\nu_1 y_1 + \nu_2 y_2) = \sum_{i,j} \lambda_i (\nu_1 \mu_j^1 + \nu_2 \mu_j^2) v_i \otimes w_j$$

and

$$\nu_1(x \otimes y_1) + \nu_2(x \otimes y_2) = \nu_1 \left(\sum_{i,j} \lambda_i \mu_j^1 (v_i \otimes w_j) \right) + \nu_2 \left(\sum_{i,j} \lambda_i \mu_j^2 (v_i \otimes w_j) \right).$$

These are equal. $\square$

Exercise. Show that given vector spaces U, V and W there is a $1-1$ correspondence

$$\{\text{linear maps } V \otimes W \to U\} \longrightarrow \{\text{bilinear maps } V \times W \to U\}$$

given by precomposition with the bilinear map $(v, w) \to v \otimes w$ above.

Lemma. *If $x_1, \ldots, x_m$ is any basis of V and $y_1, \ldots, y_n$ is any basis of W then $x_i \otimes y_j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ is a basis for $V \otimes W$. Thus the definition of $V \otimes W$ does not depend on the choice of bases.*

Proof. It suffices to prove that the set $\{x_i \otimes y_j\}$ spans $V \otimes W$ since it has size mn. But if $v_i = \sum_r A_{ri} x_r$ and $w_j = \sum_s B_{sj} y_s$ then $v_i \otimes w_j = \sum_{r,s} A_{ri} B_{sj} x_r \otimes y_s$. $\square$

Remark (for enthusiastists). In fact we could have defined $V \otimes W$ in a basis independent way in the first place: let F be the (infinite dimensional) vector space with basis $\langle v \otimes w \mid v \in V, w \in W \rangle$; and R be the subspace generated by

$$x \otimes (\nu_1 y_1 + \nu_2 y_2) - \nu_1(x \otimes y_1) - \nu_2(x \otimes y_2)$$

and

$$(\nu_1 x_1 + \nu_2 x_2) \otimes y - \nu_1(x_1 \otimes y) - \nu_2(x_2 \otimes y)$$

for all $x, x_1, x_2 \in V$, $y, y_1, y_2 \in W$ and $\nu_1, \nu_2 \in k$; then $V \otimes W \cong F/R$ naturally.

Exercise. Show that for vector spaces U, V and W there is a natural (basis independent) isomorphism

$$(U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W).$$

Definition. Suppose that V and W are vector spaces with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ and $\varphi \colon V \to V$ and $\psi \colon W \to W$ are linear maps. We can define $\varphi \otimes \psi \colon V \otimes W \to V \otimes W$ as follows:

$$(\varphi \otimes \psi)(v_i \otimes w_j) = \varphi(v_i) \otimes \psi(w_j).$$

Example. If φ is represented by the matrix A_{ij} and ψ is represented by the matrix B_{ij} and we order the basis $v_i \otimes w_j$ lexicographically (ie $v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_1 \otimes w_n, v_2 \otimes w_1, \ldots, v_m \otimes w_n$) then $\varphi \otimes \psi$ is represented by the block matrix

$$\begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

since

$$(\varphi \otimes \psi)(v_i \otimes w_j) = \left(\sum_k A_{ki} v_k \right) \otimes \left(\sum_l B_{lj} w_l \right) = \sum_{k,l} A_{ki} B_{lj} (v_k \otimes w_l).$$

Lemma. *The linear map $\varphi \otimes \psi$ does not depend on the choice of bases.*

Proof. It suffices to show that for any $v \in V$ and $w \in W$,

$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w).$$

Writing $v = \sum \lambda_i v_i$ and $w = \sum \mu_j w_j$ we see

$$(\varphi \otimes \psi)(v \otimes w) = \sum_{i,j} \lambda_i \mu_j \varphi(v_i) \otimes \psi(w_j) = \varphi(v) \otimes \psi(w)$$

as required. □

Remark. The proof really just says $V \times W \to V \otimes W$ defined by $(v, w) \mapsto \varphi(v) \otimes \psi(w)$ is bilinear and $\varphi \otimes \psi$ is its correspondent in the bijection

$$\{\text{linear maps } V \otimes W \to V \otimes W\} \to \{\text{bilinear maps } V \times W \to V \otimes W\}$$

from earlier.

Lemma. *Suppose that* $\varphi, \varphi_1, \varphi_2 \in \mathrm{Hom}_k(V, V)$ *and* $\psi, \psi_1, \psi_2 \in \mathrm{Hom}_k(W, W)$

(i) $(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2) \in \mathrm{Hom}_k(V \otimes W, V \otimes W)$*;*
(ii) $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W}$*; and*
(iii) $\mathrm{tr}(\varphi \otimes \psi) = \mathrm{tr}\,\varphi \cdot \mathrm{tr}\,\psi$.

Proof. Given $v \in V$, $w \in W$ we can use the previous lemma to compute

$$(\varphi_1\varphi_2) \otimes (\psi_1\psi_2)(v \otimes w) = \varphi_1\varphi_2(v) \otimes \psi_1\psi_2(w) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2)(v \otimes w).$$

Since elements of the form $v \otimes w$ span $V \otimes W$ and all maps are linear it follows that

$$(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2)$$

as required.

(ii) is clear.

(iii) For the formula relating traces it suffices to stare at the example above:

$$\mathrm{tr}\begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \sum_{i,j} B_{ii}A_{jj} = \mathrm{tr}\,A\,\mathrm{tr}\,B.$$

□

Definition. Given two representation (ρ, V) and (σ, W) of a group G we can define the representation $(\rho \otimes \sigma, V \otimes W)$ by $(\rho \otimes \sigma)(g) = \rho(g) \otimes \sigma(g)$.

Note that $(\rho \otimes \sigma, V \otimes W)$ is a representation of G by parts (i) and (ii) of the last lemma. Moreover $\chi_{\rho\otimes\sigma} = \chi_\rho \cdot \chi_\sigma$ by part (iii).

Lecture 12

Remarks.

(1) Tensor product of representations defined above is consistent with our earlier notion when one of the representations is one-dimensional.
(2) If X, Y are finite sets with G-action it is easy to verify that the isomorphism of vector spaces

$$\alpha_{X\times Y} : kX \otimes kY \cong kX \times Y; \;\; \delta_x \otimes \delta_y \to \delta_{(x,y)}$$

is an isomorphism of representations of G.

Definition. The *character ring* $R(G)$ of a group G is defined by

$$R(G) := \{\chi_1 - \chi_2 \mid \chi_1, \chi_2 \text{ are characters of reps of } G\} \subset \mathcal{C}_G.$$

- Since $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$, $R(G)$ is an additive subgroup of $\mathcal{C}_G$.
- Since $\mathbf{1}_G$ is a character $R(G)$ has the multiplicative unit of $\mathcal{C}_G$.
- Since $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$, $R(G)$ is closed under multiplication.

Thus $R(G)$ forms a (commutative) subring of $\mathcal{C}_G$.

If (ρ, V) is a representation of G and (σ, W) is a representation of another group H then

$$\rho \otimes \sigma \colon G \times H \to GL(V \otimes W);\quad (g,h) \mapsto \rho(g) \otimes \sigma(h)$$

defines a representation of $G \times H$ by parts (i) and (ii) of the last lemma. Part (iii) of the lemma gives that

$$(\chi_V \otimes \chi_W)(g,h) \colon = \chi_{V \otimes W}(g,h) = \chi_V(g)\chi_W(h).$$

Thus

$$R(G) \times R(H) \to R(G \times H);\quad (\chi_V, \chi_W) \mapsto \chi_{V \otimes W}$$

defines a $\mathbb{Z}$-bilinear map.

The construction of $V \otimes W$ as a representation of G last time, in the case $G = H$, comes from restricting this construction along the homomorphism

$$G \to G \times G;\quad g \mapsto (g,g).$$

Proposition. *Suppose G and H are finite groups, $(\rho_1, V_1), \ldots, (\rho_r, V_r)$ are all the simple complex representations of G and $(\sigma_1, W_1), \ldots, (\sigma_s, W_s)$ are all the simple complex representations of H.*

For each $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, $(\rho_i \otimes \sigma_j, V_i \otimes W_j)$ is an irreducible complex representation of $G \times H$. Moreover, all the irreducible representations of $G \times H$ arise in this way.

We have seen this before when G and H are abelian since then all these representations are 1-dimensional.

Proof. Let $\chi_1, \ldots, \chi_r$ be the characters of $V_1, \ldots, V_r$ and $\psi_1, \ldots, \psi_s$ the characters of $W_1, \ldots, W_s$.

The character of $V_i \otimes W_j$ is $\chi_i \otimes \psi_j \colon (g,h) \mapsto \chi_i(g)\psi_j(h)$. Then

$$\langle \chi_i \otimes \psi_j, \chi_k \otimes \psi_l \rangle_{G \times H} = \langle \chi_i, \chi_k \rangle_G \langle \psi_j, \psi_l \rangle_H = \delta_{ik}\delta_{jl}.$$

So the $\chi_i \otimes \psi_j$ are irreducible and pairwise distinct.

Now

$$\sum_{i,j} (\dim V_i \otimes W_j)^2 = \left(\sum_i (\dim V_i)^2\right)\left(\sum_j (\dim W_j)^2\right) = |G||H| = |G \times H|$$

so we must have them all.[26] $\square$

Question. If V and W are irreducible as representations of G then can $V \otimes W$ be irreducible as a representation of G?

We've seen the answer is yes is one of V and W is one-dimensional but it is not usually true.

[26]We could complete the proof by instead considering conjugacy classes in $G \times H$ to show that $\dim \mathcal{C}_{G \times H} = \dim \mathcal{C}_G \cdot \dim \mathcal{C}_H$.

Example. $G = S_3$

	1	3	2
	e	(12)	(123)
$\mathbf{1}$	1	1	1
ϵ	1	-1	1
V	2	0	-1

Clearly, $\mathbf{1} \otimes W = W$ always. $\epsilon \otimes \epsilon = \mathbf{1}$, $\epsilon \otimes V = V$ and $V \otimes V$ has character χ^2 given by $\chi^2(1) = 4$, $\chi^2(12) = 0$ and $\chi^2(123) = 1$. Thus χ^2 decomposes as $\mathbf{1} + \epsilon + \chi$.

In general if $\chi_1, \ldots, \chi_r$ are the irreducible characters then for all $1 \leqslant i, j \leqslant r$,

$$\chi_i \chi_j = \sum_{k=1}^{r} a_{i,j}^k \chi_k$$

with $a_{i,j}^k \in \mathbb{N}_0$ and these numbers $a_{i,j}^k$ completely determine the structure of $R(G)$ as a ring since $R(G) = \bigoplus_{i=1}^r \mathbb{Z}\chi_i$ as an additive group.

In fact $V \otimes V, V \otimes V \otimes V, \ldots$ are never irreducible if $\dim V > 1$. However considering them can help us build new irreducible representations.

5.2. **Symmetic and Exterior Powers.** For any vector space V, define

$$\sigma = \sigma_V : V \otimes V \to V \otimes V \text{ by } \sigma(v \otimes w) \mapsto w \otimes v \text{ for all } v, w \in V.$$

Exercise. Check this does uniquely define a linear map. Hint: Show that $(v, w) \mapsto w \otimes v$ is a bilinear map.

Notice that $\sigma^2 = \mathrm{id}$ and so, if $\mathrm{char} k \neq 2$, σ decomposes $V \otimes V$ into two eigenspaces:

$$S^2 V := \{a \in V \otimes V \mid \sigma a = a\}$$
$$\Lambda^2 V := \{a \in V \otimes V \mid \sigma a = -a\}.$$

In fact this is the isotypical decomposition of $V \otimes V$ as a rep of C_2.

Lemma. *Suppose $v_1, \ldots, v_m$ is a basis for V.*

(i) $S^2 V$ has a basis $v_i v_j := \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i)$ for $1 \leqslant i \leqslant j \leqslant m$. [27]

(ii) $\Lambda^2 V$ has a basis $v_i \wedge v_j := \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i)$ for $1 \leqslant i < j \leqslant m$.[28]

Thus $\dim S^2 V = \frac{1}{2} m(m+1)$ *and* $\dim \Lambda^2 V = \frac{1}{2} m(m-1)$.

Proof. It is easy to check that the union of the two claimed bases span $V \otimes V$ and have m^2 elements so form a basis. Moreover $v_i v_j$ do all live in $S^2 V$ and the $v_i \wedge v_j$ do all live in $\Lambda^2 V$. Everything follows.[29] □

Proposition. *Let (ρ, V) be a representation of G.*

(i) $V \otimes V = S^2 V \oplus \Lambda^2 V$ as representations of G.

(ii) for $g \in G$ such that $\rho(g)$ is diagonalisable.[30]

$$\chi_{S^2 V}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$$
$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

---

[27] $v_i v_j = v_j v_i$ if we allow $i > j$

[28] $v_i \wedge v_j = -v_j \wedge v_i$ if we allow $i \geqslant j$. In particular $v_i \wedge v_i = 0$

[29] For an alternative argument use Ex Sheet 2 Q11.

[30] This condition is merely for computational convenience.

Proof. For (i) we need to show that if $a \in V \otimes V$ and $\sigma_V(a) = \lambda a$ for $\lambda = \pm 1$ then $\sigma_V \rho_{V\otimes V}(g)(a) = \lambda \rho_{V\otimes V}(g)(a)$ for each $g \in G$. For this it suffices to prove that $\sigma g = g\sigma$ (ie $\sigma \in \mathrm{Hom}_G(V \otimes V, V \otimes V)$). But $\sigma \circ g(v \otimes w) = gw \otimes gv = g \circ \sigma(v \otimes w)$.

To compute (ii) it suffices to prove one or the other since the sum of the right-hand-sides is $\chi(g)^2 = \chi_{V\otimes V}$. Let $v_1, \ldots, v_m$ be a basis of eigenvectors for $\rho(g)$ with eigenvalues $\lambda_1, \ldots, \lambda_m$. Then $g(v_i v_j) = (\lambda_i \lambda_j) v_i v_j$.

Thus

$$\chi(g)^2 + \chi(g^2) = (\sum_i \lambda_i)^2 + \sum_i \lambda_i^2 = 2 \sum_{i \leqslant j} \lambda_i \lambda j$$

whereas $\chi_{S^2V}(g) = \sum_{i\leqslant j} \lambda_i \lambda j$. □

Exercise. Prove directly the formula for $\chi_{\Lambda^2 V}$.

