REPRESENTATION THEORY

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Lecture 1

1. Introduction

Representation Theory is the study of how symmetries occur in nature; that is the study of how groups act by linear transformations on vector spaces.

One major goal of this course will be to understand how to go about classifying all representations of a given (finite) group. For this we will need to be precise about what it means for two representations to be the same as well as how representations may decompose into smaller pieces.

We'll also use Representation Theory to better understand groups themselves. An example of the latter that we'll see later in the course is the Burnside p^aq^b-theorem which tells us that the order of a finite simple group cannot have precisely two distinct prime factors.

1.1. Linear algebra revision.

Linear algebra revision. By *vector space* we will always mean a finite dimensional vector space over a field k unless we say otherwise. This field k will usually be algebraically closed and of characteristic zero, for example $\mathbb{C}$, because this is typically the easiest case. However there are rich theories for more general fields and we will sometimes hint at them.

Given a vector space V, we define the *general linear group* of V

$$GL(V) = \mathrm{Aut}(V) = \{\alpha\colon V \to V \mid \alpha \text{ linear and invertible}\}.$$

This is a group under composition of maps.

Because all our vector spaces are finite dimensional, there is an isomorphism $k^d \xrightarrow{\sim} V$ for some $d \geqslant 0$.[1] Here d is the isomorphism invariant of V called its dimension. The choice of isomorphism determines a basis $e_1, \ldots, e_d$ for V.[2] Then

$$GL(V) \cong \{A \in \mathrm{Mat}_d(k) \mid \det(A) \neq 0\}.$$

This isomorphism is given by the map that sends the linear map α to the matrix A such that $\alpha(e_i) = A_{ji}e_j$.

Exercise. Check that this does indeed define an isomorphism of groups. ie check that α is an invertible if and only if $\det A \neq 0$; and that the given map is a bijective group homomorphism.

[1] In fact the set of such isomorphisms is in bjiection with $GL(V)$ so typically there are very many such.

[2] Here e_i is the image of the ith standard basis vector for k^d under the isomorphism.

The choice of isomorphism $k^d \xrightarrow{\sim} V$ also induces a decomposition of V as a direct sum of one-dimensional subspaces

$$V = \bigoplus_{i=1}^{d} ke_i.$$

This decomposition is not unique is general[3] but the number of summands is always $\dim V$.

1.2. **Group representations — definitions and examples.** Recall that an action of a group G on a set X is a function $\cdot\colon G \times X \to X; (g,x) \mapsto g \cdot x$ such that

(i) $e \cdot x = x$ for all $x \in X$;
(ii) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

Recall also that to define such an action is equivalent to defining a group homomorphism $\rho\colon G \to S(X)$ where $S(X)$ denotes the symmetric group on the set X; that is the set of bijections from X to itself equipped with the binary operation of composition of functions.

Definition. A *representation* ρ of a group G on a vector space V is a group homomorphism $\rho\colon G \to GL(V)$, the group of invertible linear transformations of V.

By abuse of notation we will sometimes refer to the representation by ρ, sometimes by the pair (ρ, V) and sometimes just by V with the ρ implied. This can sometimes be confusing but we have to live with it.

Defining a representation of G on V corresponds to assigning a linear map $\rho(g)\colon V \to V$ to each $g \in G$ such that

(i) $\rho(e) = \mathrm{id}_V$;
(ii) $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$;
(iii) $\rho(g^{-1}) = \rho(g)^{-1}$ for all $g \in G$.

Exercise. Show that, given condition (ii) holds, conditions (i) and (iii) are equivalent to one another in the above. Show moreover that conditions (i) and (iii) can be replaced by the condition that $\rho(g) \in GL(V)$ for all $g \in G$.

Given a basis for V a representation ρ is an assignment of a matrix $\rho(g)$ to each $g \in G$ such that (i),(ii) and (iii) hold.

Definition. The *degree* of ρ or *dimension* of ρ is $\dim V$.

Definition. We say a representation ρ is *faithful* if $\ker \rho = \{e\}$.

Examples.

(1) Let G be any group and $V = k$. Then $\rho\colon G \to \mathrm{Aut}(V); g \mapsto \mathrm{id}$ is called the *trivial representation.*
(2) Let $G = C_2 = \{\pm 1\}$, $V = \mathbb{R}^2$, then

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a group rep of G on V.

[3]that is it depends on the choice of basis up to rescaling the basis vectors so there is more than one such decomposition if $d > 1$

(3) Let $G = (\mathbb{Z}, +)$, V a vector space, and ρ a representation of G on V. Then necessarily $\rho(0) = \mathrm{id}_V$, and $\rho(1)$ is some invertible linear map α on V. Now $\rho(2) = \rho(1+1) = \rho(1)^2 = \alpha^2$. Inductively we see $\rho(n) = \alpha^n$ for all $n > 0$. Finally $\rho(-n) = (\alpha^n)^{-1} = (\alpha^{-1})^n$. So $\rho(n) = \alpha^n$ for all $n \in \mathbb{Z}$.

Notice that conversely given any invertible linear map $\alpha\colon V \to V$ we may define a representation of G on V by $\rho(n) = \alpha^n$.

Thus we see that there is a 1-1 correspondence between representations of $\mathbb{Z}$ and invertible linear transformations given by $\rho \mapsto \rho(1)$.

(4) Let $G = (\mathbb{Z}/N, +)$, and $\rho\colon G \to GL(V)$ a rep. As before we see $\rho(n) = \rho(1)^n$ for all $n \in \mathbb{Z}$ but now we have the additional constraint that $\rho(N) = \rho(0) = \mathrm{id}_V$.

Thus representations of $\mathbb{Z}/N$ correspond to invertible linear maps α such that $\alpha^N = \mathrm{id}_V$. Of course any linear map such that $\alpha^N = \mathrm{id}_V$ is invertible so we may drop the word invertible from this correspondence.

(5) Let $G = S_3$, the symmetric group of $\{1, 2, 3\}$, and $V = \mathbb{R}^2$. Take an equilateral triangle in V centred on 0; then G acts on the triangle by permuting the vertices. Each such symmetry induces a linear transformation of V. For example $g = (12)$ induces the reflection through the vertex three and the midpoint of the opposite side, and $g = (123)$ corresponds to a rotation by $2\pi/3$.

Exercise. Choose a basis for $\mathbb{R}^2$. Write the coordinates of the vertices of the triangle in this basis. For each $g \in S_3$ write down the matrix of the corresponding linear map. Check that this does define a representation of S_3 on V. Would the calculations be easier in a different basis?

Lecture 2

(6) Given a finite set X we may form the vector space kX of functions X to k with basis $\langle \delta_x \mid x \in X \rangle$ where $\delta_x(y) = \delta_{xy}$.

Then an action of G on X induces a representation $\rho\colon G \to \mathrm{Aut}(kX)$ by $(\rho(g)f)(x) = f(g^{-1} \cdot x)$ called the *permutation representation* of G on X.

It is straightforward to verify that $\rho(g)$ is linear and that $\rho(e) = \mathrm{id}_{kX}$. So to check that ρ is a representation we must show that $\rho(gh) = \rho(g)\rho(h)$ for each $g, h \in G$.

For this observe that for each $x \in X$,

$$\rho(g)(\rho(h)f)(x) = (\rho(h)f)(g^{-1}x) = f(h^{-1}g^{-1}x) = \rho(gh)f(x).$$

Notice that $\rho(g)\delta_x(y) = \delta_{x,g^{-1}\cdot y} = \delta_{g\cdot x,y}$ so $\rho(g)\delta_x = \delta_{g\cdot x}$. So by linearity $\rho(g)(\sum_{x\in X} \lambda_x \delta_x) = \sum \lambda_x \delta_{g\cdot x}$.

(7) In particular if G is finite then the action of G on itself by left multiplication induces the *regular representation* kG of G. The regular representation is always faithful because $\rho(g)\delta_e = \delta_e$ implies that $ge = e$ and so $g = e$.

(8) If $\rho\colon G \to GL(V)$ is a representation of G then we can use ρ to define a representation of G on V^*

$$\rho^*(g)(f)(v) = f(\rho(g^{-1})v); \quad \forall f \in V^*, v \in V.$$

(9) More generally, if $(\rho, V), (\rho', W)$ are representations of G then $(\alpha, \mathrm{Hom}_k(V, W))$ defined by

$$\alpha(g)(f)(v) = \rho'(g)f(\rho(g)^{-1}v); \quad \forall g \in G, f \in \mathrm{Hom}_k(V, W), v \in V$$

is a rep of G.

Note that if $W = k$ is the trivial rep. this reduces to example 8.

Exercise. Check the details.[4] Moreover show that if $V = k^n$ and $W = k^m$ with the standard bases, so that $\mathrm{Hom}_k(V, W) = \mathrm{Mat}_{m,n}(k)$, then

$$\alpha(g)(A) = \rho'(g)A\rho(g)^{-1} \text{ for all } A \in \mathrm{Mat}_{m,n}(k) \text{ and } g \in G.$$

(10) If $\rho\colon G \to GL(V)$ is a representation of G and $\theta\colon H \to G$ is a group homomorphism then $\rho\theta\colon H \to GL(V)$ is a representation of H. If H is a subgroup of G and θ is inclusion we call this the *restriction* of ρ to H.

1.3. **The category of representations.** We want to classify all representations of a group G but first we need a good notion of when two representations are the same.

Definition. We say that $\rho\colon G \to GL(V)$ and $\rho'\colon G \to GL(V')$ are *isomorphic* representations if there is a linear isomorphism $\varphi\colon V \to V'$ such that

$$\rho'(g) = \varphi \circ \rho(g) \circ \varphi^{-1} \text{ for all } g \in G$$

i.e. if $\rho'(g) \circ \varphi = \varphi \circ \rho(g)$. We say that φ *intertwines* ρ and ρ'.

Notice that id_V intertwines ρ and ρ; if φ intertwines ρ and ρ' then φ^{-1} intertwines ρ' and ρ; and if moreover φ' intertwines ρ' and ρ'' then $\varphi'\varphi$ intertwines ρ and ρ''. Thus isomorphism is an equivalence relation.

Notice that if $\rho\colon G \to GL(V)$ is a representation and $\varphi\colon V \to V'$ is a vector space isomorphism then we may define $\rho'\colon G \to GL(V')$ by $\rho'(g) = \varphi \circ \rho(g) \circ \varphi^{-1}$. Then ρ' is also a representation. In particular every representation is isomorphic to a matrix representation $G \to GL_d(k)$.

If $\rho, \rho'\colon G \to GL_d(k)$ are matrix representations of the same degree then an intertwining map $k^d \to k^d$ is an invertible matrix P and the matrices of the reps it intertwines are related by $\rho'(g) = P\rho(g)P^{-1}$. Thus matrix representations are isomorphic precisely if they represent the same family of linear maps with respect to different bases.

Examples.

(1) If $G = \{e\}$ then a representation of G is just a vector space and two vector spaces are isomorphic as representations precisely if they have the same dimension.
(2) If $G = \mathbb{Z}$ then $\rho\colon G \to GL(V)$ and $\rho'\colon G \to GL(V')$ are isomorphic reps if and only if there are bases of V and V' such that $\rho(1)$ and $\rho'(1)$ are the same matrix. In other words isomorphism classes of representations of $\mathbb{Z}$ correspond to conjugacy classes of invertible matrices. Over $\mathbb{C}$ the latter is classified by Jordan Normal Form (more generally by rational canonical form).
(3) If $G = C_2 = \{\pm 1\}$ then isomorphism classes of representations of G correspond to conjugacy classes of matrices that square to the identity. Since the minimal polynomial of such a matrix divides $X^2 - 1 = (X-1)(X+1)$ provided the field does not have characteristic 2 every such matrix is conjugate to a diagonal matrix with diagonal entries all ± 1.

   Exercise. Show that there are precisely $n+1$ isomorphism classes of representations of C_2 of dimension n.

(4) If X, Y are finite sets with a G-action and $f\colon X \to Y$ is a G-equivariant bijection i.e. f is a bijection such that $g \cdot f(x) = f(g \cdot x)$ for all $x \in X$ and $g \in G$, then

[4] This will also appear on Examples Sheet 1.

$\varphi\colon kX \to kY$ defined by $\varphi(\theta)(y) = \theta(f^{-1}y)$ intertwines kX and kY. (Note that $\varphi(\delta_x) = \delta_{f(x)}$)

Note that two isomorphic representations must have the same dimension but that the converse is not true.

Lecture 3

Definition. Suppose that $\rho\colon G \to GL(V)$ is a rep. We say that a k-linear subspace W of V is *G-invariant* if $\rho(g)(W) \subseteq W$ for all $g \in G$ (ie $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$).