Lecture 13

Example. S_4

	e	$(12)(34)$	(123)	(12)	(1234)
$\mathbf{1}$	1	1	1	1	1
ϵ	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
$\epsilon\chi_3$	3	-1	0	-1	1
χ_5	2	2	-1	0	0
χ_3^2	9	1	0	1	1
$\chi_3(g^2)$	3	3	0	3	-1
$S^2\chi_3$	6	2	0	2	0
$\Lambda^2\chi_3$	3	-1	0	-1	1

Thus $S^2\chi_3 = \chi_5 + \chi_3 + \mathbf{1}$ and $\Lambda^2\chi_3 = \epsilon\chi_3$. Notice that given $\mathbf{1}$ and ϵ and χ_3 we could've constructed the remaining two irreducible characters using $S^2\chi_3$ and $\Lambda^2\chi_3$.

More generally, for any vector space V we may consider $V^{\otimes n} = V \otimes \cdots \otimes V$. Then for any $\omega \in S_n$ we can define a linear map $\sigma(\omega)\colon V^{\otimes n} \to V^{\otimes n}$ by

$$\sigma(\omega)\colon v_1 \otimes \cdots v_n \mapsto v_{\omega^{-1}(1)} \otimes \cdots v_{\omega^{-1}(n)}$$

for $v_1, \ldots, v_n \in V$.

Exercise. Show that this defines a representation of S_n on $V^{\otimes n}$ and that if V is a representation of G then the G-action and the S_n-action on $V^{\otimes n}$ commute.

Let's suppose for now that k has characteristic 0. Thus we can decompose $V^{\otimes n}$ as a rep of S_n and each isotypical component will be a G-invariant subspace of $V^{\otimes n}$. In particular we can make the following definition.

Definition. Suppose that V is a vector space we define

(i) the n^{th} *symmetric power* of V to be

$$S^n V := \{a \in V^{\otimes n} \mid \sigma(\omega)(a) = a \text{ for all } \omega \in S_n\}$$

and

(ii) the n^{th} *exterior (or alternating) power* of V to be

$$\Lambda^n V := \{a \in V^{\otimes n} \mid \sigma(\omega)(a) = \epsilon(\omega) a \text{ for all } \omega \in S_n\}.$$

Note that, for $n \geqslant 3$,

$$S^nV \oplus \Lambda^n V = \{a \in V^{\otimes n} \mid \sigma(\omega)(a) = a \text{ for all } \omega \in A_n\} \subsetneq V^{\otimes n}.$$

We also define the following notation for $v_1, \ldots, v_n \in V$,

$$v_1 \cdots v_n := \frac{1}{n!} \sum_{\omega \in S_n} v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \in S^n V$$

and

$$v_1 \wedge \cdots \wedge v_n := \frac{1}{n!} \sum_{\omega \in S_n} \epsilon(\omega) v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \in \Lambda^n V.$$

Exercise. Show that if $v_1, \ldots, v_d$ is a basis for V then

$$\{v_{i_1} \cdots v_{i_n} \mid 1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant d\}$$

is a basis for S^nV and

$$\{v_{i_1} \wedge \cdots \wedge v_{i_n} \mid 1 \leqslant i_1 < \cdots < i_n \leqslant d\}$$

is a basis for $\Lambda^n V$. Hence given $g \in G$ acting diagonalisably on V, compute the character values $\chi_{S^nV}(g)$ and $\chi_{\Lambda^nV}(g)$ in terms of the eigenvalues of g on V.

For any vector space V, $\Lambda^{\dim V} V \cong k$ and $\Lambda^n V = 0$ if $n > \dim V$.

Exercise. Show that if (ρ, V) is a representation of G then the representation of G on $\Lambda^{\dim V} V \cong k$ is given by $g \mapsto \det \rho(g)$; ie the $\dim V^{th}$ exterior power of V is isomorphic to $\det \rho$.

We may stick these vector spaces together to form algebras.

Definition. Given a vector space V we may define the *tensor algebra* of V,

$$TV := \oplus_{n \geqslant 0} V^{\otimes n}$$

(where $V^{\otimes 0} = k$). Then TV is a (non-commutative) graded ring with the product of $v_1 \otimes \cdots \otimes v_r \in V^{\otimes r}$ and $w_1 \otimes \cdots \otimes w_s \in V^{\otimes s}$ given by

$$v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s \in V^{\otimes r+s}.$$

with graded quotient rings the *symmetric algebra* of V,

$$SV := TV/(x \otimes y - y \otimes x \mid x, y \in V),$$

and the *exterior algebra* of V,

$$\Lambda V := TV/(x \otimes y + y \otimes x \mid x, y \in V).$$

One can show that $SV \cong \bigoplus_{n \geqslant 0} S^n V$ under $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$ and $\Lambda V \cong \bigoplus_{n \geqslant 0} \Lambda^n V$ under $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \wedge \cdots \wedge x_n$.

Now SV is a commutative ring and ΛV is graded-commutative; that is if $x \in \Lambda^r V$ and $y \in \Lambda^s V$ then $x \wedge y = (-1)^{rs} y \wedge x$.

5.3. **Duality.** Recall that $\mathcal{C}_G$ has the $*$-operation given by $f^*(g) = f(g^{-1})$. This also restricts to $R(G)$.

Recall that if G is group and (ρ, V) is a representation of G then the *dual representation* (ρ^*, V^*) of G is given by

$$(\rho^*(g)\theta)(v) = \theta(\rho(g^{-1})v)$$

for $\theta \in V^*$, $g \in G$ and $v \in V$.

Lemma. $\chi_{V^*} = (\chi_V)^*$.

Proof. If $\rho(g)$ is represented by a matrix A with respect to a basis $v_1, \dots, v_d$ for V and $\epsilon_1, \dots, \epsilon_d$ is the dual basis for V^*. Then $\rho(g)^{-1}v_i = \sum (A^{-1})_{ji}v_j$.

Thus $(\rho^*(g)\epsilon_k)(v_i) = \epsilon_k \left(\sum_j (A^{-1})_{ji}v_j\right) = (A^{-1})_{ki}$ and so

$$\rho^*(g)\epsilon_k = \sum_i (A^{-1})^T_{ik}\epsilon_i$$

i.e. $\rho^*(g)$ is represented by $(A^{-1})^T$ with respect to the dual basis. Taking traces gives the result. $\square$

Definition. We say that V is *self-dual* if $V \cong V^*$ as representations of G.

When G is finite and $k = \mathbb{C}$, V is self-dual if and only if $\chi_V(g) \in \mathbb{R}$ for all $g \in G$; since this is equivalent to $\chi_{V^*} = \chi_V$.

Examples.

(1) $G = C_3 = \langle x \rangle$ and $V = \mathbb{C}$. If ρ is given by $\rho(x) = \omega = e^{\frac{2\pi i}{3}}$ then $\rho^*(x) = \omega^2 = \overline{\omega}$ and V is not self-dual.
(2) $G = S_n$: since g is always conjugate to its inverse in S_n, $\chi^* = \chi$ always and so every representation is self-dual.
(3) Permutation representations $\mathbb{C}X$ are always self-dual.

We've now got a number of ways to build representations of a group G:

- permutation representations coming from group actions;
- via representations of a group H and a group homomorphism $G \to H$ (e.g. restriction);
- tensor products;
- symmetric and exterior powers;
- decomposition of these into irreducible components;
- character theoretically using orthogonality of characters.

We're now going to discuss one more way related to restriction.

Lecture 14

6. Induction

6.1. **Construction.** Suppose that H is a subgroup of G. Restriction makes representations of G into representations of H. We would like a way of building representations of G from representations of H.

Recall that $[g]_G$ denotes the conjugacy class of $g \in G$. So $\mathbf{1}_{[g]_G} : G \to k$ given by

$$\mathbf{1}_{[g]_G}(x) = \begin{cases} 1 & \text{if } x \text{ is conjugate to } g \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$

is in $\mathcal{C}_G$.

We note that for $g \in G$,

$$[g]_G^{-1} = [g^{-1}]_G, \text{ since } (xgx^{-1})^{-1} = xg^{-1}x^{-1},$$

and so

$$(\mathbf{1}_{[g]_G})^* = \mathbf{1}_{[g^{-1}]_G}.$$

If $H \leqslant G$ then $[g]_G \cap H$ is a union of H-conjugacy classes

$$[g]_G \cap H = \bigcup_{[h]_H \subseteq [g]_G} [h]_H$$

so

$$r\colon \mathcal{C}_G \to \mathcal{C}_H; f \mapsto f|_H$$

is a well-defined linear map with

$$r(\mathbf{1}_{[g]_G}) = \sum_{[h]_H \subseteq [g]_G} \mathbf{1}_{[h]_H}.$$

Since for every finite group G, $\langle f_1, f_2\rangle_G = \frac{1}{|G|}\sum_{g\in G} f_1^*(g) f_2(g)$ defines a non-degenerate bilinear form on $\mathcal{C}_G$, the map r has an adjoint r^* characterised by

$$\langle r(f_1), f_2\rangle_H = \langle f_1, r^*(f_2)\rangle_G \text{ for } f_1 \in \mathcal{C}_G, f_2 \in \mathcal{C}_H.$$

In particular for $f \in \mathcal{C}_H$,

$$\langle \mathbf{1}_{[g^{-1}]_G}, r^*(f)\rangle_G = \langle r(\mathbf{1}_{[g^{-1}}]_G), f\rangle_H = \sum_{[h]_H \subseteq [g]_G} \frac{1}{|C_H(h)|} f(h).$$

On the other hand,

$$\langle \mathbf{1}_{[g^{-1}]_G}, r^*(f)\rangle_G = \frac{1}{|G|}\sum_{x\in[g]_G} r^*(f)(x) = \frac{1}{|C_G(g)|} r^*(f)(g).$$

Thus, by comparing these we see that

$$r^*(f)(g) \quad = \sum_{[h]_H \subseteq [g]_G} \frac{|C_G(g)|}{|C_H(h)|} f(h). \tag{1}$$

Since $x^{-1}gx = y^{-1}gy$ if and only if $xy^{-1} \in C_G(g)$ we can rewrite this as

$$r^*(f)(g) = \sum_{h \in H \cap [g]^G} \frac{|C_G(g)|}{|C_H(h)||[h]_H|} f(h) = \frac{1}{|H|}\sum_{x\in G} f^\circ(x^{-1}gx)$$

where

$$f^\circ(g) = \begin{cases} f(g) & \text{for } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

Question. Is $r^*(R(H)) \subseteq R(G)$?

Suppose that χ is a $\mathbb{C}$-character of H and ψ is an irreducible $\mathbb{C}$-character of G. Then

$$\langle \psi, r^*(\chi)\rangle_G = \langle r(\psi), \chi\rangle_H \in \mathbb{N}_0$$

by orthogonality of characters, since $r(\psi)$ is a character of H.

So writing $\mathrm{Irr}(G)$ to denote the set of irreducible $\mathbb{C}$-characters of G

$$r^*(\chi) \quad = \sum_{\psi \in \mathrm{Irr}(G)} \langle \psi|_H, \chi\rangle_H \psi \tag{2}$$

is even a character in $R(G)$. The formula (2) is only useful for actually computing $r^*(\chi)$ if we already understand $\mathrm{Irr}(G)$. Since our purpose will often be to use $\mathrm{Irr}(H)$ to understand $\mathrm{Irr}(G)$, the formula (1) will typically prove more useful.

Example. $G = S_3$ and $H = A_3 = \{1, (123), (132)\}$.

If $f \in \mathcal{C}_H$ then

$$\begin{aligned} r^*(f)(e) &= \frac{6}{3} f(e) = 2f(e), \\ r^*(f)((12)) &= 0, \text{ and} \\ r^*(f)((123)) &= \frac{3}{3} f((123)) + \frac{3}{3} f((132)) = f((123)) + f((132)). \end{aligned}$$

Thus

A_3	1	(123)	(132)
χ_1	1	1	1
χ_2	1	w	w^2
χ_3	1	w^2	w

S_3	1	(12)	(123)
$r^*(\chi_1)$	2	0	2
$r^*(\chi_2)$	2	0	-1
$r^*(\chi_3)$	2	0	-1

So $r^*(\chi_1) = \mathbf{1} + \epsilon$ and $r^*(\chi_2) = r^*(\chi_3)$ is the 2-dimensional irreducible character χ_V of S_3 consistent with the formula (2) since

$$\mathbf{1}|_{A_3} = \epsilon|_{A_3} = \chi_1$$

and

$$(\chi_V)|_{A_3} = \chi_2 + \chi_3.$$

Note that if χ is an irreducible character of H then $r^*(\chi)$ may be an irreducible character of G but need not be so. Also note that $r^*(\chi)(e) = \frac{|G|}{|H|}\chi(e)$ always.

We'd like to build a representation of G with character $r^*(\chi)$ given a representation W of H with character χ.

Suppose that X is a finite set and W is a k-vector space. We may define

$$\mathcal{F}(X, W) := \{f : G \to W\}$$

the k-vector space of functions X to W with pointwise operations. In particular $\mathcal{F}(X, k) = kX$.

We can compute $\dim \mathcal{F}(X, W) = |X| \dim W$ since if $w_1, \ldots, w_d$ is a basis for W then $(\delta_x w_i : x \in X, 1 \leqslant i \leqslant d)$ is a basis for $\mathcal{F}(X, W)$.

If K is a group, X has a K-action and W is a representation of K then $\mathcal{F}(X,W)$ can be viewed as a representation of K via

$$(k\cdot f)(x) = kf(k^{-1}x) \text{ for all } f\in\mathcal{F}(X,W), k\in K, x\in X.$$

For example if $W=k$ is the trivial representation of K then $\mathcal{F}(X,W)=kX$ as representations of K.

Now we consider a special case of this construction. Suppose $H\leqslant G$ are finite groups. Then G can be viewed as a set with $G\times H$-action via

$$(g,h)\cdot x = gxh^{-1} \text{ for all } g,x\in G, h\in H.$$

If W is a representation of H then it can be viewed as a representaion of $G\times H$ via

$$(g,h)\cdot w = hw \text{ for all } g\in G, h\in H, w\in W.$$

Now $\mathcal{F}(G,W)$ is a representation of $G\times H$ via

$$\left((g,h)\cdot f\right)(x) = h\cdot f(g^{-1}xh) \text{ for } (g,h)\in G\times H, x\in G, f\in\mathcal{F}(G,W).$$

Using this $\mathcal{F}(G,W)$ can be viewed as a representation of G and of H by restriction along the respective natural maps $G\to G\times\{e_H\}$ and $H\to\{e_G\}\times H$.