In that case we call W a *subrepresentation* of V; we may define a representation $\rho_W : G \to GL(W)$ by $\rho_W(g)(w) = \rho(g)(w)$ for $w \in W$.

We call a subrepresentation W of V *proper* if $W \neq V$ and $W \neq 0$. We say that $V \neq 0$ is *irreducible* or *simple* if it has no proper subreps.

Examples.

(1) Any one-dimensional representation of a group is irreducible.

(2) Suppose that $\rho\colon \mathbb{Z}/2 \to GL(k^2)$ is given by $-1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (char $k \neq 2$). Then there are precisely two proper subreps spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

Proof. It is easy to see that these two subspaces are G-invariant. Any proper subrep must be one dimensional and so by spanned by an eigenvector of $\rho(-1)$. But the eigenspaces of $\rho(-1)$ are precisely those already described. $\square$

(3) If G is C_2 then the only irreducible representations are one-dimensional.

Proof. Suppose $\rho\colon G \to GL(V)$ is an irreducible rep. The minimal polynomial of $\rho(-1)$ divides $X^2 - 1 = (X-1)(X+1)$. Thus $\rho(-1)$ has an eigenvector v. Now $0 \neq \langle v \rangle$ is a subrep. of V. Thus $V = \langle v \rangle$. $\square$

Notice we've shown along the way that there are precisely two simple reps of G if k doesn't have characteristic 2 and only one if it does.

(4) If $G = D_6$ then every irreducible complex representation has dimension at most 2.

Proof. Suppose $\rho\colon G \to GL(V)$ is an irred. G-rep. Let r be a non-trivial rotation and s a reflection in G. Then $\rho(r)$ has a eigenvector v, say. So $\rho(r)v = \lambda v$ for some $\lambda \neq 0$. Consider $W := \langle v, \rho(s)v \rangle \subset V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, W is G-invariant. Since V is irreducible, $W = V$. $\square$

Exercise. Show that there are precisely three irreducible reps of D_6 up to isomorphism, one of dimension 2 and two of dimension 1. (Hint: In the argument above necessarily $\lambda^3 = 1$ and we can split into cases where $\rho(s)(v) \in \langle v \rangle$ and where $\rho(s)(v) \notin \langle v \rangle$).

(5) If $G = \mathbb{Z}$ and (ρ, V) is a representation over $\mathbb{C}$ then when is V irreducible?

We can choose a basis for V so that $\rho(1)$ is in Jordan Normal Form. It is easy to see that the Jordan blocks determine invariant subspaces; so if V is irreducible then there is only one Jordan block. Say $\rho(1) = A$ then $Ae_i = \lambda e_i + e_{i-1}$ for some non-zero λ and $i = 1, \dots d$ (where by convention $e_0 = 0$).

Exercise. Show that the invariant subspaces are precisely the subspaces of the form $\langle e_1, \dots, e_k \rangle$ for $k \leqslant d$.

It follows that the only irreducible representations of $\mathbb{Z}$ are one-dimensional. $\rho\colon \mathbb{Z} \to \mathbb{C}^\times;\ 1 \mapsto \lambda$.

Proposition. *Suppose $\rho\colon G \to GL(V)$ is a rep and $W \leqslant V$. Then the following are equivalent:*

(i) W is a subrep;
(ii) there is a basis $v_1, \dots, v_d$ of V such that $v_1, \dots, v_r$ is a basis of W and the matrices $\rho(g)$ are all block upper triangular;
(iii) for every basis $v_1, \dots, v_d$ of V such that $v_1, \dots, v_r$ is a basis of W the matrices $\rho(g)$ are all block upper triangular.

Proof. Think about it! □

Definition. If W is a subrep of a rep (ρ, V) of G then we may define a *quotient representation* $\rho_{V/W}\colon G \to GL(V/W)$ by $\rho_{V/W}(g)(v + W) = \rho(g)(v) + W$. Since $\rho(g)W \subset W$ for all $g \in G$ this is well-defined.

We'll start dropping ρ now and write g for $\rho(g)$ where it won't cause confusion.

Definition. If (ρ, V) and (ρ', W) are reps of G we say a linear map $\varphi\colon V \to W$ is a *G-linear* map if $\varphi g = g\varphi$ (ie $\varphi \circ \rho(g) = \rho'(g) \circ \varphi$) for all $g \in G$. We write

$$\mathrm{Hom}_G(V, W) = \{\varphi \in \mathrm{Hom}_k(V, W) \mid \varphi \text{ is } G \text{ linear}\},$$

a k-vector space.

Remarks.

(1) $\varphi \in \mathrm{Hom}_k(V, W)$ is an intertwining map precisely if φ is a bijection and φ is in $\mathrm{Hom}_G(V, W)$.
(2) If $W \leqslant V$ is a subrep then the natural inclusion map $\iota\colon W \to V;\ w \mapsto w$ is in $\mathrm{Hom}_G(W, V)$ and the natural projection map $\pi\colon V \to V/W;\ v \mapsto v + W$ is in $\mathrm{Hom}_G(V, V/W)$.
(3) Recall that $\mathrm{Hom}_k(V, W)$ is a G-rep via $(g\varphi)(v) = g(\varphi(g^{-1}v))$ for $\varphi \in \mathrm{Hom}_k(V, W)$, $g \in G$ and $v \in V$. Then $\varphi \in \mathrm{Hom}_G(V, W)$ precisely if $g\varphi = \varphi$ for all $g \in G$.

Lemma (First isomorphism theorem for representations)**.** *Suppose (ρ, V) and (ρ', W) are representations of G and $\varphi \in \mathrm{Hom}_G(V, W)$ then*

(i) $\ker\varphi$ is a subrep of V.
(ii) $\mathrm{Im}\,\varphi$ is a subrep of W.
(iii) $V/\ker\varphi$ is isomorphic to $\mathrm{Im}\,\varphi$ as reps of G.

Proof.
(i) if $v \in \ker\varphi$ and $g \in G$ then $\varphi(gv) = g\varphi(v) = 0$
(ii) if $w = \varphi(v) \in \mathrm{Im}\,\varphi$ and $g \in G$ then $gw = \varphi(gv) \in \mathrm{Im}\,\varphi$.
(iii) We know that the linear map φ induces a linear isomorphism

$$\overline{\varphi}\colon V/\ker\varphi \to \mathrm{Im}\varphi;\ v + \ker\varphi \mapsto \varphi(v)$$

then $g\overline{\varphi}(v + \ker\varphi) = g(\varphi(v)) = \varphi(gv) = \overline{\varphi}(gv + \ker\varphi)$ □

Lecture 4

2. Complete reducibility and Maschke's Theorem

Question. Given a representation V and a G-invariant subspace W when can we find a vector space complement of W that is also G-invariant?

Example. Suppose $G = C_2$, $V = \mathbb{R}^2$ and $\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $W = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ has many vector space complements but only one of them, $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$, is a G-invariant.

Definition. We say a representation V is a *direct sum* of U and W if U and W are subreps of V such that $V = U \oplus W$ as vector spaces (ie $V = U + W$ and $U \cap W = 0$).

Given two representations (ρ_1, U) and (ρ_2, W) we may define a representation of G on $U \oplus W$ by $\rho(g)(u, w) = (\rho_1(g)u, \rho_2(g)w)$.

Examples.

(1) Suppose G acts on a finite set X and X may be written as the disjoint union of two G-invariant subsets X_1 and X_2 (i.e. $g \cdot x \in X_i$ for all $x \in X_i$ and $g \in G$). Then $kX \cong kX_1 \oplus kX_2$ under $f \mapsto (f|_{X_1}, f|_{X_2})$.

That is $kX = \{f \mid f(x) = 0\ \forall x \in X_2\} \oplus \{f \mid f(x) = 0\ \forall x \in X_1\}$.

More generally if the G-action on X decomposes into orbits as a disjoint union $X = \bigcup_{i=1}^r \mathcal{O}_i$ then

$$kX = \bigoplus_{i=1}^r \mathbf{1}_{\mathcal{O}_i}(kX) \cong \bigoplus k\mathcal{O}_i.$$

where $\mathbf{1}_{\mathcal{O}_i} \colon kX \to kX$ is given by $\mathbf{1}_{\mathcal{O}_i}(f)(x) = \begin{cases} f(x) & x \in \mathcal{O}_i \\ 0 & x \notin \mathcal{O}_i. \end{cases}$

(2) If G acts transitively on a finite set X then $U := \{f \in kX \mid \sum_{x \in X} f(x) = 0\}$ and $W := \{f \in kX \mid f \text{ is constant}\}$ are subreps of kX.

Proof. If $f \in U$ then for $g \in G$,

$$\sum_{x \in X} (g \cdot f)(x) = \sum_{x \in X} f(g^{-1}x) = 0$$

since $x \mapsto g^{-1}x$ is a bijection $X \to X$. Similarly if $f \in W$; $f(x) = \lambda$ for all $x \in X$ then for $g \in G$, $(g.f)(x) = f(g^{-1}x) = \lambda$ for all $x \in X$. □

If k is charactersitic 0 then $kX = U \oplus W$. What happens if k has characteristic $p > 0$?

(3) We saw before that every representation of $\mathbb{Z}/2$ over $\mathbb{C}$ is a direct sum of 1-dimensional subreps as we may diagonalise $\rho(-1)$. Let's think about how this might generalise:

Suppose that G is a finite abelian group, and (ρ, V) is a complex representation of G. Each element $g \in G$ has finite order so has a minimal polynomial dividing $X^n - 1$ for $n = o(g)$. In particular it has distinct roots. Thus there is a basis for V such that $\rho(g)$ is diagonal. But because G is abelian $\rho(g)$ and $\rho(h)$ commute for each pair $g, h \in G$ and so the $\rho(g)$ may be simultaneously diagonalised (Sketch proof: if each $\rho(g)$ is a scalar matrix the result is clear. Otherwise pick $g \in G$ such that $\rho(g)$ is not a scalar matrix. Each eigenspace $E(\lambda)$ of $\rho(g)$ will be G-invariant since G is abelian. By induction on $\dim V$ we may solve the problem for each subrep $E(\lambda)$ and then take the union of these bases). Thus V decomposes as a direct sum of 1-dimensional subreps

(4) We saw that if $(\rho, \mathbb{C}^n)$ is the representation of $\mathbb{Z}$ given by $\rho(1)(e_1) = e_1$ and $\rho(1)(e_i) = e_i + e_{i-1}$ for $i > 1$, then $\mathbb{C}^n$ has precisely $n-1$ proper invariant subspaces namely $\langle e_1, \ldots, e_k\rangle$ for $1 \leqslant k < n$. Thus none of these have an invariant complement.

Proposition. *Suppose $\rho\colon G \to GL(V)$ is a rep. and $V = U \oplus W$ as vector spaces. Then the following are equivalent:*

(i) $V = U \oplus W$ as reps;

(ii) there is a basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of U and $v_{r+1}, \ldots v_d$ is a basis for W and the matrices $\rho(g)$ are all block diagonal;

(iii) for every basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of U and $v_{r+1}, \ldots, v_d$ is a basis for W and the matrices $\rho(g)$ are all block diagonal.

Proof. Think about it! □

But the following example provides a warning.

Example. $\rho\colon \mathbb{Z}/2 \to GL_2(\mathbb{R})$; $1 \mapsto \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$ defines a representation (check). The representation $\mathbb{R}^2$ breaks up as $\langle e_1\rangle \oplus \langle e_1 - e_2\rangle$ as subreps even though the matrix is upper triangular but not diagonal.

We've seen by considering $G = \mathbb{Z}$ that it is not true that for every reperesentation of a group G, every subrepresentation has a G-invariant complement. However, the following remarkable theorem is true.

Theorem (Maschke's Theorem)**.** *Let G be a finite group and (ρ, V) a representation of G over a field k of characteristic zero. Suppose $W \leqslant V$ is a G-invariant subspace. Then there is a G-invariant complement to W ie a G-invariant subspace U of V such that $V = U \oplus W$.*

Corollary (Complete reducibility)**.** *If G is a finite group, (ρ, V) a representation over a field of characteristic zero. Then $V \cong W_1 \oplus \cdots W_r$ is a direct sum of representations with each W_i irreducible.*

Proof. By induction on $\dim V$. If $\dim V = 0$ or V is irreducible then the result is clear. Otherwise V has a non-trivial G-invariant subspace W.