Now

$$\begin{aligned}\mathcal{F}(G,W)^H &= \{f\in\mathcal{F}(G,W) : (e,h)\cdot f = f \text{ for all } h\in H\}\\ &= \left\{f\in\mathcal{F}(G,W) : f(xh)=h^{-1}f(x) \text{ for all } h\in H, x\in G\right\}\end{aligned}$$

is a G-invariant subspace of $\mathcal{F}(G,W)$ since the G and H actions commute; if $(e,h)\cdot f=f$ for $h\in H$, then for $g\in G$,

$$(e,h)(g,e)f = (g,e)(e,h)f = (g,e)f.$$

For example if k is the trivial representation of W then $\mathcal{F}(G,k)^H\cong kG/H$ as representations of G.

Definition. Suppose that H is a subgroup of a finite group G and W is a representation of H. We define the *induced representation* to be

$$\operatorname{Ind}_H^G W := \mathcal{F}(G,W)^H$$

as a representation of G.

Lemma. $\dim\operatorname{Ind}_H^G W = \frac{|G|}{|H|}\dim W$

Proof. Let $X=G/H$ be the set of left cosets of H in G and let $x_1,\ldots,x_{|G/H|}$ be left coset representatives then

$$\theta\colon\mathcal{F}(G,W)^H\to\mathcal{F}(X,W)$$

given by

$$\theta(f)(x_iH) = f(x_i) \text{ for } f\in\mathcal{F}(G,W)^H$$

is a k-linear map with inverse given by

$$\varphi(l)(x_ih) = h^{-1}l(x_i) \text{ for } l\in\mathcal{F}(X,W) \text{ and } h\in H.$$

Now the result follows from an earlier computation of $\dim\mathcal{F}(X,W)$. □

If V is a representation of G, we'll write $\operatorname{Res}_H^G V$ for the representation of H obtained by restriction.

Theorem (Frobenius reciprocity)**.** *Let V be a representation of G, and W a representation of H, then*

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W).$$

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Corollary. *If $k = \mathbb{C}$ then*

$$\langle \chi_V, \chi_{\operatorname{Ind}_H^G W} \rangle_G = \langle \chi_V|_H, \chi_W \rangle_H.$$

In particular $\chi_{\operatorname{Ind}_H^G W} = r^*(\chi_W)$.

Proof of Frobenius Reciprocity. We'll prove that

$$\operatorname{Hom}_k(V, W) \cong \operatorname{Hom}_G(V, \mathcal{F}(G, W))$$

as representations of H and then deduce the result by taking H-invariants.

Here the action of H on the RHS is via

$$(h \cdot \theta)(v) = h \cdot \theta(v) \text{ for all } \theta \in \operatorname{Hom}_G(V, \mathcal{F}(G, W)), v \in V \text{ and } h \in H.$$

so that $\operatorname{Hom}_G(V, \mathcal{F}(G, W))^H = \operatorname{Hom}_G(V, \mathcal{F}(G, W)^H) = \operatorname{Hom}_G(V, \operatorname{Ind}_H^H W)$. Note that this means that

$$(h \cdot \theta)(v)(x) = h\left(\theta(v)(xh)\right) = h(\theta(h^{-1}x^{-1}v)(e)) \text{ for all } x \in G \tag{3}$$

since θ is G-invariant.

We can define a linear map

$$\Psi\colon \operatorname{Hom}_G(V, \mathcal{F}(G, W)) \to \operatorname{Hom}_k(V, W)$$

by

$$\Psi(\theta)(v) = \theta(v)(e)$$

and claim that Ψ is an H-intertwining map. First we compute for $h \in H$ and $\theta \in \operatorname{Hom}_G(V, \mathcal{F}(G, W))$,

$$\begin{aligned}
(h \cdot \Psi(\theta))(v) &= h(\Psi(\theta(h^{-1}v)) \\
&= h\left(\theta(h^{-1}v)(e)\right) \\
&= (h \cdot \theta)(v)(e) \text{ by (3)} \\
&= \Psi(h \cdot \theta)(v)
\end{aligned}$$

Thus it remains to prove that Ψ is an isomorphism.

Given $\varphi \in \operatorname{Hom}_k(V, W)$ we can define $\varphi_G \in \operatorname{Hom}_k(V, \operatorname{Hom}(G, W))$ by

$$\varphi_G(v)(x) = \varphi(x^{-1}v) \text{ for } v \in V \text{and } x \in G.$$

Then for all $x, g \in G$ and $v \in V$

$$\varphi_G(gv)(x) = \varphi(x^{-1}gv) = \varphi_G(v)(g^{-1}x) = (g \cdot \varphi_G(v))(x)$$

i.e. $\varphi_G \in \operatorname{Hom}_G(V, \operatorname{Hom}(G, W))$. We can compute

$$\Psi(\varphi_G)(v) = \varphi_G(v)(e) = \varphi(v)$$

for $\varphi \in \operatorname{Hom}_k(V, W)$ and $v \in V$ and

$$\Psi(\theta)_G(v)(x) = \Psi(\theta)(x^{-1}v) = \theta(x^{-1}v)(e) = x^{-1}\theta(v)(e) = \theta(v)(x)$$

for $\theta \in \operatorname{Hom}_G(V, \mathcal{F}(G, W))$, $x \in G$ and $v \in V$. Thus $\varphi \mapsto \varphi_G$ is an inverse to Ψ. $\square$

Remark. We could've computed $\chi_{\mathrm{Ind}_H^G W}$ directly and shown that it is equal to $r^*(\chi_W)$. Frobenius reciprocity would then follow in the case $k = \mathbb{C}$. However our proof works for all fields k and gives us more information about the nature of an isomorphism.

6.2. **Mackey Theory.** This is the study of representations like $\mathrm{Res}_K^G \mathrm{Ind}_H^G W$ for H, K subgroups of G and W a representation of H. We can (and will) use it to characterise when $\mathrm{Ind}_H^G W$ is irreducible.

If H, K are subgroups of G then $H \times K$ acts on G via

$$(h, k) \cdot g = kgh^{-1}$$

An orbit of this action is called a *double coset* we write

$$KgH := \{kgh \mid k \in K, h \in H\}$$

for the orbit containing g.

Definition. $K\backslash G/H := \{KgH \mid g \in G\}$ is the set of double cosets.

The double cosets partition G.

Given any representation (ρ, W) of H and $g \in G$, we can define $({}^g\rho, {}^gW)$ to be the representation of ${}^gH := gHg^{-1} \leqslant G$ on W given by $({}^g\rho)(ghg^{-1}) = \rho(h)$ for $h \in H$.

Theorem (Mackey's Restriction Formula)**.** *If G is a finite group with subgroups H and K, and W is a representation of H then*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G W \cong \bigoplus_{g \in K\backslash G/H} \mathrm{Ind}_{K\cap {}^gH}^K \mathrm{Res}_{{}^gH\cap K}^{{}^gH} {}^gW.$$

Note that

$$\begin{aligned} \mathrm{Ind}_H^G W &= \mathcal{F}(G, W)^H \\ &= \mathcal{F}\left(\coprod_{KgH \in K\backslash G/H} KgH, W\right)^H \\ &\cong \bigoplus_{KgH \in K\backslash G/H} \mathcal{F}(KgH, W)^H \text{ as reps of } K. \end{aligned}$$

Thus it suffices to show that for each g,

$$\mathcal{F}(KgH, W)^H \cong \mathcal{F}(K, {}^gW)^{{}^gH\cap K}$$

as representations of K. We defer the proof of this to the next lecture.

Corollary (Character version of Mackey's Restriction Formula)**.** *If χ is a character of a representation of H then*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G \chi = \sum_{KgH \in K\backslash G/H} \mathrm{Ind}_{{}^gH\cap K}^K {}^g\chi.$$

where ${}^g\chi$ is the class function on ${}^gH \cap K$ given by ${}^g\chi(x) = \chi(g^{-1}xg)$.

Exercise. Prove this corollary directly with characters

Corollary (Mackey's irreducibility criterion)**.** *If H is a subgroup of G and W is a representation of H, then $\mathrm{Ind}_H^G W$ is irreducible if and only if*

(i) W is irreducible and
(ii) for each $g \in G \backslash H$, the two representations $\mathrm{Res}^{{}^gH}_{H \cap {}^g H}\, {}^gW$ and $\mathrm{Res}^{H}_{{}^gH \cap H} W$ of $H \cap^g H$ have no irreducible factors in common.

Proof.

$$
\begin{aligned}
< \mathrm{Ind}_H^G \chi_W, \mathrm{Ind}_H^G \chi_W \rangle_G &\overset{\text{Frob. recip.}}{=} \langle \chi_W, \mathrm{Res}_H^G \mathrm{Ind}_H^G \chi_W \rangle_H \\
&\overset{\text{Mackey}}{=} \sum_{g \in H\backslash G/H} \langle \chi_W, \mathrm{Ind}^H_{H \cap^g H} \mathrm{Res}^{{}^gH}_{H \cap^g H} {}^g\chi_W \rangle_H \\
&\overset{\text{Frob. recip.}}{=} \sum_{g \in H\backslash G/H} \langle \mathrm{Res}^H_{H \cap^g H} \chi_W, \mathrm{Res}^{{}^gH}_{H \cap^g H} {}^g\chi_W \rangle_{H \cap^g H}
\end{aligned}
$$

So $\mathrm{Ind}_H^G W$ is irreducible precisely if

$$
\sum_{g \in H\backslash G/H} \langle \mathrm{Res}^H_{H \cap^g H} \chi_W, \mathrm{Res}^{{}^gH}_{H \cap^g H} {}^g\chi_W \rangle_{H \cap^g H} = 1.
$$

The term corresponding to the coset $HeH = H$ is $\langle \chi_W, \chi_W \rangle_H$ which is at least 1 and equal to 1 precisely if W is irreducible. The other terms are all $\geqslant 0$ and are zero precisely if condition (ii) of the statement holds. □

Corollary. *If H is a normal subgroup of G ,and W is an irreducible rep of H then $\mathrm{Ind}_H^G W$ is irreducible if and only if ${}^g\chi_W \neq \chi_W$ for all $g \in G \backslash H$.*

Proof. Since H is normal, $gHg^{-1} = H$ for all $g \in G$. Moreover ${}^gW$ is irreducible since W is irreducible.

So by Mackey's irreducibility criterion, $\mathrm{Ind}_H^G W$ irreducible precisely if $W \not\cong {}^gW$ for all $g \in G \backslash H$. This last is equivalent to $\chi_W \neq {}^g\chi_W$ as required.

□

Examples.

(1) $H = \langle r \rangle \cong C_n$, the rotations in $G = D_{2n}$. The irreducible characters χ of H are all of the form $\chi(r^j) = e^{\frac{2\pi ijk}{n}}$. We see that $\mathrm{Ind}_H^G \chi$ is irreducible if and only if $\chi(r^j) \neq \chi(r^{-j})$ for some j. This is equivalent to χ not being real valued.

(2) $G = S_n$ and $H = A_n$. If $g \in S_n$ is a cycle type that splits into two conjugacy classes in A_n and χ is an irreducible character of A_n that takes different values of the two classes then $\mathrm{Ind}_H^G \chi$ is irreducible.

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Recall that if W is a representation of H and H, K are subgroups of G and $g \in G$ then

$$
\mathcal{F}(KgH, W)^H = \{f \colon KgH \to W \mid f(xh) = h^{-1}f(x) \text{ for all } x \in KgH, h \in H\}
$$

with K-action given by

$$
(k \cdot f)(x) = f(k^{-1}x) \text{ for all } k \in K \text{ and } x \in KgH.
$$

We reduced the proof of Mackey's Decomposition Theorem to the following Lemma.

Lemma. *There is an isomorphism of representations of K*

$$\mathcal{F}(KgH, W)^H \cong \mathcal{F}(K, {}^gW)^{K\cap {}^gH}.$$

Proof. Let $\Theta\colon \mathcal{F}(KgH, W)^H \to \mathcal{F}(K, {}^gW)$ be defined by

$$\Theta(f)(k) = f(kg).$$

If $k' \in K$ then

$$(k' \cdot \Theta(f))(k) = f(k'^{-1}kg) = (k' \cdot f)(kg) = \Theta(k' \cdot f)(k)$$

and so Θ is K-linear.

If $ghg^{-1} \in K$ for some $h \in H$,

$$\begin{aligned}\Theta(f)(kghg^{-1}) &= f(kgh)\\ &= \rho(h^{-1})f(kg)\\ &= ({}^g\rho)(ghg^{-1})^{-1}\Theta(f)(k)\end{aligned}$$

Thus $\operatorname{Im}\Theta \leqslant \mathcal{F}(K, {}^gW)^{K\cap {}^gH}$.

We try to define an inverse to Θ. Let

$$\Psi\colon \mathcal{F}(K, {}^gW)^{K\cap {}^gH} \to \mathcal{F}(KgH, W)^H$$

be given by

$$\Psi(f)(kgh) = \rho(h)^{-1}f(k)$$

If $k_1gh_1 = k_2gh_2$ then $k_2^{-1}k_1 = gh_2h_1^{-1}g^{-1} \in K \cap {}^gH$. So

$$\begin{aligned}f(k_2) &= f(k_1(k_2^{-1}k_1)^{-1})\\ &= {}^g\rho(gh_2h_1^{-1}g^{-1})f(k_1)\\ &= \rho(h_2h_1^{-1})f(k_1)\end{aligned}$$

Thus

$$\rho(h_2)^{-1}f(k_2) = \rho(h_1)^{-1}f(k_1)$$

and $\Psi(f)$ is well-defined.

Moreover for $f \in \mathcal{F}(K, {}^gW)^{K\cap {}^gH}$,

$$\Theta\Psi(f)(k) = \Psi(f)(kg) = f(k),$$

and for $f \in \mathcal{F}(KgH, W)^H$,

$$\begin{aligned}\Psi\Theta(f)(kgh) &= \rho(h^{-1})\Theta(f)(k)\\ &= \rho(h^{-1})f(kg)\\ &= f(kgh).\end{aligned}$$

Thus Ψ is inverse to Θ. □

6.3. **Frobenius groups.**

Theorem. *(Frobenius 1901) Let G be a finite group acting transitively on a set X. If each $g \in G\backslash\{e\}$ fixes at most one element of X then*

$$K = \{1\} \cup \{g \in G \mid gx \neq x \text{ for all } x \in X\}$$

is a normal subgroup of G of order $|X|$.

Definition. A *Frobenius group* is a finite group G that has a transitive action on a set X with $1 < |X| < |G|$ such that each $g \in G\backslash\{e\}$ fixes at most one $x \in X$.

Examples.