By the theorem there is a G-invariant complement U and $V \cong U \oplus W$ as G-reps. But $\dim U, \dim W < \dim V$, so by induction they can each be decomposed as a direct sum of irreducibles reps. Thus V can too. □

Lecture 5

Example. Let G act on a finite set X, and consider the real permutation representation $\mathbb{R}X = \{f\colon X \to \mathbb{R}\}$ with $(\rho(g)f)(x) = f(g^{-1}x)$.

Idea: with respect to the given basis δ_x all the matrices $\rho(g)$ are orthogonal; that is they preserve distance with respect to the standard inner product $(-,-)$. This is because $(f_1, f_2) = \sum_{x\in X} f_1(x)f_2(x)$ and so for each $g \in G$

$$(g \cdot f_1, g \cdot f_2) = \sum_{x\in X} f_1(g^{-1}x)f_2(g^{-1}x) = (f_1, f_2)$$

since g^{-1} permutes the elements of X.

In particular if W is a subrep of $\mathbb{R}X$ and

$$W^\perp := \{v \in \mathbb{R}X \mid (v, w) = 0 \text{ for all } w \in W\}$$

then if $g \in G$ and $v \in W^\perp$ and $w \in W$ we have $(w, gv) = (g^{-1}w, v) = 0$ since $g^{-1}w \in W$. Thus G preserves $W^\perp$ which is thus a G-invariant complement to W.

Let's extend this idea. Recall, if V is a complex vector space then a Hermitian inner product is a positive definite Hermitian sesquilinear map $(-,-)\colon V \times V \to \mathbb{C}$ that is a map satisfying

(i) $(ax + by, z) = \overline{a}(x, z) + \overline{b}(y, z)$ and $(x, ay + bz) = a(x, y) + b(x, z)$ for $a, b \in \mathbb{C}$, $x, y, z \in V$ (sesquilinear);
(ii) $(x, y) = \overline{(y, x)}$ (Hermitian);
(iii) $(x, x) > 0$ for all $x \in V\backslash\{0\}$ (positive definite).

If $W \subset V$ is a linear subspace of a complex vector space with a Hermitian inner product and $W^\perp = \{v \in V \mid (v, w) = 0 \ \forall w \in W\}$ then $W^\perp$ is a vector space complement to W in V.

Definition. A Hermitian inner product on a G-rep V is G-invariant if $(gx, gy) = (x, y)$ for all $g \in G$ and $x, y \in V$; equivalently if $(gx, gx) = (x, x)$ for all $g \in G$ and $x \in V$.

Lemma. *If $(-,-)$ is a G-invariant Hermitian inner product on a G-rep V and $W \subset V$ is a subrep then $W^\perp$ is a G-invariant complement to W.*

Proof. It suffices to prove that $W^\perp$ is G-invariant since $W^\perp$ is a complement to W.

Suppose $g \in G$, $x \in W^\perp$ and $w \in W$. Then $(gx, w) = (x, g^{-1}w) = 0$ since $g^{-1}w \in W$. Thus $gx \in W^\perp$ as required. □

Recall that the *unitary group* $U(n)$ is the subgroup of $GL_n(\mathbb{C})$ consisting of matrices A such that $\overline{A^T}A = I$. Equivalently

$$U(n) = \{A \in GL_n(\mathbb{C}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{C}^n\}$$

where $\langle -,- \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^n$.

It follows from the Lemma that if $\rho\colon G \to GL(V)$ is a complex representation of any group G such that V has a G-invariant inner product is completely reducible i.e. it can be written as a direct sum of simple subrepresentations. In particular if $\rho\colon G \to U(n) \leqslant GL_n(\mathbb{C})$ is a matrix representation with all matrices $\rho(g)$ in $U(n)$ then $(\rho, \mathbb{C}^n)$ is completely reducible.

Proposition (Weyl's unitary trick)**.** *If V is a complex representation of a finite group G, then there is a G-invariant Hermitian inner product on V.*

Proof. Pick any Hermitian inner product $\langle -,- \rangle$ on V (e.g. choose a basis $e_1, \ldots, e_n$ and take the standard inner product $\langle \sum \lambda_i e_i, \sum \mu_i e_i \rangle = \sum \overline{\lambda_i}\mu_i$). Then define a new inner product $(-,-)$ on V by averaging:

$$(x, y) := \frac{1}{|G|}\sum_{g \in G} \langle gx, gy \rangle.$$

It is easy to see that $(-,-)$ is a Hermitian innder product because $\langle -,-\rangle$ is so. For example if $a,b\in\mathbb{C}$ and $x,y,z\in V$, then

$$\begin{aligned}(x,ay+bz) &= \frac{1}{|G|}\sum_{g\in G}\langle gx, g(ay+bz)\rangle\\ &= \frac{1}{|G|}\sum_{g\in G}\langle gx, ag(y)+bg(z)\rangle\\ &= \frac{1}{|G|}\sum_{g\in G}(a\langle gx,gy\rangle + b\langle gx,gz\rangle)\\ &= a(x,y)+b(z,y)\end{aligned}$$

as required.

But now if $h\in G$ and $x,y\in V$ then

$$(hx,hy)=\frac{1}{|G|}\sum_{g\in G}\langle ghx,ghy\rangle = \frac{1}{|G|}\sum_{g'\in G}\langle g'x,g'y\rangle$$

and so $(-,-)$ is G-invariant. □

Corollary. *For every complex representation V of a finite group G, every subrepresentation has a G-invariant complement and so V is completely reducible i.e. it decomposes as a direct sum of simple subrepresentations.*

Proof. Apply Weyl's unitary trick and then the last Lemma. □

Corollary (of Weyl's unitary trick)**.** *Every finite subgroup G of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.*

Proof. By the unitary trick we can find a G-invariant Hermitian inner product $(-,-)$ and choose an orthonormal basis for $\mathbb{C}^n$ with respect to $(-,-)$ using Gram-Schmidt, say.

Let P be the change of basis matrix from the standard basis to the $(-,-)$-orthonormal basis. Then $(Pa,Pb)=\langle a,b\rangle$ for $a,b\in V$. So, for each $g\in G$,

$$\langle P^{-1}gPa, P^{-1}gPb\rangle = (gPa,gPb)=(Pa,Pb)=\langle a,b\rangle.$$

Thus $P^{-1}gP\in U(n)$ for each $g\in G$ as required. □

Thus studying all complex representations of a finite group G is equivalent to studying unitary (ie distance preserving) ones.

We now adapt our proof of complete reducibility to handle any field of characteristic k, even if there is no notion of inner product.

Theorem (Maschke's Theorem)**.** *Let G be a finite group and V a representation of G over a field k of characteristic zero. Then every subrep W of V has a G-invariant complement.*

Proof. Idea: if $\pi\colon V\to V$ is a projection i.e. $\pi^2=\pi$ then $V=\operatorname{Im}\pi\oplus\ker\pi$ as vector spaces. If π is G-linear then $\ker\pi$ and $\operatorname{Im}\pi$ are both G-invariant. So we pick a projection $V\to V$ with image W and average it.

Let $\pi\colon V\to V$ be any k-linear projection with $\pi(w)=w$ for all $w\in W$ and $\operatorname{Im}\pi=W$.

Recall that $\mathrm{Hom}_k(V,V)$ is a rep of G via $(g\varphi)(v) = g(\varphi(g^{-1}v))$. Let $\pi' \colon V \to V$ be defined by

$$\pi' := \frac{1}{|G|}\sum_{g\in G}(g\pi)$$

Then $\mathrm{Im}\,\pi' \leqslant W$ and $\pi'(w) = \frac{1}{|G|}\sum_{g\in G} g(\pi(g^{-1}w)) = w$ since $g(\pi(g^{-1}w)) = w$ for all $g \in G$ and $w \in W$.

Moreover for $h \in G$, $(h\pi') = \frac{1}{|G|}\sum_{g\in G}(hg)\pi = \frac{1}{G}\sum_{g'\in G} g'\pi = \pi'$.

Thus $\pi' \in \mathrm{Hom}_G(V,W)$ and π' is a G-invariant projection $V \to V$ with image W. So $\ker\pi'$ is the required G-invariant complement to W. □

Lecture 6

Remarks (on the Proof of Maschke's Theorem).

(1) We can explicitly compute π' and $\ker\pi'$ given (ρ, V) and W via the formula

$$\pi' = \frac{1}{|G|}\sum_{g\in G} g\cdot\pi.$$

(2) Notice that we only used that char $k = 0$ when we inverted $|G|$. So in fact we only need that the characteristic of k does not divide $|G|$.

(3) For any G-rep V (with char k not dividing $|G|$), the map

$$v \mapsto \frac{1}{|G|}\sum_{g\in G} g\cdot v$$

is a k-linear projection onto $V^G := \{v \in V \mid g\cdot v = v\}$. As a foreshadowing of what is coming soon, notice that

$$\dim V^G = \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}(g)$$

since tr is linear and for $\pi \colon V \to V$ any projection onto W, $\mathrm{tr}\,\pi = \dim W$.

3. Schur's Lemma

Recall that if V is a vector space of dimension d then $\mathrm{Aut}(V) \cong GL_d(k)$. This group parameterises the set of bases of V. The decompositions $V = \bigoplus_{i=1}^d V_i$ with each $\dim V_i = 1$ are parameterised by $GL_d(k)/T$ where T is the subgroup of $GL_d(k)$ consisting of diagonal matrices if we remember the order of the V_i; and by $GL_d(k)/N(T)$ where $N(T)$ is the subgroup of $GL_d(k)$ consisting of matrices with precisely one non-zero entry in each row and in each column if we only consider the decompositon up to permuting the factors.[5]

Theorem (Schur's Lemma)**.** *Suppose that V and W are irreducible reps of G over k. Then*

(i) every element of $\mathrm{Hom}_G(V, W)$ is either 0 or an isomorphism,
(ii) if k is algebraically closed then $\dim_k \mathrm{Hom}_G(V, W)$ is either 0 or 1 .

In other words irreducible representations are rigid in the same sense that one-dimensional vector spaces are rigid.

Proof. (i) Let φ be a non-zero G-linear map from V to W. Then $\ker\varphi \lneq V$ is a G-invariant subspace of V. So as V is simple, $\ker\varphi = 0$. Similarly $0 \neq \mathrm{Im}\,\varphi \leqslant W$ so $\mathrm{Im}\,\varphi = W$ since W is simple. Thus φ is both injective and surjective, so an isomorphism.

(ii) Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$ are non-zero. Then by (i) they are both isomorphisms. Consider $\varphi = \varphi_1^{-1}\varphi_2 \in \mathrm{Hom}_G(V, V)$. Since k is algebraically closed we may find λ an eigenvalue of φ then $\varphi - \lambda\,\mathrm{id}_V$ has non-zero (and G-invariant) kernel and so the map is zero. Thus $\varphi_1^{-1}\varphi_2 = \lambda\,\mathrm{id}_V$ and $\varphi_2 = \lambda\varphi_1$ as required. □

Proposition. *If V, V_1 and V_2 are k-representations of G then*

$$\mathrm{Hom}_G(V, V_1 \oplus V_2) \cong \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$$

and

$$\mathrm{Hom}_G(V_1, \oplus V_2, V) \cong \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V).$$

Proof. There are natural inclusion maps $\mathrm{Hom}_k(V, V_i) \to \mathrm{Hom}_k(V, V_1 \oplus V_2)$ for $i = 1, 2$ given by postcomposition with the natural map $V_i \to V_1 \oplus V_2$. These induce a linear isomorphism

$$\mathrm{Hom}_k(V, V_1) \oplus \mathrm{Hom}_k(V, V_2) \to \mathrm{Hom}_k(V, V_1 \oplus V_2)$$

given by $(f_1, f_2) \mapsto f_1 + f_2$. This is an intertwining map i.e. $g \cdot (f_1, f_2) = g \cdot f_1 + g \cdot f_2$. Since in general, $\mathrm{Hom}^G(U, W)$ consists of the G-fixed points of $\mathrm{Hom}_k(U, W)$, it follows that there is an induced map

$$\mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2) \to \mathrm{Hom}_G(V, V_1 \oplus V_2)$$

that is an isomorphism.

Similarly there is a G-linear isomorphism

$$\mathrm{Hom}_k(V_1 \oplus V_2, V) \to \mathrm{Hom}_k(V_1, V) \oplus \mathrm{Hom}_k(V_2, V)$$

given by $f \mapsto (f|_{V_1}, f|_{V_2})$ and again it follows that there is an induced map

$$\mathrm{Hom}_G(V_1 \oplus V_2, V) \to \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V)$$

that is an isomorphism. □

[5]This is also the normaliser of T in $GL_d(k)$.