(1) $G = D_{2n}$ with n odd acting naturally on the vertices of an n-gon. The reflection fix precisely one vertex and the non-trivial rotations fix no vertices.

(2) $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_p, a \neq 0 \right\}$ acting on $X = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \mid x \in \mathbb{F}_p \right\}$ by matrix multiplication.

It follows that no Frobenius group can be simple. The normal subgroup K is called the *Frobenius kernel* and any of the groups $\mathrm{Stab}_G(x)$ for $x \in X$ is called a *Frobenius complement*. No proof of the theorem is known that does not use representation theory.

Proof of Theorem. For $x \in X$, let $H = \mathrm{Stab}_G(x)$ so $|G| = |X||H|$ by the orbit-stabiliser theorem.

By hypothesis if $g \in G \backslash H$ then

$$gHg^{-1} \cap H = \mathrm{Stab}_G(gx) \cap \mathrm{Stab}_G(x) = \{e\}.^{31}$$

Thus

(i) $|\bigcup_{g \in G} gHg^{-1}| = |\bigcup_{x \in X} \mathrm{Stab}_G(x)| = (|H| - 1)|X| + 1$;
(ii) If $h_1, h_2 \in H$ then $[h_1]_H = [h_2]_H$ if and only if $[h_1]_G = [h_2]_G$; and
(iii) $C_G(h) = C_H(h)$ for $e \neq h \in H$

By (i) $|K| = |\{e\} \cup \left(|G| \backslash \bigcup_{x \in X} \mathrm{Stab}_G(x)\right)| = |H||X| - (|H| - 1)|X| = |X|$ as required.

We must show that $K \lhd G$. Our strategy will be to prove that it is the kernel of some representation of G.

If χ is a character of H we can compute $\mathrm{Ind}_H^G \chi$:

$$\begin{aligned} \mathrm{Ind}_H^G \chi(g) &= \sum_{[h]_H \subseteq [g]_G} \frac{|C_G(g)|}{|C_H(h)|} \chi(h) \\ &= \begin{cases} |X|\chi(e) & \text{if } g = e \\ \chi(h) & \text{if } [g]_G = [h]_G \neq \{e\} \text{ by (ii) and (iii)} \\ 0 & \text{if } g \in K \backslash \{e\}. \end{cases} \end{aligned}$$

Suppose now that $\mathrm{Irr}(H) = \{\chi_1, \ldots, \chi_r\}$ and let

$$\theta_i = \mathrm{Ind}_H^G \chi_i + \chi_i(e) \mathbf{1}_G - \chi_i(e) \mathrm{Ind}_H^G \mathbf{1}_H \in R(G) \text{ for } i = 1, \ldots, r$$

so

$$\theta_i(g) = \begin{cases} \chi_i(e) & \text{if } g = e \\ \chi_i(h) & \text{if } [g]_G = [h]_G \\ \chi_i(e) & \text{if } g \in K \end{cases}$$

If θ_i were a character then the corresponding representation would have kernel containing K. Since $\theta_i \in R(G)$ we can write it as a $\mathbb{Z}$-linear combination of irreducible characters $\theta_i = \sum n_i \psi_i$, say, for $\psi_i \in \mathrm{Irr}(G)$ and $n_i \in \mathbb{Z}$.

---

[31] It is straightforward to verify that a group is Frobenius if and only if there is a non-trivial proper subgroup H of G such that $gHg^{-1} \cap H = \{e\}$ for all $g \in G \backslash H$. To go one way take $X = G/H$, to go the other take $H = \mathrm{Stab}_G(x)$ for some $x \in X$.

Now we can compute

$$\begin{aligned}\langle\theta_i,\theta_i\rangle_G &= \frac{1}{|G|}\sum_{g\in G}|\theta_i(g)|^2\\ &= \frac{1}{|G|}\left(\sum_{h\in H\backslash\{e\}}|X||\chi_i(h)|^2+\sum_{k\in K}\chi_i(e)^2\right)\\ &= \frac{|X|}{|G|}\left(\sum_{h\in H}|\chi_i(h)|^2\right)\\ &= \langle\chi_i,\chi_i\rangle_H = 1.\end{aligned}$$

But on the other hand it must be $\sum n_i^2$. Thus θ_i is $\pm\psi$ for some character ψ of G. Since $\theta_i(e) > 0$ it must actually be an irreducible character.

To finish we write $\theta = \sum \chi_i(e)\theta_i$ and so $\theta(h) = \sum_{i=1}^r \chi_i(e)\chi_i(h) = 0$ for $h \in H\backslash\{e\}$ by column orthogonality, and $\theta(k) = \sum \chi_i(e)^2 = |H|$ for $k \in K$. Thus $K = \ker\theta$ is a normal subgroup of G. □

In his thesis John Thompson proved, amongst other things, that the Frobenius kernel must be the direct product of its Sylow subgroups.

LECTURE 17

7. ARITHMETIC PROPERTIES OF CHARACTERS

In this section we'll investigate how arithmetic properties of characters produce a suprising interplay between the structure of the group and properties of the character table. The highlight of this will be the proof of Burnside's famous p^aq^b-theorem that says that the order of a simple group cannot have precisely two distinct prime factors.

We'll continue with our assumption that $k = \mathbb{C}$ and also assume that our groups are finite.

7.1. **Arithmetic results.**

We'll need to quote some results about arithmetic without proof; proofs should be provided in the Number Fields course (or in one later case Galois Theory).

Definition. $x \in \mathbb{C}$ is an *algebraic integer* if it is a root of a monic polynomial with integer coefficients.

Facts.

Fact 1 The algebraic integers form a subring $\mathcal{O}$ of $\mathbb{C}$. (see Groups, Rings and Modules 2023 Examples Sheet 4 Q13)

Fact 2 Any subring of $\mathbb{C}$ that is finitely generated as an abelian group is contained in $\mathcal{O}$. (see Groups, Rings and Modules 2023 Examples Sheet 4 Q13)

Fact 3 If $x \in \mathbb{Q} \cap \mathcal{O}$ then $x \in \mathbb{Z}$. (see Numbers and Sets 2021 Example Sheet 3 Q12)

Lemma. *If χ is the character of G, then χ takes values in $\mathcal{O}$.*

Proof. We know that $\chi(g)$ is a sum of n^{th} roots of unity for $n = |G|$. Each n^{th} root of unity is by defintion a root of $X^n - 1$ and so an algebraic integer. The lemma follows from Fact 1. □

7.2. **The group algebra.**

Before we go further we need to explain how to make the vector space kG into a ring. There are in fact two sensible ways to do this. The first of these is by pointwise multiplication. This makes kG into a commutative ring. But more usefully for our immediate purposes we have the convolution product

$$(f_1f_2)(g) := \sum_{x\in G} f_1(gx)f_2(x^{-1}) = \sum_{\substack{x,y\in G\\ xy=g}} f_1(x)f_2(y)$$

that makes kG into a (typically) non-commutative ring. With this product

$$\delta_{g_1}\delta_{g_2} = \delta_{g_1g_2} \text{ for all } g_1, g_2 \in G$$

and so we may rephrase the multiplication as

$$\left(\sum_{g\in G} \lambda_g\delta_g\right)\left(\sum_{h\in G} \mu_h\delta_h\right) = \sum_{k\in G}\left(\sum_{gh=k} \lambda_g\mu_h\right)\delta_k.$$

From now on this will be the product we have in mind when we think of kG as a ring.

Notice that a (finitely generated) kG-module is the 'same' as a representation of G: given a representation (ρ, V) of G we can make it into a kG-module via

$$fv = \sum_{g\in G} f(g)\rho(g)(v).$$

for $f \in kG$ and $v \in V$. Conversely, given a finitely generated kG-module M we can view M as a representation of G via $\rho(g)(m) = \delta_g m$. Moreover G-linear maps correspond to kG-module homomorphisms under this correspondence.

Exercise. Suppose that kX is a permutation representation of G. Calculate the action of $f \in kG$ on kX under this correspondence.

It will prove useful understand the *centre* $Z(kG)$ of kG; that is the subring of $f \in kG$ such that $fh = hf$ for all $h \in kG$. This is because for every $f \in Z(kG)$ then $\sum f(g)\rho(g) \in \mathrm{Hom}_G(V,V)$ for every representation (ρ, V) of G.

Lemma. *Suppose that $f \in kG$. Then f is in $Z(kG)$ if and only if f is in $\mathcal{C}_G$, the set of class functions on G. In particular $\dim_k Z(kG)$ is the number of conjugacy classes in G.*

Proof. Suppose $f \in kG$. Notice that $fh = hf$ for all $h \in kG$ if and only if $f\delta_g = \delta_g f$ for all $g \in G$: the forward direction is clear and for the backward direction if $f\delta_g = \delta_g f$ for all $g \in G$ then

$$fh = \sum_{g\in G} fh(g)\delta_g = \sum_{g\in G} h(g)\delta_g f = hf.$$

But $\delta_g f = f\delta_g$ if and only if $\delta_g f \delta_{g^{-1}} = f$ and

$$(\delta_g f \delta_{g^{-1}})(x) = (\delta_g f)(xg) = f(g^{-1}xg).$$

So if $f \in Z(kG)$ if and only if $f \in \mathcal{C}_G$ as required. $\square$

Remark. The multiplication on $Z(kG)$ is not the same as the multiplication on $\mathcal{C}_G$ that we have seen before even though both have the same additive groups and both are commutative rings.

Definition. Given $g \in G$ define the *class sum*

$$C_{[g]_G}(x) = \begin{cases} 1 & x \in [g]_G \\ 0 & x \notin [g]_G. \end{cases}.$$

Then if $[g_1]_G = \{e\}, \ldots, [g_r]_G$ are all the conjugacy classes of G, write

$$C_i = C_{[g_i]_G} \text{ for } i = 1, \ldots r.$$

We called $C_i = \mathbf{1}_{[g_i]_G}$ before but have changed notation to remind ourselves that the multiplication is not pointwise. $C_1, \ldots, C_r$ form a basis for $Z(kG)$.

Proposition. *There are non-negative integers a_{ij}^l such that $C_iC_j = \sum_k a_{ij}^l C_l$ for $i, j, l \in \{1, \ldots, r\}$. Indeed*

$$a_{ij}^l = |\{(x, y) \in [g_i]_G \times [g_j]_G \mid xy = g_l\}|.$$

The a_{ij}^l are called the *structure constants* for $Z(kG)$.

Proof. Since $Z(kG)$ is a ring, we can certainly write $C_iC_j = \sum a_{ij}^l C_l$ for some $a_{ij}^l \in k$.

However, we can explicitly compute

$$a_{ij}^l = (C_iC_j)(g_l) = \sum_{\substack{x,y\in G\\ xy=g_l}} C_i(x)C_j(y) = |\{(x,y) \in [g_i]_G \times [g_j]_G \mid xy = g_l\}|$$

as claimed. □

Suppose now that (ρ, V) is an irreducible representation of G. Then if $z \in Z(kG)$ we've seen that $z\colon V \to V$ given by $zv = \sum_{g\in G} z(g)\rho(g)v \in \mathrm{Hom}_G(V,V)$.

By Schur's Lemma it follows that z acts by a scalar $\lambda_z \in k$ on V. In this way we get a k-algebra homomorphism $w_\rho\colon Z(kG) \to k; z \mapsto \lambda_z$.

Taking traces we see that

$$\dim V \cdot \lambda_z = \sum_{g\in G} z(g)\chi_V(g).$$

So

$$w_\rho(C_i) = \frac{\chi(g_i)}{\chi(e)}|[g_i]_G|. \tag{4}$$

We now see that w_ρ only depends on χ_ρ (and so on the isomorphism class of ρ) and we write $w_\chi = w_\rho$.

Lemma. *The values $w_\chi(C_i)$ are in $\mathcal{O}$*

Note this isn't *a priori* obvious since $\frac{1}{\chi(e)}$ will not be an algebraic integer for $\chi(e) \neq 1$.

Proof. Since w_χ is an algebra homomorphism $Z(kG) \to k$,

$$w_\chi(C_i)w_\chi(C_j) = \sum_l a_{ij}^l w_\chi(C_l). \tag{5}$$

So the subring of $\mathbb{C}$ generated by $w_\chi(C_i)$ for $i = 1, \ldots, r$ is a finitely generated abelian group spanned by $w_\chi(C_1), \ldots, w_\chi(C_r)$. The result follows from Fact 2 above. □

Lemma.

$$a_{ij}^l = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{\chi\in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_l^{-1})}{\chi(e)}.$$

In particular the a_{ij}^l are determined by the character table.

Proof. By (4) and (5), for each irreducible character χ,

$$\frac{\chi(g_i)}{\chi(e)}|[g_i]_G|\frac{\chi(g_j)}{\chi(e)}|[g_j]_G| = \sum_{l=1}^r a_{ij}^l \frac{\chi(g_l)}{\chi(e)}|[g_l]_G|$$

Multiplying both sides by $\frac{\chi(e)\chi(g_m^{-1})}{|G|}$, using $|[g]_G| = \frac{|G|}{|C_G(g)|}$ for all $g \in G$, and summing over $\chi \in \mathrm{Irr}(G)$ we obtain

$$\frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{\chi\in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_m^{-1})}{\chi(e)} = \sum_{l=1}^r \frac{a_{ij}^l}{|C_G(g_l)|} \sum_{\chi\in \mathrm{Irr}(G)} \chi(g_l)\chi(g_m^{-1}) = a_{ij}^m$$

by column orthogonality. □

Lecture 18

7.3. Degrees of irreducibles.

Theorem. *If V is an irreducible representation of a group G then* $\dim V$ *divides* $|G|$.

Proof. Let χ be the character of V. We'll show that $\frac{|G|}{\chi(e)} \in \mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$ by Fact 3 from §7.1.

$$\begin{aligned}\frac{|G|}{\chi(e)} &= \frac{1}{\chi(e)} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= \sum_{i=1}^{r} \frac{1}{\chi(e)} |[g_i]_G| \, \chi(g_i)\chi(g_i^{-1}) \\ &= \sum_{i=1}^{r} w_\chi(C_i)\chi(g_i^{-1})\end{aligned}$$

But $\mathcal{O}$ forms a ring (by Fact 1 in §7.1) and each $w_\chi(C_i)$ and each $\chi(g_i^{-1})$ is in $\mathcal{O}$ so $\frac{|G|}{\chi(e)}$ is in $\mathcal{O} \cap \mathbb{Q}$ as required. □

Examples.

(1) If G is a p-group and χ is an irreducible character then $\chi(e)$ is always a power of p. In particular if $|G| = p^2$ then, since $\sum_\chi \chi(e)^2 = p^2$, every irreducible rep is 1-dimensional and so G is abelian.

(2) If $G = A_n$ or S_n and $p > n$ is a prime, then p cannot divide the dimension of an irreducible rep.

In fact a stronger result is true:

Theorem (Burnside (1904))**.** *If (ρ, V) is an irreducible representation then* $\dim V$ *divides $|G/Z(G)|$.*

You could compare this with $||g]_G| = \frac{|G|}{|C_G(g)|}$ divides $|G/Z(G)|$.

Proof. If $Z = Z(G)$ then, by Schur's Lemma, the image of $\rho|_Z \colon Z \to GL(V)$ is contained in $k^\times \operatorname{id}_V$; $\rho(z) = \lambda_z \operatorname{id}_V$ for $z \in Z$, say.

For each $m \geqslant 2$, consider the irreducible representation of G^m given by

$$\rho^{\otimes m} \colon G^m \to GL(V^{\otimes m}).$$

If $z = (z_1, \ldots, z_m) \in Z^m$ then z acts on $V^{\otimes m}$ via $\prod_{i=1}^m \lambda_{z_i} \mathrm{id} = \lambda_{\prod_1^m z_i} \mathrm{id}$. Thus if $\prod_1^m z_i = 1$ then $z \in \ker \rho^{\otimes m}$.

Let

$$Z' = \{(z_1, \ldots, z_m) \in Z^m \mid \prod_{i=1}^m z_i = 1\}$$

so $|Z'| = |Z|^{m-1}$. We may view $\rho^{\otimes m}$ as a degree $(\dim V)^m$ irreducible representation of G^m/Z'.

Since $|G^m/Z'| = |G|^m/|Z|^{m-1}$ we can use the previous theorem to deduce that $(\dim V)^m$ divides $|G|^m/|Z|^{m-1}$.

Suppose that p is a prime such that p^a divides $\dim V$. Then p^{am} divides $|G/Z|^m|Z|$. By taking m to be large, in particular so that p^m does not divide $|Z|$, we see that p^a divides $|G/Z|$. Thus $\dim V$ divides $|G/Z|$ as claimed. $\square$

Proposition. *If G is a simple group then G has no irreducible representations of degree 2.*

Proof. If G is cyclic then G has no irreducible representations of degree bigger than 1, so we may assume G is non-abelian.

If $|G|$ is odd then we may apply the theorem above.

If $|G|$ is even then G has an element x of order 2. By example sheet 2 Q2, for every irreducible χ, $\chi(x) \equiv \chi(e) \mod 4$. So if $\chi(e) = 2$ then $\chi(x) = \pm 2$, and $\rho(x) = \pm I$. Thus $\rho(x) \in Z(\rho(G))$, a contradiction since G is non-abelian simple. $\square$

Remark. In 1963 Feit and Thompson published a 255 page paper proving that there is no non-abelian simple group of odd order.

7.4. **Burnside's $p^a q^b$ Theorem.**

Lemma. *Suppose $\alpha \in \mathcal{O}\backslash 0$ is of the form $\alpha = \frac{1}{m}\sum_{i=1}^m \lambda_i$ with $\lambda_i^n = 1$ for all i. Then $|\alpha| = 1$.*[32]

Sketch proof (non-examinable). By assumption $\alpha \in \mathbb{Q}(\epsilon)$ where $\epsilon = e^{2\pi i/n}$.

Let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$. It is known that $\{\beta \in \mathbb{Q}(\epsilon) \mid \sigma(\beta) = \beta \text{ for all } \sigma \in \mathcal{G}\} = \mathbb{Q}$.

Consider $N(\alpha) := \prod_{\sigma \in \mathcal{G}} \sigma(\alpha)$. Since $N(\alpha)$ is fixed by every element of $\mathcal{G}$, $N(\alpha) \in \mathbb{Q}$. Moreover $N(\alpha) \in \mathcal{O}$ since the Galois conjugates of a root of an integer polynomial is a root of the same polynomial. Thus $N(\alpha) \in \mathbb{Z}$.

But for each $\sigma \in \mathcal{G}$, $|\sigma(\alpha)| = |\frac{1}{m}\sum \sigma(\lambda_i)| \leqslant 1$. Thus $N(\alpha) = \pm 1$, and $|\alpha| = 1$ as required. $\square$

Lemma. *Suppose χ is an irreducible character of G, and $g \in G$ such that $\chi(e)$ and $|[g]_G|$ are coprime. Then $|\chi(g)| = \chi(e)$ or 0.*

Note if $\chi = \chi_V$ this is saying that under the given hypothesis either g acts as a scalar on V[33] or $\chi(g) = 0$.

Proof. By Bezout, we can find $a, b \in \mathbb{Z}$ such that $a\chi(e) + b|[g]_G| = 1$. Define

$$\alpha := \frac{\chi(g)}{\chi(e)} = a\chi(g) + b\frac{\chi(g)}{\chi(e)}|[g]_G|$$

Then, since $\chi(g)$ is a sum of $|G|$th roots of unity, α satisfies the conditions of the previous lemma (or is zero) and so this lemma follows. $\square$

Proposition. *If G is a non-abelian finite group with an element $g \neq e$ such that $|[g]_G|$ has prime power order then G is not simple.*

Proof. Suppose for contradiction that G is simple and has an element $g \in G\backslash\{e\}$ such that $|[g]_G| = p^r$ for some prime p.

If χ is a non-trivial irreducible character of G then $|\chi(g)| < \chi(1)$ since otherwise $\rho(g)$ is a scalar matrix and so lies in $Z(\rho(G)) \cong Z(G)$.

[32] i.e. all the λ_i are equal.

[33] and so $\rho(g) \in Z(\rho(G))$

Thus by the last lemma, for every non-trivial irreducible character, either p divides $\chi(e)$ or $|\chi(g)| = 0$. By column orthogonality,

$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi(e)\chi(g).$$

Thus $\frac{-1}{p} = \sum_{\chi \neq \mathbf{1}} \frac{\chi(e)}{p}\chi(g) \in \mathcal{O} \cap \mathbb{Q}$. That is $\frac{1}{p}$ in $\mathbb{Z}$ giving the desired contradiction. □

Theorem (Burnside (1904))**.** *Let p, q be primes and G a group of order $p^a q^b$ with a, b non-negative integers such that $a + b \geqslant 2$, then G is not simple.*

Proof. Without loss of generality $b > 0$. Let Q be a Sylow-q-subgroup of G. Since $Z(Q) \neq 1$ we can find $e \neq g \in Z(Q)$. Then q^b divides $|C_G(g)|$, so the conjugacy class containing g has order p^r for some $0 \leqslant r \leqslant a$. The theorem now follows immediately from the Proposition. □

Remarks.

(1) It follows that every group of order $p^a q^b$ is soluble. That is, there is a chain of subgroups $G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_r = \{e\}$ with G_{i+1} normal in G_i and G_i/G_{i+1} abelian for all i.

(2) Note that $|A_5| = 2^2 \cdot 3 \cdot 5$ so the order of a simple group can have precisely 3 prime factors.

(3) The first purely group theoretic proof of the $p^a q^b$-theorem appeared in 1972.

Lecture 19

8. Topological groups

In this section k will be $\mathbb{C}$ always.

8.1. **Defintions and examples.**

Definition. A *topological group* G is a group G which is also a topological space such that the multiplication map $G \times G \to G; (g,h) \mapsto gh$ and the inverse map $G \to G; g \mapsto g^{-1}$ are continuous maps.

Examples.

(1) $GL_n(\mathbb{C})$ with the subspace topology from $\mathrm{Mat}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ since
$$(AB)_{ij} = \sum_k A_{ik}B_{ki} \text{ and } A^{-1} = \frac{1}{\det A}\mathrm{adj}A.$$
More generally $GL(V)$ for V a vector space over $\mathbb{C}$, where $GL(V)$ is given the the topology that makes an isomorphism $GL(V) \to GL_n(\mathbb{C})$ given by choosing a basis a homeomorphism. Since
$$GL_n(\mathbb{C}) \to GL_n(\mathbb{C}); A \mapsto P^{-1}AP$$
is a homeomorphism for every $P \in GL_n(\mathbb{C})$, this topology does not depend on the choice of basis.

(2) G finite — with the discrete topology — since all maps $G\times G \to G$ and $G \to G$ are continuous.

(3) $O(n) = \{A \in GL_n(\mathbb{R}) \mid A^TA = I\}$; $SO(n) = \{A \in O(n) \mid \det A = 1\}$.

(4) $U(n) = \{A \in GL_n(\mathbb{C}) \mid \overline{A^T}A = I\}$; $SU(n) = \{A \in U(n) \mid \det A = 1\}$. Note that
$$U(1) = S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

(5) *G profinite such as $\mathbb{Z}_p$, the completion of $\mathbb{Z}$ with respect to the p-adic metric.

Definition. A *representation* of a topological group G on a vector space V is a continuous group homomorphism $\rho\colon G \to GL(V)$.

Remarks.

(1) If X is a topological space then $\alpha\colon X \to GL_n(\mathbb{C})$ is continuous if and only if the maps $x \mapsto \alpha_{ij}(x) = \alpha(x)_{ij}$ are continuous for all i, j.

(2) If G is a (finite) group with the discrete topology. Then continuous function $G \to X$ just means function $G \to X$.

8.2. **Compact Groups.**

Our most powerful idea for studying representations of finite groups was averaging over the group; that is the operation $\frac{1}{|G|}\sum_{g\in G}$. When considering more general topological groups we should replace $\sum$ by $\int$.

Definition. For G a topological group and $C(G,\mathbb{R}) = \{f\colon G \to \mathbb{R} \mid f \text{ is continuous}\}$, a linear map $\int_G\colon C(G,\mathbb{R}) \to \mathbb{R}$ (we write $\int_G f = \int_G f(g)\,\mathrm{d}g$) is called a *Haar integral* if

(i) $\int_G 1 = 1$ (so $\int_G$ is normalised so total volume is 1);

(ii) $\int_G f(xg)\,\mathrm{d}g = \int_G f(g)\,\mathrm{d}g = \int_G f(gx)\,\mathrm{d}g$ for all $x \in G$ (so $\int_G$ is translation invariant).[34]

[34] For example $f(xg)$ means the continuous function $G \to \mathbb{R}$ given by $g \mapsto f(xg)$ and $\int_G f(xg)\,\mathrm{d}g$ means the value of $\int_G$ evaluated at this function.

(iii) $\int_G f \geqslant 0$ if $f(g) \geqslant 0$ for all $g \in G$ (positivity).

Examples.

(1) If G finite, then $\int_G f = \frac{1}{|G|}\sum_{g\in G} f(g)$.
(2) If $G = S^1$, $\int_G f = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\mathrm{d}\theta$.

Note that, for any $\mathbb{R}$-vector space V, $\int_G$ induces a linear map also written

$$\int_G : C(G, V) \to V :$$

under the natural identification $V \to V^{**}$ for $\theta \in V^*$, $f \in C(G, V)$,

$$\theta\left(\int_G f\right) = \int_G \theta(f(g))\,\mathrm{d}g.$$

More concretely, if $v_1, \ldots, v_n$ is a basis for V then $f \in C(G, V)$ is uniquely of the form

$$f = \sum_{i=1}^n f_i v_i \text{ with } f_1, \ldots, f_n \in C(G, \mathbb{R})$$

and

$$\int_G f = \sum_{i=1}^n \left(\int_G f_i\right) v_i.$$

This map is also translation invariant and sends a constant function to its unique value.

Moreover if $\alpha \colon V \to W$ is a linear map and $f \in C(G, V)$ then $\alpha\left(\int_G f\right) = \int_G (\alpha f)$. In particular if V is a $\mathbb{C}$-vector space then $V \to V$; $v \mapsto iv$ is $\mathbb{R}$-linear and so $\int_G$ is $\mathbb{C}$-linear.

Theorem. *If G is a compact Hausdorff group, then there is a unique Haar integral on G.*

Proof. Omitted □

All the examples of topological groups from last time are compact Hausdorff except $GL_n(\mathbb{C})$ which is not compact. We'll follow standard practice and write 'compact group' to mean 'compact Hausdorff group'.

Corollary (Weyl's Unitary Trick)**.** *If G is a compact group then every representation (ρ, V) is unitary.*

Proof. Same as for finite groups: let $\langle -, - \rangle$ be any inner product on V, then

$$(v, w) = \int_G \langle \rho(g)v, \rho(g)w \rangle\,\mathrm{d}g$$

is the required G-invariant inner product since, for $x \in G$ and for $v, w \in V$,

$$(\rho(x)v, \rho(x)w) = \int_G \langle \rho(gx)v, \rho(gx)w \rangle\,\mathrm{d}g = (v, w).$$

Checking that $(-, -)$ is an inner product is straightforward using that $\int_G$ is $\mathbb{C}$-linear together with its positivity. □

Remark. It follows that every compact subgroup of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.

Corollary. *All representations of a compact group are completely reducible.*

If $\rho\colon G \to GL(V)$ is a representation then $\chi_\rho := \operatorname{tr}\rho$ is a continuous class function since each $\rho(g)_{ii}$ is continuous.

Lemma. *If U is a representation of G then*

$$\dim U^G = \int_G \chi_U.$$

Proof. Let $\pi\colon U \to U$ be defined by $\pi = \int_G \rho_U \in \operatorname{Hom}_k(U, U)$. If $x \in G$ then

$$\rho_U(x)\pi = \rho_U(x)\left(\int_G \rho_U(g)\,\mathrm{d}g\right) = \int_G \rho_U(xg)\,\mathrm{d}g = \pi$$

since $\int_G$ is translation invariant. Thus $\operatorname{Im}\pi \leqslant U^G$.

If $u \in U^G$ then

$$\pi(u) = \int_G \rho_U(g)(u)\,\mathrm{d}g = \int_G u = u.$$

Thus π is a projection onto U^G and

$$\dim U^G = \operatorname{tr}\pi = \operatorname{tr}\left(\int_G \rho_U\right) = \int_G \chi_U. \qquad \square$$

We can use the Haar integral to put an inner product on the space $\mathcal{C}_G$ of (continuous)[35] class functions:

$$\langle f, f'\rangle := \int_G \overline{f(g)} f'(g)\,\mathrm{d}g.$$

Corollary (Orthogonality of Characters)**.** *If G is a compact group and V and W are irreducible reps of G then*

$$\langle \chi_V, \chi_W\rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Proof. Same as for finite groups:

$$\begin{aligned} \langle \chi_V, \chi_W\rangle &= \int_G \overline{\chi_V(g)}\chi_W(g)\,\mathrm{d}g \\ &= \int_G \chi_{\operatorname{Hom}_k(V,W)} \\ &= \dim \operatorname{Hom}_G(V, W). \end{aligned}$$

Then apply Schur's Lemma.