Corollary. *If $V \cong \bigoplus_{i=1}^r V_i$ and $W \cong \bigoplus_{j=1}^s W_j$ then*

$$\operatorname{Hom}_G(V,W) \cong \bigoplus_{i=1}^r \bigoplus_{j=1}^s \operatorname{Hom}_G(V_i, W_j).$$

Proof. This follows from the Proposition by a straighforward induction argument. □

Corollary. *Suppose k is algebraically closed and*

$$V \cong \bigoplus_{i=1}^r V_i$$

is a decomposition of a k-rep. of G into irreducible components.

Then for each irreducible representation W of G,

$$|\{i \mid V_i \cong W\}| = \dim \operatorname{Hom}_G(W,V).$$

Proof. By the last result

$$\operatorname{Hom}_G(W,V) = \bigoplus_{i=1}^r \operatorname{Hom}_G(W,V_i)$$

and so

$$\dim \operatorname{Hom}_G(W,V) = \sum_{i=1}^r \dim \operatorname{Hom}_G(W,V_i).$$

Thus is suffices to show that

$$\dim \operatorname{Hom}_G(W,V_i) = \begin{cases} 1 & \text{if } W \cong V_i \\ 0 & \text{if } W \not\cong V_i \end{cases}$$

and this is precisely the statement of Schur's Lemma. □

Important question: How can we compute these numbers $\dim \operatorname{Hom}_G(V,W)$?

Corollary. *(of Schur's Lemma) If a finite group G has a faithful complex irreducible representation then the centre of G, $Z(G)$ is cyclic.*

Proof. Let V be a faithful complex irreducible rep of G, and let $z \in Z(G)$. Then let $\varphi_z \colon V \to V$ be defined by $\varphi_z(v) = zv$. Since $gz = zg$ for all $g \in G$, $\varphi_z \in \operatorname{Hom}_G(V,V) = \mathbb{C}\,\mathrm{id}_V$ by Schur, $\varphi_z = \lambda_z \mathrm{id}_V$, say.

Now $Z(G) \to \mathbb{C}; z \mapsto \lambda_z$ is a representation of $Z(G)$ that must be faithful since V is faithful. In particular $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$. But every such subgroup is cyclic. □

Corollary. *(of Schur's Lemma) Every irreducible complex representation of a finite abelian group G is one-dimensional.*

Proof. Let (ρ, V) be a complex irred. rep of G. For each $g \in G$, $\rho(g) \in \operatorname{Hom}_G(V,V)$. So by Schur, $\rho(g) = \lambda_g \mathrm{id}_V$ for some $\lambda_g \in \mathbb{C}$. Thus for $v \in V$ non-zero, $\langle v \rangle$ is a subrep of V. □

Examples. We can list all the irreducible. representations of C_4 and $C_2 \times C_2$

$G = C_4 = \langle x \rangle$.

	1	x	x^2	x^3
ρ_1	1	1	1	1
ρ_2	1	i	-1	$-i$
ρ_3	1	-1	1	1
ρ_4	1	$-i$	-1	i

$G = C_2 \times C_2 = \langle x, y \rangle$.

	1	x	y	xy
ρ_1	1	1	1	1
ρ_2	1	-1	1	-1
ρ_3	1	1	-1	-1
ρ_4	1	-1	-1	1

Lecture 7

Proposition. *Every finite abelian group G has precisely $|G|$ complex irreducible representations.*

Proof. Let ρ be an irred. complex rep of G. By the last corollary, $\dim \rho = 1$. So $\rho \colon G \to \mathbb{C}^\times$ is a group homomorphism.

Since G is a finite abelian group $G \cong C_{n_1} \times \cdots \times C_{n_k}$ some $n_1, \ldots, n_k$. Now if $G = G_1 \times G_2$ is the direct product of two groups then there is a 1-1 correspondance between the set of group homomorphisms $G \to \mathbb{C}^\times$ and the of pairs $(G_1 \to \mathbb{C}^\times, G_2 \to \mathbb{C}^\times)$ given by restriction $\varphi \mapsto (\varphi|_{G_1}, \varphi|_{G_2})$. Thus we may reduce to the case $G = C_n = \langle x \rangle$ is cyclic.

Now ρ is determined by $\rho(x)$ and $\rho(x)^n = 1$ so $\rho(x)$ must be an nth root of unity. Moreover we may choose $\rho(x)$ however we like amongst the nth roots of 1. □

Lemma. *If (ρ_1, V_1) and (ρ_2, V_2) are non-isomorphic one-dimensional representations of a finite group G then $\sum_{g \in G} \overline{\rho_1(g)} \rho_2(g) = 0$*

Proof. We've seen that $\mathrm{Hom}_k(V_1, V_2)$ is a G-rep under $g\varphi(v) = \rho_2(g)\varphi\rho_1(g^{-1})$ and $\sum_{g \in G} g\varphi \in \mathrm{Hom}_G(V_1, V_2) = 0$ by Schur. Since $\rho_1(g)$ is always a root of unity, $\rho_1(g^{-1}) = \overline{\rho_1(g)}$. Pick an isomorphism $\varphi \in \mathrm{Hom}_k(V_1, V_2)$. Then $0 = \sum_{g \in G} \rho_2(g)\varphi\rho_1(g^{-1}) = \sum_{g \in G} \overline{\rho_1(g)}\rho_2(g)\varphi$ as required. □

If V is a representation of a group G that is completely reducible and W is any irreducible representation of G then the *W-isotypic component of V* is the smallest subrepresentation of V containing all simple subrepresentations isomorphic to W. This exists since if $(V_i)_{i \in I}$ are subrepresentations of V containing all simple subrepresentations isomorphic to W then so is $\bigcap_{i \in I} V_i$.[6]

We say that V has a *unique isotypical decomposition* if V is the direct sum of its W-isotypic components as W varies over all simple representations of V (up to isomorphism).

Corollary. *Suppose G is a finite abelian group then every complex representation V of G has a unique isotypical decomposition.*

Proof. For each homomorphism $\theta_i \colon G \to \mathbb{C}^\times$ $(i = 1, \ldots, |G|)$ we can define W_i to be the subspace of V defined by

$$W_i = \{v \in V \mid \rho(g)v = \theta_i(g)v \text{ for all } g \in G\}.$$

Since V is completely reducible and every irreducible rep of G is one dimensional $V = \sum W_i$. We need to show that for each i $W_i \cap \sum_{j \neq i} W_j = 0$. It is equivalent to show that $\sum w_i = 0$ with $w_i \in W_i$ implies $w_i = 0$ for all i.

[6]It can also be realised as the vector space sum of all subrepresentations isomorphic to W.

But $\sum w_i = 0$ with w_i in W_i certainly implies $0 = \rho(g) \sum w_i = \sum \theta_i(g) w_i$. By choosing an ordering $g_1, \dots, g_{|G|}$ of G we see that the $|G| \times |G|$ matrix $\theta_i(g_j)$ is invertible by the lemma. Thus $w_i = 0$ for all i as required. $\square$

You will extend this result to all finite groups on Example Sheet 2.

4. Characters

Summary so far. We want to classify all representations of groups G. We've seen that if G is finite and k has characteristic zero then every representation V decomposes as $V \cong \bigoplus n_i V_i$ with V_i irreducible and $n_i \geqslant 0$. Moreover if k is also algebraically closed, we've seen that $n_i = \dim \mathrm{Hom}_G(V_i, V)$.

Our next goals are to classify all irreducible representations of a finite group and understand how to compute the n_i given V. We're going to do this using character theory.

4.1. **Definitions.**

Definition. Given a representation $\rho\colon G \to GL(V)$, the *character* of ρ is the function $\chi = \chi_\rho = \chi_V\colon G \to k$ given by $g \mapsto \operatorname{tr} \rho(g)$.

Since for matrices $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, the character does not depend on the choice of basis for V [$\operatorname{tr}(X^{-1}AX) = \operatorname{tr}(AXX^{-1}) = \operatorname{tr}(A)$]. By the same argument we also see that equivalent reps have the same character.

Example. Let $G = D_6 = \langle s, t \mid s^2 = 1, t^3 = 1, sts^{-1} = t^{-1} \rangle$, the dihedral group of order 6. This acts on $\mathbb{R}^2$ by symmetries of the triangle; with t acting by rotation by $2\pi/3$ and s acting by a reflection. To compute the character of this rep we just need to know the eigenvalues of the action of each element. Each reflection (element of the form st^i) will act by a matrix with eigenvalues ± 1. Thus $\chi(st^i) = 0$ for all i. The rotations t^r act by matrices $\begin{pmatrix} \cos 2\pi r/3 & -\sin 2\pi r/3 \\ \sin 2\pi r/3 & \cos 2\pi r/3 \end{pmatrix}$ thus $\chi(t^r) = 2\cos 2\pi r/3 = -1$ for $r = 1, 2$.

Proposition. *Let (ρ, V) be a rep of G with character χ*

(i) $\chi(e) = \dim V$;
(ii) $\chi(g) = \chi(hgh^{-1})$ for all $g, h \in G$;
(iii) If χ' is the character of (ρ', V') then $\chi + \chi'$ is the character of $V \oplus V'$.
(iv) If $k = \mathbb{C}$ and $o(g) < \infty$, $\chi(g^{-1}) = \overline{\chi(g)}$;

Proof.
(i) $\chi(e) = \operatorname{tr} \mathrm{id}_V = \dim V$.
(ii) $\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1}$. Thus $\rho(hgh^{-1})$ and $\rho(g)$ are conjugate and so have the same trace.
(iii) is clear.
(iv) if $\rho(g)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ (with multiplicity) then $\chi(g) = \sum \lambda_i$. But as $o(g)$ is finite each λ_i must be a root of unity. Thus $\overline{\chi(g)} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1}$ but, of course, the λ_i^{-1} are the eigenvalues of g^{-1}. $\square$

The proposition tells us that the character of ρ contains very little data; an element of k for each conjugacy class in G. The extraordinary thing that we will see is that, at least when G is finite and $k = \mathbb{C}$, it contains all we need to know to reconstruct ρ up to isomorphism.

Definition. We say a function $f\colon G \to \mathbb{C}$ is a *class function* if $f(hgh^{-1}) = f(g)$ for all $g, h \in G$. We'll write $\mathcal{C}_G$ for the complex vector space of class functions on G.

Notice that if $\mathcal{O}_1, \ldots, \mathcal{O}_r$ is a list of the conjugacy classes of G then the indicator functions $\mathbf{1}_{\mathcal{O}_i} \colon G \to \mathbb{C}$ given by $g \mapsto 1$ if $g \in \mathcal{O}_i$ and $g \mapsto 0$ otherwise form a basis for $\mathcal{C}_G$. In particular $\dim \mathcal{C}_G$ is the number of conjugacy classes in G.

Lecture 8

4.2. **Orthogonality of characters.** We'll now assume that G is a finite group and $k = \mathbb{C}$ unless we say otherwise.

We can make $\mathcal{C}_G$, the space of class functions, into a Hermitian inner product space by defining

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum \overline{f_1(g)} f_2(g).$$

It is easy to check that this does define an Hermitian inner product[7] and that the functions $\delta_{\mathcal{O}_i}$ are pairwise orthogonal. Notice that $\langle \delta_{\mathcal{O}_i}, \delta_{\mathcal{O}_i} \rangle = \frac{|\mathcal{O}_i|}{|G|} = \frac{1}{|C_G(x_i)|}$ for any $x_i \in \mathcal{O}_i$.

Thus if $x_1, \ldots, x_r$ are conjugacy class representatives, then we can write

$$\langle f_1, f_2 \rangle = \sum_{i=1}^{r} \frac{1}{|C_G(x_i)|} \overline{f_1(x_i)} f_2(x_i).$$

Example. $G = D_6 = \langle s, t \mid s^2 = t^3 = e, sts = t^{-1} \rangle$ has conjugacy classes $\{e\}, \{t, t^{-1}\}, \{s, st, st^2\}$ and

$$\langle f_1, f_2 \rangle = \frac{1}{6} \overline{f_1(e)} f_2(e) + \frac{1}{2} \overline{f_1(s)} f_2(s) + \frac{1}{3} \overline{f_1(t)} f_2(t).$$

Theorem (Orthogonality of characters)**.** *If V and V' are complex irreducible representations of a finite group G then $\langle \chi_V, \chi_{V'} \rangle$ is 1 if $V \cong V'$ and 0 otherwise.*

This should remind you of Schur's Lemma and in fact the similarity is no coincidence. It is a corollary of Schur. Before we prove it we need a couple of lemmas.