Note along the way we require that $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ which follows from the fact that V is unitary. $\square$

It is also possible to make sense of 'the characters are a basis for the space of (square integrable) class functions' but this requires a little knowledge of Hilbert spaces.

[35] or better still square integrable

Lecture 20

8.3. A worked example: S^1. We want to understand representations of S^1. Since

$$f \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\mathrm{d}\theta$$

is a Haar integral they are all completely reducible so it suffices to understand irreducible representations. By Schur's Lemma all such representations have degree 1 and by Weyl's unitary trick they all have image in S^1; that is they are continuous group homomorphisms $S^1 \to S^1$. Since

$$\mathbb{R} \to S^1;\ \ x \mapsto e^{2\pi i x}$$

induces an isomorphism of topological groups $\mathbb{R}/\mathbb{Z} \xrightarrow{\cong} S^1$,[36] there is a 1-1 correspondence between representations of S^1 and continuous group homomorphisms $\mathbb{R} \to S^1$ with kernel containing $\mathbb{Z}$.

Fact. If $f\colon \mathbb{R} \to S^1$ is any continuous function with $f(0) = 1$ then there is a unique continuous function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $f(x) = e^{2\pi i\alpha(x)}$.[37]

Sketch proof of Fact: locally $\alpha(x) = \frac{1}{2\pi i}\log f(x)$ and we can choose the branches of log to make the pieces glue together continuously.

Lemma. *If $\theta\colon (\mathbb{R},+) \to S^1$ is a continuous group homomorphism then there is a continuous homomorphism $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $\theta(x) = e^{2\pi i\psi(x)}$ for all $x \in \mathbb{R}$.*

Proof. The fact gives a unique continuous function ψ satisfying the defining equation and $\psi(0) = 0$. We must show ψ is a group homomorphism. To this end, let Δ be the continuous function $\mathbb{R}^2 \to \mathbb{R}$ given by

$$\Delta(a,b) := \psi(a+b) - \psi(a) - \psi(b).$$

Since $e^{2\pi i\Delta(a,b)} = \theta(a+b)\theta(a)^{-1}\theta(b)^{-1} = 1$, Δ only takes values in $\mathbb{Z}$. Thus as $\mathbb{R}^2$ is connected, Δ is constant. Since $\Delta(0,0) = 0$ we see that $\Delta \equiv 0$ and so ψ is a group homomorphism. □

Lemma. *If $\psi\colon (\mathbb{R},+) \to (\mathbb{R},+)$ is a continuous group homomorphism then there is some $\lambda \in \mathbb{R}$ such that $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$.*

Proof. Let $\lambda = \psi(1)$. Since ψ is a group homomorphism, $\psi(n) = \lambda n$ for all $n \in \mathbb{Z}$. Then $\mathrm{m}\psi(n/m) = \psi(n) = \lambda n$ and so $\psi(n/m) = \lambda n/m$. That is $\psi(x) = \lambda x$ for all $x \in \mathbb{Q}$. But $\mathbb{Q}$ is dense in $\mathbb{R}$ and ψ is continuous so $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$. □

Theorem. *Every irreducible representation of S^1 has degree 1 and is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.*

Proof. We've seem that if $\rho\colon S^1 \to GL_d(\mathbb{C})$ is an irreducible representation then $d = 1$ and $\rho(S^1) \leqslant S^1$. Moreover ρ induces a continuous homomorphism $\theta\colon \mathbb{R} \to S^1$ via $\theta(x) = \rho(e^{2\pi i x})$.

By the last two Lemmas, there is $\lambda \in \mathbb{R}$ such that

$$\theta(x) = e^{2\pi i\lambda x} \text{ for all } x \in \mathbb{R}.$$

Since $\theta(1) = 1$, $\lambda \in \mathbb{Z}$ and $\rho(e^{2\pi i x}) = e^{2\pi i\lambda x}$ for all $x \in \mathbb{R}$. □

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[36] i.e. a group isomorphism that is also a homeomorphism.

[37] In the language of algebraic topology $\mathbb{R} \to S^1$; $x \mapsto e^{2\pi i x}$ is a covering map and so paths in S^1 lift uniquely to paths in $\mathbb{R}$ after choosing the lift of the starting point. In fact $\mathbb{R}$ is the universal cover of S^1 via this map.

The theorem tell us that the 'character table' of S^1 has rows χ_n indexed by $\mathbb{Z}$ with $\chi_n(e^{i\theta}) = e^{in\theta}$.[38]

Notation. *Let*

$$\mathbb{Z}[z, z^{-1}] := \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathbb{Z}_0 \text{ with } \sum_{n\in\mathbb{Z}} |a_n| < \infty \right\}$$

Now if V is any rep of S^1 then by Machke's Theorem V breaks up as a direct sum of one dimensional subreps and so its character $\chi_V = \sum a_n z^n$ lies in $\mathbb{Z}[z, z^{-1}]$ with all $a_n \geqslant 0$ and $\sum a_n = \dim V$. Thus $\mathbb{Z}[z, z^{-1}]$ is the character ring of S^1.

As usual a_n is the number of copies of $\rho_n \colon z \mapsto z^n$ in the decomposition of V as a direct sum of simple subrepresentations. Thus, by orthogonality of characters,[39] we can compute

$$a_n = \langle \chi_n, \chi_V \rangle_{S^1} = \frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\phi}) e^{-in\phi} \, \mathrm{d}\phi$$

and

$$\chi_V(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\phi}) e^{-in\phi} \, \mathrm{d}\phi \right) e^{in\theta}.$$

So Fourier decomposition gives the decomposition of χ_V into irreducible characters and the Fourier mode is the multiplicity of an irreducible character.

Remark. In fact by the theory of Fourier series any continuous function on S^1 can be uniformly approximated by a finite $\mathbb{C}$-linear combination of the χ_n.

Moreover the χ_n form a complete orthonormal set in the Hilbert space

$$L^2(S^1) = \left\{ f \colon S^1 \to \mathbb{C} \;\middle|\; \int_0^{2\pi} |f(e^{i\theta})|^2 \, \mathrm{d}\theta \text{ exists and is finite} \right\} / \sim$$

of square-integrable complex-valued functions on S^1. That is every $f \in L^2(S^1)$ has a unique series expansion

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta'}) e^{-in\theta'} \, \mathrm{d}\theta' \right) e^{in\theta}$$

converging with respect to the norm $||f||^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, \mathrm{d}\theta$.

We can phrase this as

$$L^2(S^1) = \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}\chi_n \text{[40]}$$

which is an analogue of

$$\mathbb{C}G = \bigoplus_{V\in\mathrm{Irr}(G)} (\dim V) V$$

for finite groups.[41]

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[38]As an aside the unitary irreducible characters of $\mathbb{Z}$ are indexed by S^1 giving a duality between $\mathbb{Z}$ and S^1.

[39]in this case this simply says $\langle \chi_n, \chi_m \rangle_{S^1} = \frac{1}{2\pi} \int_0^{2\pi} e^{(m-n)i\theta} \, \mathrm{d}\theta = \delta_{n,m}$

[40]$\widehat{\bigoplus}$ is supposed to mean a completed direct sum or more precisely a direct sum in the category of Hilbert spaces.

[41]cf the Peter-Weyl theorem.

8.4. **Second worked example: $SU(2)$.**

Recall that $SU(2) = \{A \in GL_2(\mathbb{C}) \mid \overline{A^T}A = I, \det A = 1\}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ then since $\det A = 1$, $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Thus $d = \overline{a}$ and $c = -\overline{b}$. Moreover $a\overline{a} + b\overline{b} = 1$. In this way we see that

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}$$

which is homeomorphic to $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$.

More precisely if

$$\mathbb{H} := \mathbb{R} \cdot SU(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \middle| w, z \in \mathbb{C} \right\} \subset M_2(\mathbb{C})$$

then $||A||^2 = \det A$ defines a norm on $\mathbb{H} \cong \mathbb{R}^4$ and $SU(2)$ is the unit sphere in $\mathbb{H}$. If $A \in SU(2)$ and $X \in \mathbb{H}$ then $||XA = ||AX|| = ||X||$ since $||A|| = 1$. So, $SU(2)$ acts by isometries on $\mathbb{H}$ on both the left and the right and, after normalisation, usual integration of functions on S^3 defines a Haar integral on $SU(2)$. i.e.

$$\int_{SU(2)} f = \frac{1}{2\pi^2} \int_{S^3} f.$$

Here $2\pi^2$ is the volume of S^3 in $\mathbb{R}^4$ with respect to the usual measure.

We now try to compute the conjugacy classes in $SU(2)$.

Definition. Let $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in S^1 \right\} \cong S^1$.

Proposition.

(i) Every conjugacy class $\mathcal{O}$ in $SU(2)$ contains an element of T.

(ii) More precisely. if $\mathcal{O}$ is a conjugacy class then $\mathcal{O} \cap T = \{t, t^{-1}\}$ for some $t \in T$ and $t = t^{-1}$ if and only if $t = \pm I$ when $\mathcal{O} = \{t\}$.

(iii) There is a bijection

$$\{\textit{conjugacy classes in } SU(2)\} \to [-1, 1]$$

given by $[A]_{SU(2)} \mapsto \frac{1}{2} \operatorname{tr} A$.

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Proof. (i) Every unitary matrix has an orthonormal basis of eigenvectors. That is, if $A \in SU(2)$, there is a unitary matrix P such that PAP^{-1} is diagonal. Then if $Q = \frac{1}{\sqrt{\det P}}P$. $P^{-1}AP = Q^{-1}AQ \in T$ ie $[A]_{SU2} \cap T \neq \emptyset$.

(ii) If $\pm I \in \mathcal{O}$ the result is clear.

$$[t]_{SU(2)} = \{gtg^{-1} \mid g \in SU(2)\}.$$

But $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$ and $sts^{-1} = t^{-1}$ for all $t \in T$ so $[t]_{SU(2)} \cap T \supset \{t, t^{-1}\}$.

Conversely, if $t' \in \mathcal{O} \cap T$ then t' and t must have the same eigenvalues since they are conjugate. This suffices to see that $t' \in \{t^{\pm 1}\}$.

(iii) To see the given function is injective, suppose that $\frac{1}{2} \operatorname{tr} A = \frac{1}{2} \operatorname{tr} B$. Then since $\det A = \det B = 1$, A and B must have the same eigenvalues. By part (i) they are both diagonalisable and by the proof of part (ii) this suffices to see that they are conjugate.

To see that it is surjective notice that $\frac{1}{2}\operatorname{tr}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} = \cos\theta$. Since $\cos\colon \mathbb{R}\to\mathbb{R}$ has image $[-1,1]$ the given function is surjective. □

Corollary. *A (continuous) class function $f\colon SU(2)\to\mathbb{C}$ is determined by its restriction to T and $f|_T$ is even ie $f\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right) = f\left(\begin{pmatrix} z^{-1} & 0 \\ 0 & z\end{pmatrix}\right)$ for all $z\in S^1$.*[42] □

Notation. *Let*

$$\begin{aligned}\mathbb{Z}[z,z^{-1}]^{ev} &= \{f\in\mathbb{Z}[z,z^{-1}] \mid f(z)=f(z^{-1})\}\\ &= \left\{\sum a_n z^n : a_n\in\mathbb{Z},\ a_n=a_{-n} \text{ for all } n\in\mathbb{Z}\right\}.\end{aligned}$$

Lemma. *If χ is a character of a representation of $SU(2)$ then $\chi|_T\in\mathbb{Z}[z,z^{-1}]^{ev}$.*

Proof. If V is a representation of $SU(2)$ then $\operatorname{Res}_T^{SU(2)} V$ is a representation of T and $\chi_{\operatorname{Res}_T V}$ is the restriction of χ_V to T. Since every character of T is in $\mathbb{Z}[z,z^{-1}]$[43] and $\chi|_T$ is even we're done. □

It follows that $R(SU(2))\leqslant\mathbb{Z}[z,z^{-1}]^{ev}$. In fact we'll see that we have equality.

Let's write $\mathcal{O}_x=\{A\in SU(2)\mid \frac{1}{2}\operatorname{tr}A=x\}$ for $x\in[-1,1]$. We've proven that the $\mathcal{O}_x$ are the conjugacy classes in $SU(2)$. Clearly $\mathcal{O}_1=\{I\}$ and $\mathcal{O}_{-1}=\{-I\}$. For $-1<x<1$ there is some $\theta\in(0,\pi)$ such that $\cos\theta=x$ then

$$\mathcal{O}_x=\left\{\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}\,\middle|\, (\operatorname{Im}a)^2+|b|^2=\sin^2\theta\right\}$$

since $Re\,a=x=\cos\theta$. That is $\mathcal{O}_x$ is a 2-sphere of radius $|\sin\theta|$.

Thus if f is a class-function on $SU(2)$, since f is constant on each $\mathcal{O}_{\cos\theta}$,

$$\int_{SU(2)} f(g)\,\mathrm{d}g=\frac{1}{2\pi^2}\int_0^\pi f(e^{i\theta})4\pi\sin^2\theta\,\mathrm{d}\theta=\frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\sin^2\theta\,\mathrm{d}\theta.$$

Note this is normalised correctly, since $\frac{1}{\pi}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta=1$.

8.5. Representations of $SU(2)$.

Let V_n be the complex vector space of homogeneous polynomials in two variables x,y. So $\dim V_n=n+1$. Then $GL_2(\mathbb{C})$ acts on V_n via

$$\rho_n\colon GL_2(\mathbb{C})\to GL(V_n)$$

given by

$$\rho_n\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)f(x,y)=f(ax+cy,bx+dy).$$

i.e.

$$\rho_n\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)x^iy^j=(ax+cy)^i(bx+dy)^j.$$

Examples.
$V_0=\mathbb{C}$ has the trivial action.

[42] We'll write $f(z)$ for $f\left(\begin{pmatrix} z & 0\\ 0 & z^{-1}\end{pmatrix}\right)$ identifying T with S^1.

[43] As $T\cong S^1$. For the same reason we also know the coefficients a_n in $\chi|_{\operatorname{Res}_T V}(z)=\sum a_n z^n$ are non-negative.

$V_1 = \mathbb{C}^2$ is the standard representation of $GL(\mathbb{C}^2)$ on $\mathbb{C}^2$ with basis x, y.
$V_2 = \mathbb{C}^3$ has basis x^2, xy, y^2 then

$$\rho_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

In general $V_n \cong S^n V_1$ as representations of $GL_2(\mathbb{C})$.