Lemma. *If V and W are reps of a finite group G then*

$$\chi_{\mathrm{Hom}_k(V,W)}(g) = \overline{\chi_V(g)} \chi_W(g)$$

for each $g \in G$.

Proof. Given $g \in G$ we may choose bases $v_1, \ldots, v_n$ for V and $w_1, \ldots, w_m$ for W such that $gv_i = \lambda_i v_i$ and $gw_j = \mu_j w_j$. Then the functions $\alpha_{ij}(v_k) = \partial_{jk} w_i$ extend to linear maps that form a basis for $\mathrm{Hom}_k(V, W)$[8] and

$$(g \cdot \alpha_{ij})(v_k) = g \cdot (\alpha_{ij}(g^{-1} \cdot v_k)) = \delta_{jk} \lambda_k^{-1} \mu_i w_i$$

thus $g \cdot \alpha_{ij} = \lambda_j^{-1} \mu_i \alpha_{ij}$ and

$$\chi_{\mathrm{Hom}(V,W)}(g) = \sum_{i,j} \lambda_j^{-1} \mu_i = \chi_V(g^{-1}) \chi_W(g) = \overline{\chi_V(g)} \chi_W(g)$$

as claimed. □

[7] In fact it even defines an inner product on $\mathbb{C}G$ with pairwise orthogonal basis $\langle \delta_g \mid g \in G \rangle$ and $\mathcal{C}_G$ is a subspace.

[8] α_{ij} is represented by the matrix with a 1 in entry ij and 0s elsewhere with respect to the given bases

Lemma. *If U is a rep of G then*

$$\dim U^G = \dim\{u \in U \mid gu = u\ \forall g \in G\} = \langle 1, \chi_U\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_U(g).$$

Proof. Define $\pi\colon U \to U$ by $\pi(u) = \frac{1}{|G|}\sum_{g\in G} gu$. Then $\pi(u) \in U^G$ for all $u \in U$. Moreover $\pi_{U^G} = \mathrm{id}_{U^G}$ by direct calculation. Thus

$$\dim U^G = \operatorname{tr}\mathrm{id}_{U^G} = \operatorname{tr}\pi = \frac{1}{|G|}\sum_{g\in G}\chi_U(g)$$

as required. □

We can use these two lemmas to prove the following.

Proposition. *If V and W are representations of G then*

$$\dim \operatorname{Hom}_G(V, W) = \langle \chi_V, \chi_W\rangle.$$

Proof. By the lemmas $\dim\operatorname{Hom}_G(V,W) = \langle \mathbf{1}, \overline{\chi_V}\chi_W\rangle$. But it is easy to compute that $\langle \mathbf{1}, \overline{\chi_V}\chi_W\rangle = \langle\chi_V, \chi_W\rangle$ as required. □

Corollary (Orthogonality of characters)**.** *If χ, χ' are characters of irreducible reps then $\langle\chi, \chi'\rangle = \delta_{\chi,\chi'}$.*

Proof. Apply the Proposition and Schur's Lemma.[9] □

Notice that this tells us that the characters of irreducible reps form part of an orthonormal basis for $\mathcal{C}_G$. In particular the number of irreducible representations is bounded above by the number of conjugacy classes of G. In fact we'll see that the characters span the space of class functions and so that the number of irreps is precisely the number of conjugacy classes in G. We saw this when G is abelian last time.

Suppose now that $V_1, \dots, V_k$ is the list of all irreducible complex reps of G up to isomorphism and the corresponding characters are $\chi_1, \dots, \chi_k$. Our main goal at this point is to investigate how we might produce such a record of the irreducible characters. This is because the following result is true.

Corollary. *If ρ and ρ' are reps of G, then they are isomorphic if and only if they have the same character.*

Proof. We have already seen that isomorphic reps have the same character.

Suppose (ρ, V) decomposes as $\bigoplus_{i=1}^k n_iV_i$ and (ρ', V') decomposes as $\bigoplus_{i=1}^k m_iV_i$ where $m_i, n_i \geqslant 0$ for all i — Maschke's Theorem tells us that such decompositions exist. It suffices to establish that if

$$\sum n_i\chi_i = \chi_\rho = \chi_{\rho'} = \sum m_i\chi_i$$

then $n_i = m_i$ for all $i = 1, \dots, k$; that is $\chi_1, \dots, \chi_k$ are $\mathbb{Z}$-linearly independent in $\mathcal{C}_G$. In fact, they are even $\mathbb{C}$-linearly independent.

Indeed we've seen that $n_i = \dim\operatorname{Hom}_G(V_i, V) = \langle\chi_i, \chi_\rho\rangle$. Thus if $\chi_{\rho'} = \chi_\rho$ then for each $i = 1, \dots, k$, $n_i = \langle\chi_i, \chi_\rho\rangle = m_i$ as required. □

[9] Note that if $\chi_V = \chi_{V'}$, with V and V' irreducible, then $\dim\operatorname{Hom}_G(V, V') = \langle\chi_V, \chi_V\rangle > 0$ and so $V \cong V'$.

We see in the proof that the multiplicity of the factors of a complete decomposition of V can be computed purely from its character.

Notice that complete irreducibility was a key part of the proof of this corollary, as well as orthogonality of characters. For example the two reps of $\mathbb{Z}$ given by $1 \mapsto id_{\mathbb{C}^2}$ and $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are not isomorphic but have the same trace. Indeed they both have trivial subrepresentations with trivial quotient. Complete irreducibility tells us we don't need to worry about gluing.

Corollary. *If ρ is a complex representation of G with character χ then ρ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Proof. One direction follows immediately from the theorem on orthogonality of characters. For the other direction, assume that $\langle \chi, \chi \rangle = 1$. Then we may write $\chi = \sum n_i \chi_i$ for some non-negative integers n_i. By orthogonality of characters $1 = \langle \chi, \chi \rangle = \sum n_i^2$. Thus $\chi = \chi_j$ for some j, and χ is irreducible. □

This is a good way of calcuating whether a representation is irreducible.

Example.
Consider the action of S_3 on $\mathbb{C}^2$ by extending the symmetries of a triangle. $\chi(1) = 2$, $\chi(12) = \chi(23) = \chi(13) = 0$, and $\chi(123) = \chi(132) = -1$. Now

$$\langle \chi, \chi \rangle = \frac{1}{6}(2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1$$

so this rep is irreducible. Of course we had already established this by hand in (an exercise in) Lecture 3.

Lecture 9

Theorem (The character table is square)**.** *The irreducible characters of a finite group G form a orthonormal basis for the space of class functions $\mathcal{C}_G$ with respect to $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$.*

Proof. We already know that the irreducible characters form an orthonormal set. So it remains to show that they span $\mathcal{C}_G$.

Let $I = \langle \chi_1, \dots, \chi_k \rangle$ be the span of the irreducible characters. We need to show that $I^\perp = 0$.

Suppose $f \in \mathcal{C}_G$. For each representation (ρ, V) of G we may define $\varphi = \varphi_{f,V} \in \mathrm{Hom}_k(V, V)$ by $\varphi = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \rho(g)$.

Now,

$$\rho(h)^{-1} \varphi \rho(h) = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \rho(h^{-1} g h) = \frac{1}{|G|} \sum_{g' \in G} \overline{f(g')} \rho(g')$$

since f is a class function, and we see that in fact $\varphi_{f,V} \in \mathrm{Hom}_G(V, V)$.

Moreover, if $f \in I^\perp$ and V is an irreducible representation then $\varphi_{f,V} = \lambda \, \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$ by Schur's Lemma and

$$\lambda \dim V = \mathrm{tr}\, \varphi_{f,V} = \langle f, \chi_V \rangle = 0$$

so $\varphi_{f,V} = 0$.

But every representation breaks up as a direct sum of irreducible representations $V = \bigoplus V_i$ and $\varphi_{f,V}$ breaks up as $\bigoplus \varphi_{f,V_i}$. So $\varphi_{f,V} = 0$ whenever $f \in I^\perp$.

But if we take V to be the regular representation $\mathbb{C}G$ then

$$\varphi_{f,\mathbb{C}G}\partial_e = |G|^{-1}\sum_{g\in G}\overline{f(g)}\partial_g = |G|^{-1}\overline{f}.$$

Thus $f = 0$. $\square$

Corollary. *The number of irreducible representations is the number of conjugacy classes in the group.*

Corollary. *For each $g \in G$, $\chi(g)$ is real for every character χ if and only if g is conjugate to g^{-1}.*

Proof. Since $\chi(g^{-1}) = \overline{\chi(g)}$, $\chi(g)$ is real for every character χ if and only if $\chi(g) = \chi(g^{-1})$ for every character χ. Since the irreducible characters span the space of class functions and $\mathbf{1}_{\mathcal{O}_i}$ is a class function for each $i = 1, \ldots, r$, this is equivalent to g and g^{-1} living in the same conjugacy class. $\square$

4.3. **Character tables.** We now want to classify all the irreducible representations of a given finite group and we know that it suffices to write down the characters of each one.

The character table of a group is defined as follows: we list the conjugacy classes of G, $\mathcal{O}_1, \ldots, \mathcal{O}_k$ (by convention always $\mathcal{O}_1 = \{e\}$) and choose $g_i \in \mathcal{O}_i$ we then list the irreducible characters $\chi_1, \ldots, \chi_k$ (by convention $\chi_1 = \chi_{\mathbb{C}}$ the character of the trivial rep. Then we write the matrix

	e	g_2	$\cdots$	g_i	$\cdots$	g_k
χ_1	1	1	$\cdots$	1	$\cdots$	1
$\vdots$				$\vdots$		
χ_j	$\cdots$	$\cdots$	$\cdots$	$\chi_j(g_i)$	$\cdots$	$\cdots$
$\vdots$				$\vdots$		
χ_k				$\vdots$		

We sometimes write the size of the conjugacy class $\mathcal{O}_i$ above g_i and sometimes the equivalent data $|C_G(g_i)|$.

Examples.

(1) $C_3 = \langle x\rangle$ and let $\omega = e^{\frac{2\pi i}{3}}$ so $\omega^2 = \overline{\omega}$.

	e	x	x^2
χ_1	1	1	1
χ_2	1	ω	$\overline{\omega}$
χ_3	1	$\overline{\omega}$	ω

Notice that the rows are indeed pairwise orthogonal. The columns are too in this case.

(2) S_3

There are three conjugacy classes: $\mathcal{O}_1 = \{e\}$; $\mathcal{O}_2 = \{(12), (23), (13)\}$; and $\mathcal{O}_3 = \{(123), (132)\}$. Thus there are also three irreducible representations. We know that the trivial representation $\mathbf{1}$ has character $\mathbf{1}(g) = 1$ for all $g \in G$. We also know another 1-dimensional representation $\epsilon\colon S_3 \to \{\pm 1\}$ given by $g \mapsto 1$ if g is even and $g \mapsto -1$ if g is odd.

To compute the character χ of the last representation we may use orthogonality of characters. Let $\chi(e) = a$, $\chi((12)) = b$ and $\chi((123)) = c$ (a, b and c are each real since each g is conjugate to its inverse). We know that

$$\begin{aligned} 0 = \langle \mathbf{1}, \chi \rangle &= \frac{1}{6}(a + 3b + 2c), \\ 0 = \langle \epsilon, \chi \rangle &= \frac{1}{6}(a - 3b + 2c) \text{ and} \\ 1 = \langle \chi, \chi \rangle &= \frac{1}{6}(a^2 + 3b^2 + 2c^2). \end{aligned}$$

Thus we see quickly that $b = 0$, $a + 2c = 0$ and $a^2 + 2c^2 = 6$. We also know that a is a positive integer. Thus $a = 2$ and $c = -1$.

$\lvert\mathcal{O}_i\rvert$	1	3	2
	e	(12)	(123)
$\mathbf{1}$	1	1	1
ϵ	1	-1	1
χ	2	0	-1

In fact we already knew about this 2-dimensional representation; it is the one coming from the symmetries of a triangle inside $\mathbb{R}^2$.

The rows are orthogonal under $\langle f_1, f_2 \rangle = \sum_1^3 \frac{1}{|C_G(g_i)|} \overline{f_1(g_i)} f_2(g_i)$.

But the columns are also orthogonal with respect to the standard inner product. If we compute their length we get:

$$\begin{aligned} 1^2 + 1^2 + 2^2 &= 6 = |S_3| \\ 1^2 + (-1)^2 + 0^2 &= 2 = |C_{S_3}((12))| \\ 1^2 + 1^2 + (-1)^2 &= 3 = |C_{S_3}((123))|. \end{aligned}$$

This is an instance of a more general phenomenon.

Proposition (Column Orthogonality)**.** *If G is a finite group and $\chi_1, \ldots, \chi_r$ is a complete list of the irreducible characters of G then for each $g, h \in G$,*

$$\sum_{i=1}^r \overline{\chi_i(g)} \chi_i(h) = \begin{cases} 0 & \text{if } g \text{ and } h \text{ are not conjugate in } G \\ |C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate in } G. \end{cases}$$

In particular

$$\sum_{i=1}^r (\dim V_i)^2 = \sum_{i=1}^r \chi_i(e)^2 = |G|.$$

Proof. Let X be character table thought of as a matrix; $X_{ij} = \chi_i(g_j)$ and let D be the diagonal matrix whose diagonal entries are $|C_G(g_i)|$

Orthogonality of characters tell us that

$$\sum_k |C_G(g_k)|^{-1} \overline{X_{ik}} X_{jk} = \partial_{ij}$$

ie $\overline{X} D^{-1} X^T = I$.

Since X is square we may write this as $D^{-1} \overline{X}^T = X^{-1}$. Thus $\overline{X}^T X = D$. That is

$$\sum_k \overline{\chi_k(g_i)} \chi_k(g_j) = \partial_{ij} |C_G(g_i)|$$

as required. □

Examples.

(1) $G = S_4$: the character table is as follows

$\lvert C_G(x_i)\rvert$	24	8	3	4	4
$\lvert[x_i]\rvert$	1	3	8	6	6
	e	$(12)(34)$	(123)	(12)	(1234)
$\mathbf{1}$	1	1	1	1	1
ϵ	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
$\epsilon\chi_3$	3	-1	0	-1	1
χ_5	2	2	-1	0	0

Proof. The trivial $\mathbf{1}$ and sign ϵ characters may be constructed in the same way as for S_3.

Consider the action of S_4 on $\mathbb{C}X$ for $X = \{1, 2, 3, 4\}$ induced from the natural action of S_4 on X.

We can compute that the character of this rep is given by

$$\chi_{\mathbb{C}X}(g) = |\{\text{fixed points of } g\}|.^{10}$$

So $\chi(1) = 4$, $\chi((12)) = 2$, $\chi((123)) = 1$ and $\chi((12)(34)) = \chi((1234)) = 0$. Thus

$$\langle \chi, \chi \rangle = \frac{4^2}{24} + \frac{0^2}{8} + \frac{1^2}{3} + \frac{2^2}{4} + \frac{0^2}{4} = 2.$$

Lecture 10

(Examples continued)

So if we decompose $\chi = \sum n_i\chi_i$ into irreducibles we know $\sum n_i^2 = 2$ then we must have $\chi = \chi' + \chi''$ with χ' and χ'' non-isomorphic irreducible characters.

Notice that

$$\langle \mathbf{1}, \chi \rangle = \frac{4}{24} + \frac{0}{8} + \frac{1}{3} + \frac{2}{4} + \frac{0}{4} = 1$$

so one of the irreducible constituents is the trivial character The other has character $\chi - \mathbf{1}$.

In fact we have seen this decomposition of $\mathbb{C}X$ explicitly. The constant functions gives a trivial subrep and the orthogonal complement with respect to the standard inner product (that is the set of functions that sum to zero) gives the other subrep.

We saw on Example Sheet 1 (Q2) that given a 1-dimensional represntation θ and an irreducible representation ρ we may form another irreducible representation $\theta\otimes\rho$ by $\theta\otimes\rho(g) = \theta(g)\rho(g)$. It is not hard to see that $\chi_{\theta\otimes\rho}(g) = \theta(g)\chi_\rho(g)$. Thus we get another irreducible character $\epsilon\chi_3$.

We can then complete the character table using column orthogonality: We note that $24 = 1^2 + 1^2 + 3^2 + 3^2 + \chi_5(e)^2$ thus $\chi_5(e) = 2$. Then using $\sum_1^5 \chi_i(1)\chi_i(g) = 0$ we can construct the remaining values in the table. □

Notice that the two dimensional representation corresponding to χ_5 may be obtained by composing the surjective group homomorphism $S_4 \to S_3$ (with kernel the Klein-4-group) with the irreducible two dimension rep of S_3.

[10] To see this write down the matrices. Or wait to see a more general version of this statement next time.

(2) $G = A_4$. Each irreducible representation of S_4 may be restricted to A_4 and its character values on elements of A_4 will be unchanged. In this way we get three characters of A_4: $\mathbf{1}$, $\psi_2 = \chi_3|_{A_4}$ and $\psi_3 = \chi_5|_{A_4}$. Of course $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. Similarly,

$$\langle \psi_2, \psi_2 \rangle = \frac{1}{12}(3^2 + 3(-1)^2 + 8(0^2)) = 1$$

so ψ_2 also remains irreducible. However

$$\langle \psi_3, \psi_3 \rangle = \frac{1}{12}(2^2 + 3(2^2) + 8(-1)^2) = 2$$

so ψ_3 breaks up into two non-isomorphic irreducible reps of A_4.

Exercise. Use this information to construct the whole character table of A_4.

4.4. **Permuation representations.** Recall that if X is a finite set with G-action then $\mathbb{C}X = \{f \colon X \to \mathbb{C}\}$ is a representation of G via $gf(x) = f(g^{-1}x)$.

Lemma. *If χ is the character of $\mathbb{C}X$ then $\chi(g) = |\{x \in X \mid gx = x\}|$.*

Proof. If $X = \{x_1, \dots, x_d\}$ and $gx_i = x_j$ then $g\partial_{x_i} = \partial_{x_j}$ so the ith column of g has a 1 in the jth entry and zeros elsewhere. So it contributes 1 to the trace precisely if $x_i = x_j$. □

Corollary. *If $V_1, \dots, V_k$ is a complete list of irreducible reps of a finite group G then the regular representation decomposes as*

$$\mathbb{C}G \cong n_1 V_1 \oplus \cdots \oplus n_k V_k$$

with $n_i = \dim V_i = \chi_i(e) > 0$.

In particular every irreducible representation is isomorphic to a subrepresentation of the regular representation and

$$|G| = \sum (\dim V_i)^2.$$

Proof. $\chi_{\mathbb{C}G}(e) = |G|$ and $\chi_{kG}(g) = 0$ for $g \neq e$. Thus if we decompose kG we obtain

$$n_i = \langle \chi_{\mathbb{C}G}, \chi_i \rangle = \frac{1}{|G|}|G|\chi_i(e) = \chi_i(e)$$

as required. □

Proposition (Burnside's Lemma)**.** *Let G be a finite group and X a finite set with a G-action and χ the character of $\mathbb{C}X$. Then $\langle \mathbf{1}, \chi \rangle$ is the number of orbits of G on X.*

Proof. If we decompose X into a disjoint of orbits $X_1 \cup \cdots \cup X_k$ then we've seen that $\mathbb{C}X = \bigoplus_{i=1}^k \mathbb{C}X_i$. So $\chi_X = \sum_{i=1}^k \chi_{X_i}$ and we may reduce to the case that G-acts transitively on X.

Now

$$\begin{aligned}|G|\langle\chi_X,1\rangle &= \sum_{g\in G}\chi_X(g) = \sum_{g\in G}|\{x\in X \mid gx=x\}\\ &= |\{(g,x)\in G\times X \mid gx=x\}| = \sum_{x\in X}|\{g\in G \mid gx=x\}\\ &= \sum_{x\in X}|\operatorname{Stab}_G(x)|\\ &= |X|\left(\frac{|G|}{|X|}\right) \text{ (by the Orbit-Stabiliser Theorem)}\\ &= |G|\end{aligned}$$

as required. □

If X is a set with a G-action we may view $X\times Y$ as a set with a G-action via $(g,(x,y))\mapsto(gx,gy)$.

Corollary. *If G is a finite group and X is a finite set with a G-action and χ is the character of the permutation representation $\mathbb{C}X$ then $\langle\chi,\chi\rangle$ is the number of G-orbits on $X\times X$.*

Proof. Notice that (x,y) is fixed by $g\in G$ if and only if both x and y are fixed. Thus $\chi_{X\times X}(g)=\chi_X(g)\chi_X(g)$ by the Lemma.

Now $\langle\chi_X,\chi_X\rangle=\frac{1}{|G|}\sum_{g\in G}\chi_X(g)\chi_X(g)=\langle\mathbf{1},\chi_{X\times X}\rangle$ and the result follows from Burnside's Lemma. □

Remark. If X is any set with a G-action with $|X|>1$ then $\{(x,x)|x\in X\}\subset X\times X$ is G-stable and so is the complement $\{(x,y)\in X\times X \mid x\neq y\}$.

We say that G acts 2-*transitively* on X if G has only two orbits on $X\times X$. Given a 2-transitive action of G on X we've seen that the character χ of the permutation representation satisfies $\langle\chi,\chi\rangle=2$ and $\langle\mathbf{1},\chi\rangle=1$. Thus $\mathbb{C}X$ has two non-isomorphic irreducible summands — the constant functions and the functions f such that $\sum_{x\in X}f(x)=0$.

Exercise. If $G=GL_2(\mathbb{F}_p)$ then decompose the permutation rep of G coming from the action of G on $\mathbb{F}_p\cup\{\infty\}$ by Mobius transformations.

5. The character ring

Given a finite group G, the set of class functions $\mathcal{C}_G$ comes equipped with certain algebraic structures: it is a commutative ring under pointwise addition and multiplication — ie $(f_1+f_2)(g) = f_1(g)+f_2(g)$ and $f_1f_2(g) = f_1(g)f_2(g)$ for each $g \in G$, the additive identity is the constant function value 0 and the multiplicative identity constant value 1; there is a ring automorphism $*$ of order two given by $f^*(g) = f(g^{-1})$; and there is a non-degenerate bilinear form given by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|}\sum_{g\in G} f_1^*(g)f_2(g).$$

We will see that all this structure is related to structure on the category of representations: we have already seen some of this. If V_1 and V_2 are representations with characters χ_1 and χ_2 then $\chi_1+\chi_2 = \chi_{V_1\oplus V_2}$ and $\langle \chi_1, \chi_2\rangle = \dim \operatorname{Hom}_G(V_1, V_2)$.

Lecture 11

Definition. The *character ring* $R(G)$ of a group G is defined by

$$R(G) := \{\chi_1 - \chi_2 \mid \chi_1, \chi_2 \text{ are characters of reps of } G\} \subset \mathcal{C}_G.$$

We'll see that the character ring inherits all the algebraic structure of $\mathcal{C}_G$ mentioned above.

5.1. **Tensor products.** We've seen that $\chi_{\mathbb{C}X\times Y} = \chi_{\mathbb{C}X}\cdot\chi_{\mathbb{C}Y}$. We want to generalise this.

Suppose that V and W are vector spaces over a field k, with bases $v_1,\ldots,v_m$ and $w_1,\ldots,w_n$ respectively. We may view $V\oplus W$ either as the vector space with basis $v_1,\ldots,v_m,w_1,\ldots,w_n$ (so $\dim V\oplus W = \dim V + \dim W$) or more abstractly as the vector space of pairs (v,w) with $v\in V$ and $w \in W$ and pointwise operations.

Definition. The *tensor product* $V\otimes W$ of V and W is the k-vector space with basis given by symbols $v_i\otimes w_j$ for $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$ and so

$$\dim V\otimes W = \dim V\cdot \dim W.$$

Example. If X and Y are sets then $kX\otimes kY$ has basis $\partial_x\otimes\partial_y$ for $x\in X$ and $y\in Y$. Notice that $kX\otimes kY$ is isomorphic to $kX\times Y$ under $\partial_x\otimes\partial_y\mapsto \partial_{x,y}$.

Notation. *If $v = \sum \lambda_i v_i \in V$ and $w = \sum \mu_j w_j \in W$,*

$$v\otimes w\colon = \sum_{i,j}\lambda_i\mu_j(v_i\otimes w_j)\in V\otimes W.$$

Note that, in general, not every element of $V\otimes W$ may be written in the form $v\otimes w$ (eg $v_1\otimes w_1 + v_2\otimes w_2$). The smallest number of summands that are required is known as the *rank* of the tensor.

Lemma. *The map $V\times W\to V\otimes W$ given by $(v,w)\mapsto v\otimes w$ is bilinear.*

Proof. First, we should prove that if $x, x_1, x_2\in V$ and $y, y_1, y_2 \in W$ and $\nu_1,\nu_2\in k$ then

$$x\otimes(\nu_1y_1+\nu_2y_2) = \nu_1(x\otimes y_1)+\nu_2(x\otimes y_2)$$

and

$$(\nu_1x_1+\nu_2x_2)\otimes y = \nu_1(x_1\otimes y)+\nu_2(x_2\otimes y).$$

We'll just do the first; the second follows by symmetry.