Since $SU(2)$ is a subgroup of $GL_2(\mathbb{C})$ we can view V_n as a representation of $SU(2)$ by restriction. In fact as we'll see, the V_n are precisely the irreducible reps of $SU(2)$ (up to isomorphism).

Let's compute the character $\chi_{V_n}|_T$ of (ρ_n, V_n):

$$\rho_n\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right)(x^i y^j) = (zx)^i (z^{-1}y)^j = z^{i-j} x^i y^j.$$

So for each $0 \leqslant j \leqslant n$, $\mathbb{C}x^j y^{n-j}$ a T-subrepresentation with character z^{2j-n} and

$$\chi_{V_n}\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \in \mathbb{Z}[z, z^{-1}]^{ev}.$$

Theorem. *V_n is irreducible as a reperesentation of $SU(2)$.*

Proof. Let $0 \neq W \leqslant V_n$ be a $SU(2)$-invariant subspace. We want to show that $W = V_n$.

W is T-invariant so as $\mathrm{Res}_T^{SU(2)} V_n = \bigoplus_{j=0}^n \mathbb{C}x^j y^{n-j}$ is a direct sum of non-isomorphic representations of T,

(6) W has as a basis a subset of $\{x^j y^{n-j} \mid 0 \leqslant j \leqslant n\}$.

Thus $x^j y^{n-j} \in W$ for some $0 \leqslant j \leqslant n$. Since

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x^j y^{n-j} = \frac{1}{\sqrt{2}}((x-y)^j (x+y)^{n-j}) \in W$$

so by (6) we can deduce that $x^n \in W$. Repeating the same calculation for $j = n$, we see that $(x+y)^n \in W$ and so, by (6) again, $x^i y^{n-i} \in W$ for all i.

Thus $W = V_n$. □

Exercise. Alternative proof:

$$\langle \chi_{V_n}, \chi_{V_n} \rangle_{SU(2)} = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{e^{(n+1)i\theta} - e^{-(n+1)i\theta}}{e^{i\theta} - e^{-i\theta}}\right)^2 \sin^2\theta \,\mathrm{d}\theta = 1.$$

Theorem. *Every irreducible representation of $SU(2)$ is isomorphic to V_n for some $n \geqslant 0$.*

Proof. Let V be an irreducible representation of $SU(2)$ so $\chi_V \in \mathbb{Z}[z, z^{-1}]^{ev}$.

Now $\chi_0 = 1, \chi_1 = z + z^{-1}, \chi_2 = z^2 + 1 + z^{-2}, \ldots$. Thus $\chi_V = \sum_{i=0}^n \lambda_i \chi_i$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$.

Now by orthogonality of characters

$$\lambda_i = \langle \chi_{V_i}, \chi_V \rangle_{SU(2)} = \begin{cases} 1 & \text{if } V \cong V_i \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi_V \neq 0$ there is some i such that $\lambda_i = 1$ and $V \cong V_i$. □

We also want to understand $\otimes$ for representations of $SU(2)$. Recall that if G is a group and V, W are representations of G then $\chi_{V\otimes W} = \chi_V \chi_W$.

Let's compute some examples for $SU(2)$:

$$\chi_{V_1\otimes V_1}(z) = (z + z^{-1})^2 = z^2 + 1 + z^{-2} + 1 = \chi_{V_2} + \chi_{V_0}$$

and

$$\chi_{V_2\otimes V_1}(z) = (z^2 + 1 + z^{-2})(z + z^{-1}) = z^3 + 2z + 2z^{-1} + z^{-3} = \chi_{V_3} + \chi_{V_1}.$$

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Proposition (Clebsch–Gordan rule)**.** *For $n, m \in \mathbb{N}$,*

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|+2} \oplus V_{|n-m|}.$$

Proof. Without loss of generality, $n \geqslant m$. Then

$$\begin{aligned}(\chi_n \cdot \chi_m)(z) &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \cdot (z^m + z^{m-2} + \cdots + z^{-m}) \\ &= \sum_{j=0}^{m} \frac{z^{n+m+1-2j} - z^{-(n+m+1-2j)}}{z - z^{-1}} \\ &= \sum_{j=0}^{m} \chi_{n+m-2j}(z)\end{aligned}$$

as required. □

8.6. **Representations of $SO(3)$.**

Proposition. *The action by conjugation of $SU(2)$ on the three-dimensional normed $\mathbb{R}$-vector space of 2×2 $\mathbb{C}$-matrices*

$$\left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : a + \overline{a} = 0 \right\}$$

with $||A||^2 = \det A$ induces an isomorphism of topological groups

$$SU(2)/\{\pm I\} \xrightarrow{\sim} SO(3).$$

Proof. See Example Sheet 4 Q4.[44] □

Corollary. *Every irreducible representation of $SO(3)$ is of the form V_{2n} for some $n \geqslant 0$.*

Proof. It follows from the Proposition that irreducible representations of $SO(3)$ correspond to irreducible representations of $SU(2)$ such that $-I$ acts trivially. But it is easy to verify that $-I$ acts on V_n as $(-1)^n$ □

[44] If you get stuck then consult my notes from 2012 for some hints.

9. Character table of $GL_2(\mathbb{F}_q)$

9.1. $\mathbb{F}_q$**.** Let $p > 2$ be a prime, $q = p^a$ a power of p for some $a > 0$, and $\mathbb{F}_q$ be the field with q elements. We know that $\mathbb{F}_q^\times \cong C_{q-1}$.

Notice that $\mathbb{F}_q^\times \to \mathbb{F}_q^\times; x \mapsto x^2$ is a group homomorphism with kernel ± 1. Thus half the elements of $\mathbb{F}_q^\times$ are squares and half are not. Moreover $x \mapsto x^{\frac{q-1}{2}}$ is a group homomorphism that sends squares to 1 and non-squares to -1.

Let $\epsilon \in \mathbb{F}_q^\times$ be a fixed non-square, so $\epsilon^{\frac{q-1}{2}} = -1$, and let

$$\mathbb{F}_{q^2} := \{a + b\sqrt{\epsilon} \mid a, b \in \mathbb{F}_q\},$$

the field extension of $\mathbb{F}_q$ with q^2 elements under the obvious operations.

Every element of $\mathbb{F}_q$ has a square root in $\mathbb{F}_{q^2}$ since if λ is non-square then $\lambda/\epsilon = \mu^2$ is a square, and $(\sqrt{\epsilon}\mu)^2 = \lambda$. It follows by completing the square that every quadratic polynomial in $\mathbb{F}_q$ factorizes in $\mathbb{F}_{q^2}$.

Notice that $(a + b\sqrt{\epsilon})^q = a^q + b^q \epsilon^{\frac{q-1}{2}}\sqrt{\epsilon} = (a - b\sqrt{\epsilon})$.[45] Thus the roots of an irreducible quadratic over $\mathbb{F}_q$ are of the form λ, λ^q.[46]

9.2. $GL_2(\mathbb{F}_q)$ **and its conjugacy classes.** We want to compute the character table of the group

$$G := GL_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{F}_q \text{ and } ad - bc \neq 0 \right\}.$$

The order of G is the number of bases for $\mathbb{F}_q^2$ over $\mathbb{F}_q$. This is $(q^2-1)(q^2-q)$.

First, we compute the conjugacy classes in G. We know from linear algebra (rational canonical form) that 2×2-matrices are determined by their minimal polynomials up to conjugation. By Cayley–Hamilton each element A of $GL_2(\mathbb{F}_q)$ has minimal polynomial $m_A(X)$ of degree at most 2 and $m_A(0) \neq 0$.

There are four cases.

Case 1: $m_A = X - \lambda$ for some $\lambda \in \mathbb{F}_q^\times$. Then $A = \lambda I$. So $C_G(A) = G$, and

$$|[A]_G| = |\{\lambda I\}| = 1.$$

There are $q-1$ such classes corresponding the possible choices of λ.

Case 2: $m_A = (X-\lambda)^2$ for some $\lambda \in \mathbb{F}_q^\times$ so $[A]_G = \left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right]_G$. Now

$$C_G\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in \mathbb{F}_q, a \neq 0 \right\}$$

so

$$|[A]_G| = \frac{(q-1)^2(q+1)q}{(q-1)q} = (q-1)(q+1).$$

There are $q-1$ such classes.

Case 3: $m_A = (X-\lambda)(X-\mu)$ for some distinct $\lambda, \mu \in \mathbb{F}_q^\times$. Then

$$[A]_G = \left[\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right]_G = \left[\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}\right]_G.$$

Moreover

$$C_G\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a, d \in \mathbb{F}_q^\times \right\} =: T.$$

---

[45]Since $p \mid \binom{q}{i}$ for $i = 1, \ldots, q-1$.

[46]$\lambda \mapsto \lambda^q$ should be viewed as an analogue of complex conjugation.

So
$$|[A]_G| = \frac{q(q-1)(q^2-1)}{(q-1)^2} = q(q+1).$$
There are $\binom{q-1}{2}$ corresponding to each possible choice of the pair $\{\lambda, \mu\}$.

Case 4: $m_A(X)$ is irreducible over $\mathbb{F}_q$ of degree 2 so
$$\begin{aligned} m_A(X) &= (X-\alpha)(X-\alpha^q) \in \mathbb{F}_{q^2}[X] \\ &= (X^2 - (\alpha+\alpha^q)X + \alpha^{q+1}) \\ &= (X^2 - (\operatorname{tr} A)X + \det A) \end{aligned}$$
for some $\alpha = \lambda + \mu\sqrt{\epsilon}$ with $\lambda, \mu \in \mathbb{F}_q$, $\mu \neq 0$. Then
$$[A]_G = \left[\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}\right]_G = \left[\begin{pmatrix} \lambda & -\epsilon\mu \\ -\mu & \lambda \end{pmatrix}\right]_G$$
since both these matrices have trace $2\lambda = \alpha + \alpha^q$ and determinant
$$(\lambda + \sqrt{\epsilon}\mu)(\lambda - \sqrt{\epsilon}\mu) = \alpha\alpha^q.$$
Now
$$C_G\left(\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}\right) = \left\{\begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} \middle| a^2 - \epsilon b^2 \neq 0\right\} =: K.$$
If $a^2 = \epsilon b^2$ then ϵ is a square or $a = b = 0$. So $|K| = q^2 - 1$ and so
$$|[A]_G| = \frac{q(q-1)(q^2-1)}{q^2-1} = q(q-1).$$
There are $q(q-1)/2$ such classes corresponding to the choices of the pair $\{\alpha, \alpha^q\}$.

In summary

Representative A	C_G	$\lvert[A]_G\rvert$	No of such classes
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	G	1	$q-1$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$	$(q-1)(q+1)$	$q-1$
$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	T	$q(q+1)$	$\binom{q-1}{2}$
$\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}$	K	$q(q-1)$	$\binom{q}{2}$

The groups T and K are both *maximal tori*. That is they are maximal subgroups of G subject to the fact that they are conjugate to a subgroup of the group of diagonal matrices over some field extension of $\mathbb{F}_q$. T is called *split* and K is called *non-split*.

Some other important subgroups of G are
$$Z := \{\lambda I \mid \lambda \in \mathbb{F}_q^{\times}\}$$
which is the subgroup of scalar matrices (the centre);
$$N := \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F}_q\right\}$$
a Sylow p-subgroup of G; and
$$B := \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| b \in \mathbb{F}_q, a, d \in \mathbb{F}_q^{\times}\right\}$$

a *Borel subgroup* of G. Then N is normal in B and

$$B/N \cong T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times \cong C_{q-1} \times C_{q-1}.$$

9.3. **The character table of B.**

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Let's warm ourselves up by computing the character table of B.

Recall

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : b \in \mathbb{F}_q, a, d \in \mathbb{F}_q{}^\times \right\}$$

and

$$N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\} \lhd B \leqslant G = GL_2(\mathbb{F}_q).$$

The conjugacy classes in B are

Representative	C_B	No of elts	No of such classes
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	B	1	$q-1$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	ZN	$q-1$	$q-1$
$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	T	q	$(q-1)(q-2)$

Moreover $B/N \cong T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. So if $\Theta_q := \{\text{reps } \theta\colon \mathbb{F}_q^\times \to \mathbb{C}^\times\}$, then Θ_q is a cyclic group of order $q-1$ under pointwise operations. Moreover, for each pair $\theta, \phi \in \Theta_q$, we have a 1-dimensional representation of B (factoring through B/N) given by

$$\chi_{\theta,\phi}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \theta(a)\phi(d)$$

giving $(q-1)^2$ 1-dimensional reps.

Suppose $\gamma\colon (\mathbb{F}_q, +) \to \mathbb{C}^\times$ is a degree 1 representation and $\theta \in \Theta_q$, we can define a 1-dimensional representation of $ZN \cong \mathbb{F}_q^\times \times \mathbb{F}_q$; $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto (a, a^{-1}b)$ by

$$\rho_{\theta,\gamma}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \theta(a)\gamma(a^{-1}b).$$

Now $ZN \lhd B$ so by Mackey's irreducibility criterion $\operatorname{Ind}_{ZN}^B \rho_{\theta,\gamma}$ is irreducible if and only if ${}^g\rho_{\theta,\gamma} \neq \rho_{\theta,\gamma}$ for all $g \notin ZN$. Since $\left\{ t_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda \in \mathbb{F}_q{}^\times \right\}$ is a family of left coset reps of ZN in B and

$$\left({}^{t_\lambda}\rho_{\theta,\gamma}\right)\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \rho_{\theta,\gamma}\left(\begin{pmatrix} a & \lambda b \\ 0 & a \end{pmatrix}\right) = \theta(a)\gamma(a^{-1}\lambda b),$$

$${}^{t_\lambda}\rho_{\theta,\gamma} = \rho_{\theta,\gamma} \text{ if and only if } \gamma(a^{-1}\lambda b) = \gamma(a^{-1}b) \text{ for all } b \in \mathbb{F}_q.$$

The latter is equivalent to $\gamma((\lambda-1)b) = 1$ for all $b \in \mathbb{F}_q$ i.e. either $\lambda = 1$ or $\gamma = \mathbf{1}_{\mathbb{F}_q}$. So $\operatorname{Ind}_{ZN}^B \rho_{\theta,\gamma}$ is irreducible if and only if $\gamma \neq \mathbf{1}_{\mathbb{F}_q}$.