Write $x = \sum_i \lambda_i v_i$, $y_k = \sum_j \mu_j^k w_j$ for $k = 1, 2$. Then

$$x \otimes (\nu_1 y_1 + \nu_2 y_2) = \sum_{i,j} \lambda_i(\nu_1 \mu_j^1 + \nu_2 \mu_j^2) v_i \otimes w_j$$

and

$$\nu_1(x \otimes y_1) + \nu_2(x \otimes y_2) = \nu_1 \left(\sum_{i,j} \lambda_i \mu_j^1 (v_i \otimes w_j) \right) + \nu_2 \left(\sum_{i,j} \lambda_i \mu_j^2 (v_i \otimes w_j) \right).$$

These are equal. □

Exercise. Show that given vector spaces U, V and W there is a $1-1$ correspondence

$$\{\text{linear maps } V \otimes W \to U\} \longrightarrow \{\text{bilinear maps } V \times W \to U\}$$

given by precomposition with the bilinear map $(v, w) \to v \otimes w$ above.

Lemma. *If $x_1, \ldots, x_m$ is any basis of V and $y_1, \ldots, y_n$ is any basis of W then $x_i \otimes y_j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ is a basis for $V \otimes W$. Thus the definition of $V \otimes W$ does not depend on the choice of bases.*

Proof. It suffices to prove that the set $\{x_i \otimes y_j\}$ spans $V \otimes W$ since it has size mn. But if $v_i = \sum_r A_{ri} x_r$ and $w_j = \sum_s B_{sj} y_s$ then $v_i \otimes w_j = \sum_{r,s} A_{ri} B_{sj} x_r \otimes y_s$. □

Remark (for enthusiastists). In fact we could have defined $V \otimes W$ in a basis independent way in the first place: let F be the (infinite dimensional) vector space with basis $\langle v \otimes w \mid v \in V, w \in W \rangle$; and R be the subspace generated by

$$x \otimes (\nu_1 y_1 + \nu_2 y_2) - \nu_1(x \otimes y_1) + \nu_2(x \otimes y_2)$$

and

$$(\nu_1 x_1 + \nu_2 x_2) \otimes y - \nu_1(x_1 \otimes y) + \nu_2(x_2 \otimes y)$$

for all $x, x_1, x_2 \in V$, $y, y_1, y_2 \in W$ and $\nu_1, \nu_2 \in k$; then $V \otimes W \cong F/R$ naturally.

Exercise. Show that for vector spaces U, V and W there is a natural (basis independent) isomorphism

$$(U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W).$$

Definition. Suppose that V and W are vector spaces with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ and $\varphi\colon V \to V$ and $\psi\colon W \to W$ are linear maps. We can define $\varphi \otimes \psi\colon V \otimes W \to V \otimes W$ as follows:

$$(\varphi \otimes \psi)(v_i \otimes w_j) = \varphi(v_i) \otimes \psi(w_j).$$

Example. If φ is represented by the matrix A_{ij} and ψ is represented by the matrix B_{ij} and we order the basis $v_i \otimes w_j$ lexicographically (ie $v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_1 \otimes w_n, v_2 \otimes w_1, \ldots, v_m \otimes w_n$) then $\varphi \otimes \psi$ is represented by the block matrix

$$\begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Lemma. *The linear map $\varphi \otimes \psi$ does not depend on the choice of bases.*

Proof. It suffices to show that for any $v \in V$ and $w \in W$,

$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w).$$

Writing $v = \sum \lambda_i v_i$ and $w = \sum \mu_j w_j$ we see

$$(\varphi \otimes \psi)(v \otimes w) = \sum_{i,j} \lambda_i \mu_j \varphi(v_i) \otimes \psi(w_j) = \varphi(v) \otimes \psi(w)$$

as required. □

Remark. The proof really just says $V \times W \to V \otimes W$ defined by $(v, w) \mapsto \varphi(v) \otimes \psi(w)$ is bilinear and $\varphi \otimes \psi$ is its correspondent in the bijection

$$\{\text{linear maps } V \otimes W \to V \otimes W\} \to \{\text{bilinear maps } V \times W \to V \otimes W\}$$

from earlier.

Lemma. *Suppose that $\varphi, \varphi_1, \varphi_2 \in \operatorname{Hom}_k(V, V)$ and $\psi, \psi_1, \psi_2 \in \operatorname{Hom}_k(W, W)$*

(i) $(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2) \in \operatorname{Hom}_k(V \otimes W, V \otimes W)$*;*
(ii) $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V\otimes W}$*; and*
(iii) $\operatorname{tr}(\varphi \otimes \psi) = \operatorname{tr} \varphi \cdot \operatorname{tr} \psi$*.*

Proof. Given $v \in V$, $w \in W$ we can use the previous lemma to compute

$$(\varphi_1\varphi_2) \otimes (\psi_1\psi_2)(v \otimes w) = \varphi_1\varphi_2(v) \otimes \psi_1\psi_2(w) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2)(v \otimes w).$$

Since elements of the form $v \otimes w$ span $V \otimes W$ and all maps are linear it follows that

$$(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2)$$

as required.

(ii) is clear.

(iii) For the formula relating traces it suffices to stare at the example above:

$$\operatorname{tr} \begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \sum_{i,j} B_{ii}A_{jj} = \operatorname{tr} A \operatorname{tr} B.$$

□

Definition. Given two representation (ρ, V) and (ρ', W) of a group G we can define the representation $(\rho \otimes \rho', V \otimes W)$ by $(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$.

Proposition. *If (ρ, V) and (ρ', W) are representations of G then $(\rho \otimes \rho', V \otimes W)$ is a representation of G and $\chi_{\rho\otimes\rho'} = \chi_\rho \cdot \chi_{\rho'}$.*

Proof. That $\rho \otimes \rho'$ is a representation follows from (i) and (ii) of the last lemma. The formula for characters is a straightforward consequence of part (iii). □

Remarks.

(1) It follows that $R(G)$ is closed under multiplication.

Lecture 12

(2) Tensor product of representations defined above is consistent with our earlier notion when one of the representations is one-dimensional.

(3) It follows from the lemma that if (ρ, V) is a representation of G and (ρ', W) is a representation of another group H then we may make $V \otimes W$ into a representation of $G \times H$ via

$$\rho_{V\otimes W}(g, h) = \rho(g) \otimes \rho'(h).$$

That this does define a representation of $G \times H$ follows from parts (i) and (ii) of the last lemma. Part (iii) of the lemma gives that

$$(\chi_V \otimes \chi_W)(g, h) \colon = \chi_{V\otimes W}(g, h) = \chi_V(g)\chi_W(h).$$

In the last proposition we take the case $G = H$ and then restrict this representation to the diagonal subgroup $G \cong \{(g, g) \mid g \in G\} \subset G \times G$.

(4) If X, Y are finite sets with G-action it is easy to verify that the isomorphism of vector spaces $kX \otimes kY \cong kX \times Y$; $\delta_x \otimes \delta_y \to \delta_{x,y}$ is an isomorphism of representations of G (or even of $G \times G$).

Now return to our assumption that $k = \mathbb{C}$.

Proposition. *Suppose G and H are finite groups, $(\rho_1, V_1), \ldots, (\rho_r, V_r)$ are all the simple complex representations of G and $(\rho'_1, W_1), \ldots, (\rho'_s, W_s)$ are all the simple complex representations of H.*

For each $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, $(\rho_i \otimes \rho'_j, V_i \otimes W_j)$ is an irreducible complex representation of $G \times H$. Moreover, all the irreducible representations of $G \times H$ arise in this way.

We have seen this before when G and H are abelian since then all these representations are 1-dimensional.

Proof. Let $\chi_1, \ldots, \chi_r$ be the characters of $V_1, \ldots, V_r$ and $\psi_1, \ldots, \psi_s$ the characters of $W_1, \ldots, W_s$.

The character of $V_i \otimes W_j$ is $\chi_i \otimes \psi_j \colon (g, h) \mapsto \chi_i(g)\psi_j(h)$. Then

$$\langle \chi_i \otimes \psi_j, \chi_k \otimes \psi_l\rangle_{G\times H} = \langle \chi_i, \chi_k\rangle_G \langle \psi_j, \psi_l\rangle_H = \partial_{ik}\partial_{jl}.$$

So the $\chi_i \otimes \psi_j$ are irreducible and pairwise distinct.

Now

$$\sum_{i,j} (\dim V_i \otimes W_j)^2 = \left(\sum_i (\dim V_i)^2\right)\left(\sum_j (\dim W_j)^2\right) = |G||H| = |G \times H|$$

so we must have them all.[11] $\square$

Question. If V and W are irreducible then must $V \otimes W$ be irreducible?

We've seen the answer is yes is one of V and W is one-dimensional but it is not usually true.

Example. $G = S_3$

[11] We could complete the proof by instead considering conjugacy classes in $G \times H$ to show that $\dim \mathcal{C}_{G\times H} = \dim \mathcal{C}_G \cdot \dim \mathcal{C}_H$.

	1	3	2
	e	(12)	(123)
$\mathbf{1}$	1	1	1
ϵ	1	-1	1
V	2	0	-1

Clearly, $\mathbf{1} \otimes W = W$ always. $\epsilon \otimes \epsilon = \mathbf{1}$, $\epsilon \otimes V = V$ and $V \otimes V$ has character χ^2 given by $\chi^2(1) = 4$, $\chi^2(12) = 0$ and $\chi^2(123) = 1$. Thus χ^2 decomposes as $\mathbf{1} + \epsilon + \chi$.

Of course in general $\chi_i\chi_j = \sum_k a_{i,j}^k \chi_k$ with $a_{i,j}^k \in \mathbb{N}_0$ for all i, j, k and these numbers $a_{i,j}^k$ completely determine the structure of $R(G)$ as a ring.

In fact $V \otimes V, V \otimes V \otimes V, \ldots$ are never irreducible if $\dim V > 1$.

5.2. **Symmetic and Exterior Powers.** For any vector space V, define

$$\sigma = \sigma_V : V \otimes V \to V \otimes V \text{ by } \sigma(v \otimes w) \mapsto w \otimes v \text{ for all } v, w \in V.$$

Exercise. Check this does uniquely define a linear map. Hint: Show that $(v, w) \mapsto v \otimes w$ is a bilinear map.

Notice that $\sigma^2 = \mathrm{id}$ and so, if *char* $k \neq 2$, σ decomposes $V \otimes V$ into two eigenspaces:

$$S^2V := \{a \in V \otimes V \mid \sigma a = a\}$$
$$\Lambda^2 V := \{a \in V \otimes V \mid \sigma a = -a\}.$$

In fact this is the isotypical decomposition of $V \otimes V$ as a rep of C_2.

Lemma. *Suppose $v_1, \ldots, v_m$ is a basis for V.*

(i) S^2V has a basis $v_iv_j := \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i)$ for $1 \leqslant i \leqslant j \leqslant d$. [12]

(ii) Λ^2V has a basis $v_i \wedge v_j := \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i)$ for $1 \leqslant i < j \leqslant d$.[13]

Thus $\dim S^2V = \frac{1}{2}m(m+1)$ *and* $\dim \Lambda^2 V = \frac{1}{2}m(m-1)$.

Proof. It is easy to check that the union of the two claimed bases span $V \otimes V$ and have m^2 elements so form a basis. Moreover v_iv_j do all live in S^2V and the $v_i \wedge v_j$ do all live in Λ^2V. Everything follows.[14] □

Proposition. *Let (ρ, V) be a representation of G over $\mathbb{C}$.*[15]

(i) $V \otimes V = S^2V \oplus \Lambda^2V$ as representations of G.

(ii) for $g \in G$,

$$\chi_{S^2V}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$$
$$\chi_{\Lambda^2V}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

Proof. For (i) we need to show that if $a \in V \otimes V$ and $\sigma_V(a) = \lambda a$ for $\lambda = \pm 1$ then $\sigma_V \rho_{V \otimes V}(g)(a) = \lambda \rho_{V \otimes V}(g)(a)$ for each $g \in G$. For this it suffices to prove that $\sigma g = g\sigma$ (ie $\sigma \in \mathrm{Hom}_G(V \otimes V, V \otimes V)$). But $\sigma \circ g(v \otimes w) = gw \otimes gv = g \circ \sigma(v \otimes w)$.