Now since
$$\operatorname{Ind}_{ZN}^{B}\chi(b)=\sum_{[g]_{ZN}\subseteq[b]_B}\frac{|C_B(b)|}{|C_{ZN}(g)|}\chi(g)$$
We see that
$$\begin{aligned}
\operatorname{Ind}_{ZN}^{B}\rho_{\theta,\gamma}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right)&=(q-1)\theta(\lambda),\\
\operatorname{Ind}_{ZN}^{B}\rho_{\theta,\gamma}\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right)&=\sum_{b\in{\mathbb{F}_q}^\times}\theta(\lambda)\gamma(b)\\
&=\theta(\lambda)\left(\sum_{b\in\mathbb{F}_q}\gamma(b)\right)-\theta(\lambda)\\
&=\theta(\lambda)(q\langle\mathbf{1}_{\mathbb{F}_q},\gamma\rangle_{\mathbb{F}_q}-1)\\
&=\begin{cases}-\theta(\lambda) & \text{if } \gamma\neq\mathbf{1}_{\mathbb{F}_q}\\ (q-1)\theta(\lambda) & \text{if } \gamma=\mathbf{1}_{\mathbb{F}_q}\end{cases}\\
\operatorname{Ind}_{ZN}^{B}\rho_{\theta,\gamma}\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right)&=0
\end{aligned}$$

Let $\mu_\theta:=\operatorname{Ind}_{ZN}^{B}\rho_{\theta,\gamma}$ for $\gamma\neq\mathbf{1}_{\mathbb{F}_q}$ noting that this does not then depend on γ. Then each μ_θ is irreducible by the discussion above and we have $(q-1)$ irreducible representations of degree $q-1$. Thus the character table of B is

	$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$
$\chi_{\theta,\phi}$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu)$
μ_θ	$(q-1)\theta(\lambda)$	$-\theta(\lambda)$	0

Remarks.

(1) The 0 in the bottom right corner appears in $q-1$ rows and $(q-1)(q-2)$ columns. But they are forced to be 0 by a Lemma in §7.4 since the order of these conjugacy classes are all q, the degree of the irreducible representations are all $(q-1)$ which is coprime to q, and these elements can't act by scalars because the representations are faithful and the elements are not in the centre.

(2) $B=Z\times\left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\,\middle|\, a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q\right\}$ and the second factor is a Frobenius group. So Example Sheet 3 Q10, together with our construction of irreducible representations of a direct product as the tensor product of the irreducible representations of the factors, tells us that we should expect to be able to construct all the irreducible representation of B in the manner that we have done so.

9.4. **The character table of G.** As $\det\colon G\to{\mathbb{F}_q}^\times$ is a surjective group homomorphism, for each $\theta\in\Theta_q$ we have a 1-dimensional representation of G via $\chi_\theta:=\theta\circ\det$ giving $q-1$ representations of degree 1.

Next we'll do some induction from B. Writing $s=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ we see that
$$\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}s\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}=\begin{pmatrix}b & a+b\beta\\ d & \beta d\end{pmatrix}$$

and these elements are all distinct. Hence BsN contains $q|B|$ elements so must be $G\backslash B$.[47] Thus $BsN = BsB$ and $B\backslash G/B$ has two elements $G = B \coprod BsB$ (this is called Bruhat decomposition).

By the proof of Mackey's irreducibility criterion if χ is a character of B then

$$\langle \operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi \rangle_G = \langle \chi, \chi \rangle_B + \langle \operatorname{Res}^B_{B\cap {}^sB} \chi, \operatorname{Res}^{{}^sB}_{B\cap {}^sB} {}^s\chi \rangle_{B\cap {}^sB}$$

Now

$$s\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} s^{-1} = \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}$$

so $B \cap {}^sB = T$ and

$$\langle \operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi \rangle_G = \langle \chi, \chi \rangle_B + \langle \chi|_T, {}^s\chi|_T \rangle_T$$

where

$${}^s\chi\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = \chi\left(\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}\right)$$

Thus $W_{\theta,\phi} := \operatorname{Ind}_B^G \chi_{\theta,\phi}$ is irreducible for $\theta \neq \phi \in \Theta_q$. These are called *principal series representations.*

We can also compute that $W_{\theta,\theta}$ has two irreducible factors and

$$\langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_B^G \mu_\theta \rangle_G = 1 + \frac{1}{|T|}\left(\sum_{\lambda\in\mathbb{F}_q{}^\times} |(q-1)\theta(\lambda)|^2\right) = 1 + (q-1) = q.$$

Now for any character χ of B

$$\operatorname{Ind}_B^G \chi(g) = \sum_{[b]_B \subseteq [g]_G} \frac{|C_G(g)|}{|C_B(b)|}\chi(b).$$

So

$$\begin{aligned}
\operatorname{Ind}_B^G \chi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) &= (q+1)\chi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right), \\
\operatorname{Ind}_G^B \chi\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) &= \chi\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right), \\
\operatorname{Ind}_B^G \chi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) &= \chi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) + \chi\left(\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}\right) \text{ and} \\
\operatorname{Ind}_B^G \chi\left(\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}\right) &= 0.
\end{aligned}$$

Notice that $W_{\theta,\phi} \cong W_{\phi,\theta}$ so we get $\binom{q-1}{2}$principal series representations.

We also notice that $W_{\theta,\theta} \cong \chi_\theta \otimes W_{\mathbf{1},\mathbf{1}}$ and

$$W_{\mathbf{1},\mathbf{1}} = \operatorname{Ind}_B^G \mathbf{1} = \mathbb{C}G/B$$

is a permutation representation. Thus $W_{\mathbf{1},\mathbf{1}}$ decomposes as $\mathbf{1} \oplus V_{\mathbf{1}}$ with $V_{\mathbf{1}}$ an irreducible representation of degree q. This representation is known as the *Steinberg representation.* Then $W_{\theta,\theta} \cong \chi_\theta \oplus V_\theta$ with $V_\theta = \chi_\theta \otimes V_{\mathbf{1}}$ is also irreducible of degree q a *twisted Steinberg.*

We have explicitly constructed $(q-1)+\binom{q-1}{2}+(q-1)$ irreducible representations i.e. not just their characters. We have $\binom{q}{2}$ characters to go. It will turn out that

[47] As $|G| = (q+1)|B|$.

they are indexed by pairs $\{\varphi, \varphi^q\}$ degree 1 representations of K such that $\varphi \neq \varphi^q$ but we won't we able to explicitly construct the representations.

Lecture 24

So far we have

# classes	$q-1$ $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$	$q-1$ $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$	$\binom{q-1}{2}$ $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$	$\binom{q}{2}$ $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$	# of reps
χ_θ	$\theta(\lambda)^2$	$\theta(\lambda)^2$	$\theta(\lambda)\theta(\mu)$	$\theta(\lambda^2-\epsilon\mu^2)$	$q-1$
V_θ	$q\theta(\lambda)^2$	0	$\theta(\lambda)\theta(\mu)$	$-\theta(\lambda^2-\epsilon\mu^2)$	$q-1$
$W_{\theta,\phi}$	$(q+1)\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu)+\phi(\lambda)\theta(\mu)$	0	$\binom{q-1}{2}$

It follows from calculations from last time that

$$\operatorname{Ind}_B^G \mu_\theta(g) = \begin{cases} (q+1)(q-1)\theta(\lambda) & \text{if } [g]_G = \left[\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right]_G \\ -\theta(\lambda) & \text{if } [g]_G = \left[\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right]_G \\ 0 & \text{otherwise}\end{cases}$$

and that $\langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_B^G \mu_\theta\rangle_G = q$.

Our next strategy is to induce characters from K. The map $\mathbb{F}_{q^2} \to M_2(\mathbb{F}_q)$ given by

$$\lambda + \mu\sqrt{\epsilon} \mapsto \begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$$

is an isomorphism of rings $\mathbb{F}_{q^2} \to K \cup \{0\}$ and we will identify these. Notice that $\mathbb{F}_q^\times$ corresponds to $Z \leqslant K$ with $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} = \lambda$. Moreover

$$\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}^q = \begin{pmatrix}\lambda & -\epsilon\mu\\ -\mu & \lambda\end{pmatrix}$$

since $(\lambda + \sqrt{\epsilon}\mu)^q = (\lambda - \sqrt{\epsilon}\mu)$.

We want to understand $\operatorname{Ind}_K^G \varphi$ for a character φ of K. First we understand the double cosets $K\backslash G/K$ and then we can apply Mackey to compute $\langle \operatorname{Ind}_K^G \varphi, \operatorname{Ind}_K^G \varphi\rangle_G$.

Note that for $k \in K$ and $g \in G$, $kgK = gK$ if and only if $g^{-1}kg \in K$. Since $[k]_G \cap K = \{k, k^q\}$ we see that this is in turn equivalent to $g^{-1}kg \in \{k, k^q\}$. Writing $t = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ we can compute that

$$t^{-1}\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix} t = \begin{pmatrix}\lambda & -\epsilon\mu\\ -\mu & \lambda\end{pmatrix}$$

so $kgK = gK$ if and only if $g^{-1}kg = k$ or $(tg)^{-1}k(tg) = k$.

Furthermore since

$$C_G\left(\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\right) = \begin{cases} G & \text{if } \mu = 0\\ K & \text{if } \mu \neq 0,\end{cases}$$

we see that $kgK = gK$ if and only if either $gK \in \{K, tK\}$ or $k \in Z$. It follows that

$$|KgK| = \begin{cases} |K| & \text{if } g \in K \cup tK \\ |K||K/Z| & \text{otherwise.} \end{cases}$$

Since $|K||K/Z| = (q^2-1)(q+1)$, there are

$$\frac{|G| - 2|K|}{|K||K/Z|} = \frac{|G/K| - 2}{|K/Z|} = \frac{q(q-1) - 2}{q+1} = q - 2$$

double cosets of size $|K||K/Z|$.

Now $K \cap {}^tK = K$ and for $g \in G\backslash K \cup tK$, $K \cap {}^gK = \mathrm{Stab}_K(gK) = Z$ so by Mackey

$$\langle \mathrm{Ind}_K^G \varphi, \mathrm{Ind}_K^G \varphi \rangle_G = \langle \varphi, \varphi \rangle_K + \langle \varphi, {}^t\varphi \rangle_K + \sum_{g \in K\backslash G/K \backslash \{K, tK\}} \langle \varphi|_Z, {}^g\varphi|_Z \rangle_Z$$

Since ${}^g\varphi|_Z = \varphi|_Z$ for all $g \in G$ and ${}^t\varphi = \varphi^q$,

$$\langle \mathrm{Ind}_K^G \varphi, \mathrm{Ind}_K^G \varphi \rangle_G = \begin{cases} q - 1 & \text{if } \varphi \neq \varphi^q \\ q & \text{if } \varphi = \varphi^q. \end{cases}$$

Suppose now that $\varphi\colon K \to \mathbb{C}^\times$ is a 1-dimensional character of K. Then

$$\mathrm{Ind}_K^G \varphi(g) = \begin{cases} q(q-1)\varphi(\lambda) & \text{if } [g]_G = [\lambda]_G \text{ for } \lambda \in \mathbb{F}_q^\times \\ \varphi(\alpha) + \varphi^q(\alpha) & \text{if } [g]_G = [\alpha]_G \text{ for } \alpha \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We can thus compute

$$\begin{aligned} \langle \mathrm{Ind}_B^G \mu_\theta, \mathrm{Ind}_K^G \varphi \rangle &= \frac{1}{|G|} \sum_{\lambda \in Z} (q^2 - 1)\overline{\theta(\lambda)} q(q-1)\varphi(\lambda) \\ &= (q-1)\langle \theta, \mathrm{Res}_Z^K \varphi \rangle_Z \end{aligned}$$

Thus $\mathrm{Ind}_B^G \mu_\theta$ and $\mathrm{Ind}_K^G \varphi$ have many factors in common when $\phi|_Z = \theta$.

Now, for each φ such that $\varphi \neq \varphi^q$ then our calculations tell us that if $\beta_\varphi = \mathrm{Ind}_B^G \mu_{\varphi|_Z} - \mathrm{Ind}_K^G \varphi \in R(G)$ then

$$\langle \beta_\varphi, \beta_\varphi \rangle_G = q - 2(q-1) + (q-1) = 1.$$

Since also $\beta_\varphi(1) = q - 1 > 0$ it follows that β_φ is an irreducible character. Since $\beta_\varphi = \beta_{\varphi^q}$, $\varphi^{q^2} = \varphi$ and $|\{\varphi : \varphi = \varphi^q\}| = q - 1$ we get $\frac{(q^2-1)-(q-1)}{2} = \binom{q}{2}$ characters in this way and the character table of $GL_2(\mathbb{F}_q)$ is complete.

# classes	$q-1$	$q-1$	$\binom{q-1}{2}$	$\binom{q}{2}$	
rep	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	α, α^q	# of reps
χ_θ	$\theta(\lambda)^2$	$\theta(\lambda)^2$	$\theta(\lambda)\theta(\mu)$	$\theta(\alpha^{q+1})$	$q-1$
V_θ	$q\theta(\lambda)^2$	0	$\theta(\lambda)\theta(\mu)$	$-\theta(\alpha^{q+1})$	$q-1$
$W_{\theta,\phi}$	$(q+1)\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu) + \phi(\lambda)\theta(\mu)$	0	$\binom{q-1}{2}$
β_φ	$(q-1)\varphi(\lambda)$	$-\varphi(\lambda)$	0	$-(\varphi + \varphi^q)(\alpha)$	$\binom{q}{2}$

The representations corresponding to the β_φ known as *discrete series representations* have not been computed explicitly. Drinfeld found these representations in l-adic étale cohomology groups of an algebraic curve X over $\mathbb{F}_q$. These cohomology groups should be viewed as generalisations of 'functions on X'. This work was generalised by Deligne and Lusztig for all finite groups of Lie type.

This construction also enables us to compute the character table of $PGL_2(\mathbb{F}_q) := GL_2(\mathbb{F}_q)/Z(GL_2(\mathbb{F}_q))$ as its irreducible representations are the irreducible representations of $GL_2(\mathbb{F}_q)$ such that the scalar matrices act trivially. i.e. the χ_θ and V_θ such that $\theta^2 = 1$, the $W_{\theta,\theta^{-1}}$ such that $\theta^2 \neq 1$ and the β_φ such that $\varphi|_Z = \mathbf{1}_Z$ i.e. $\varphi^{q+1} = 1$ as well as $\varphi^{q-1} \neq 1$.

We can also then compute the character table of $PSL_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)/Z(SL_2(\mathbb{F}_q))$ which has index 2 in $PGL_2(\mathbb{F}_q)$ by restriction. These groups are all simple when $q \geqslant 5$ and this can be seen from the character table.