---

[12] $v_iv_j = v_jv_i$ if we allow $i > j$

[13] $v_i \wedge v_j = -v_j \wedge v_i$ if we allow $i \geqslant j$. In particular $v_i \wedge v_i = 0$

[14] For an alternative argument use Ex Sheet 2 Q11.

[15] We don't strictly need this assumption here. Characteristic not 2 suffices.

To compute (ii) it suffices to prove one or the other since the sum of the right-hand-sides is $\chi(g)^2 = \chi_{V\otimes V}$. Let $v_1, \ldots, v_m$ be a basis of eigenvectors for $\rho(g)$ with eigenvalues $\lambda_1, \ldots, \lambda_m$. Then $g(v_i v_j) = (\lambda_i\lambda_j)v_i v_j$.

Thus

$$\chi(g)^2 + \chi(g^2) = (\sum_i \lambda_i)^2 + \sum_i \lambda_i^2 = 2\sum_{i\leqslant j} \lambda_i \lambda j$$

whereas $\chi_{S^2V}(g) = \sum_{i\leqslant j} \lambda_i\lambda j$. □

Exercise. Prove directly the formula for $\chi_{\Lambda^2 V}$.

Example. S_4

	1	3	8	6	6
	e	(12)(34)	(123)	(12)	(1234)
$\mathbf{1}$	1	1	1	1	1
ϵ	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
$\epsilon\chi_3$	3	-1	0	-1	1
χ_5	2	2	-1	0	0
χ_3^2	9	1	0	1	1
$\chi_3(g^2)$	3	3	0	3	-1
$S^2\chi_3$	6	2	0	2	0
$\Lambda^2\chi_3$	3	-1	0	-1	1

Thus $S^2\chi_3 = \chi_5 + \chi_3 + \mathbf{1}$ and $\Lambda^2\chi_3 = \epsilon\chi_3$. Notice that given $\mathbf{1}$ and ϵ and χ_3 we could've constructed the remaining two irreducible characters using $S^2\chi_3$ and $\Lambda^2\chi_3$.

More generally, for any vector space V we may consider $V^{\otimes n} = V\otimes\cdots\otimes V$. Then for any $\omega\in S_n$ we can define a linear map $\sigma(\omega)\colon V^{\otimes n}\to V^{\otimes n}$ by

$$\sigma(\omega)\colon v_1\otimes\cdots v_n \mapsto v_{\omega^{-1}(1)}\otimes\cdots v_{\omega^{-1}(n)}$$

for $v_1, \ldots, v_n \in V$.

Exercise. Show that this defines a representation of S_n on $V^{\otimes n}$ and that if V is a representation of G then the G-action and the S_n-action on $V^{\otimes n}$ commute.

Lecture 13

Thus we can decompose $V^{\otimes n}$ as a rep of S_n and each isotypical component will be a G-invariant subspace of $V^{\otimes n}$. In particular we can make the following definition.

Definition. Suppose that V is a vector space we define

(i) the n^{th} *symmetric power* of V to be

$$S^nV := \{a\in V^{\otimes n} \mid \sigma(\omega)(a) = a \text{ for all } \omega\in S_n\}$$

and

(ii) the n^{th} *exterior (or alternating) power* of V to be

$$\Lambda^nV := \{a\in V^{\otimes n} \mid \sigma(\omega)(a) = \epsilon(\omega)a \text{ for all } \omega\in S_n\}.$$

Note that $S^nV \oplus \Lambda^n V = \{a \in V^{\otimes n} \mid \sigma(\omega)(a) = a \text{ for all } \omega \in A_n\} \subsetneq V^{\otimes n}$.

We also define the following notation for $v_1, \ldots, v_n \in V$,

$$v_1 \cdots v_n := \frac{1}{n!} \sum_{\omega \in S_n} v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \in S^n V$$

and

$$v_1 \wedge \cdots \wedge v_n := \frac{1}{n!} \sum_{\omega \in S_n} \epsilon(\omega) v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \in \Lambda^n V.$$

Exercise. Show that if $v_1, \ldots, v_d$ is a basis for V then

$$\{v_{i_1} \cdots v_{i_n} \mid 1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant d\}$$

is a basis for S^nV and

$$\{v_{i_1} \wedge \cdots \wedge v_{i_n} \mid 1 \leqslant i_1 < \cdots < i_n \leqslant d\}$$

is a basis for $\Lambda^n V$. Hence given $g \in V$, compute the character values $\chi_{S^nV}(g)$ and $\chi_{\Lambda^n V}$ in terms of the eigenvalues of g on V.

For any vector space V, $\Lambda^{\dim V} \cong k$ and $\Lambda^n V = 0$ if $n > \dim V$.

Exercise. Show that if (ρ, V) is a representation of G then the representation of G on $\Lambda^{\dim V} V \cong k$ is given by $g \mapsto \det \rho(g)$; ie the $\dim V^{th}$ exterior power of V is isomorphic to $\det \rho$.

In characteristic zero, we may stick these vector spaces together to form algebras.

Definition. Given a vector space V we may define the *tensor algebra* of V,

$$TV := \oplus_{n \geqslant 0} V^{\otimes n}$$

(where $V^{\otimes 0} = k$). Then TV is a (non-commutative) graded ring with the product of $v_1 \otimes \cdots \otimes v_r \in V^{\otimes r}$ and $w_1 \otimes \cdots \otimes w_s \in V^{\otimes s}$ given by

$$v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s \in V^{\otimes r+s}.$$

with graded quotient rings the *symmetric algebra* of V,

$$SV := TV/(x \otimes y - y \otimes x \mid x, y \in V),$$

and the *exterior algebra* of V,

$$\Lambda V := TV/(x \otimes y + y \otimes x \mid x, y \in V).$$

One can show that $SV \cong \bigoplus_{n \geqslant 0} S^n V$ under $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$ and $\Lambda V \cong \bigoplus_{n \geqslant 0} \Lambda^n V$ under $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \wedge \cdots \wedge x_n$.

Now SV is a commutative ring and ΛV is graded-commutative; that is if $x \in \Lambda^r V$ and $y \in \Lambda^s V$ then $x \wedge y = (-1)^{rs} y \wedge x$.

5.3. **Duality.** Recall that $\mathcal{C}_G$ has the $*$-operation given by $f^*(g) = f(g^{-1})$. This also restricts to $R(G)$.

Definition. If G is group and (ρ, V) is a representation of G then the *dual representation* (ρ^*, V^*) of G is given by $(\rho^*(g)\theta)(v) = \theta(\rho(g^{-1})v)$ for $\theta \in V^*$, $g \in G$ and $v \in V$.

Lemma. $\chi_{V^*} = (\chi_V)^*$.

Proof. If $\rho(g)$ is represented by a matrix A with respect to a basis $v_1, \dots, v_d$ for V and $\epsilon_1, \dots, \epsilon_d$ is the dual basis for V^*. Then $\rho(g)^{-1}v_i = \sum (A^{-1})_{ji} v_j$.

Thus $(\rho^*(g)\epsilon_k)(v_i) = \epsilon_k \left(\sum_j (A^{-1})_{ji} v_j \right) = (A^{-1})_{ki}$ and so

$$\rho^*(g)\epsilon_k = \sum_i (A^{-1})^T_{ik} \epsilon_i$$

i.e. $\rho^*(g)$ is represented by $(A^{-1})^T$ with respect to the dual basis. Taking traces gives the result. □

Definition. We say that V is *self-dual* if $V \cong V^*$ as representations of G.

When G is finite and $k = \mathbb{C}$, V is self-dual if and only if $\chi_V(g) \in \mathbb{R}$ for all $g \in G$; since this is equivalent to $\chi_{V^*} = \chi_V$.

Examples.

(1) $G = C_3 = \langle x \rangle$ and $V = \mathbb{C}$. If ρ is given by $\rho(x) = \omega = e^{\frac{2\pi i}{3}}$ then $\rho^*(x) = \omega^2 = \overline{\omega}$ and V is not self-dual.
(2) $G = S_n$: since g is always conjugate to its inverse in S_n, $\chi^* = \chi$ always and so every representation is self-dual.
(3) Permutation representations $\mathbb{C}X$ are always self-dual.

Exercise. Show both directly and using characters that if U, V, W are complex representations of G then

$$V \otimes W \cong \mathrm{Hom}_k(V^*, W) \text{ and } \mathrm{Hom}_k(V \otimes W, U) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(W, U))$$

as representations of G. Deduce that if V is self-dual then either $\langle \mathbf{1}, \chi_{S^2V} \rangle \neq 0$ or $\langle \mathbf{1}, \chi_{\Lambda^2 V} \rangle \neq 0$. Hint:

$$\Theta \colon V^* \otimes W \to \mathrm{Hom}_k(V, W); \Theta(\epsilon \otimes w)(v) = \epsilon(v)w$$

and

$$\Psi \colon \mathrm{Hom}_k(V \otimes W, U) \to \mathrm{Hom}_k(V, \mathrm{Hom}_k(W, U)); \Psi(\alpha)(v)(w) = \alpha(v \otimes w)$$

characterise the required isomorphisms.

We've now got a number of ways to build representations of a group G:

- permutation representations coming from group actions;
- via representations of a group H and a group homomorphism $G \to H$ (e.g. restriction);
- tensor products;
- symmetric and exterior powers;
- decomposition of these into irreducible components;
- character theoretically using orthogonality of characters.

We're now going to discuss one more way related to restriction.

Lecture 14

6. Induction

Suppose that H is a subgroup of G. Restriction turns representations of G into representations of H. We would like a way of building representations of G from representations of H. There is a good way of doing so called induction although it is a little more delicate than restriction.

Notation. *Given a group G we'll write $[g]_G$ for the conjugacy class of $g \in G$. So $\mathbf{1}_{[g]_G} : G \to k$ given by*

$$\mathbf{1}_{[g]_G}(x) = \begin{cases} 1 & \text{if } x \text{ is conjugate to } g \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$

is in $\mathcal{C}_G$.

We note that for $g \in G$,

$$[g]_G^{-1} = [g^{-1}]_G,$$

since $(xgx^{-1})^{-1} = xg^{-1}x^{-1}$, and so $(\mathbf{1}_{[g]_G})^* = \mathbf{1}_{[g^{-1}]_G}$.

If $H \leqslant G$ then $[g]_G \cap H$ is a union of H-conjugacy classes

$$[g]_G \cap H = \bigcup_{[h]_H \subset [g]_G} [h]_H$$

so

$$r\colon \mathcal{C}_G \to \mathcal{C}_H; f \mapsto f|_H$$

is a well-defined linear map with

$$r^*(\mathbf{1}_{[g]_G}) = \sum_{[h]_H \subseteq [g]_G} \mathbf{1}_{[h]_H}$$

.

Since for every finite group G, $\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} f_1^*(g) f_2(g)$ defines a non-degenerate bilinear form on $\mathcal{C}_G$, the map r has an adjoint r^* characterised by

$$\langle r(f_1), f_2 \rangle_H = \langle f_1, r^*(f_2) \rangle_G \text{ for } f_1 \in \mathcal{C}_G, f_2 \in \mathcal{C}_H.$$

In particular for $f \in \mathcal{C}_H$,

$$\langle \mathbf{1}_{[g^{-1}]_G}, r^*(f) \rangle_G = \langle r(\mathbf{1}_{[g^{-1}]}_G, f \rangle_H = \frac{1}{|H|} \sum_{[h]_H \subseteq [g]_G} |[h]_H| f(h).$$

On the other hand,

$$\langle \mathbf{1}_{[g^{-1}]_G}, r^*(f) \rangle_G = \frac{1}{|G|} \sum_{x \in [g]_G} r^*(f)(x) = \frac{|[g]_G|}{|G|} r^*(f)(g)$$

Thus, by comparing these we see that

$$r^*(f)(g) \quad = \quad \sum_{[h]_H \subseteq [g]_G} \frac{|C_G(g)|}{|C_H(h)|} f(h) \tag{1}$$

Question. Is $r^*(R(H)) \subseteq R(G)$?

Suppose that χ is a $\mathbb{C}$-character of H and ψ is an irreducible $\mathbb{C}$-character of G. Then

$$\langle \psi, r^*(\chi)\rangle_G = \langle r(\psi), \chi\rangle_H \in \mathbb{N}_0$$

by orthogonality of characters, since $r(\psi)$ is a character of H.

So writing $\mathrm{Irr}(G)$ to denote the set of irreducible $\mathbb{C}$-characters of G

$$r^*(\chi) \quad = \sum_{\chi\in \mathrm{Irr}(G)} \langle \psi|_H, \chi\rangle_H \psi \tag{2}$$

is even a character in $R(G)$. The formula (2) is only useful for actually computing $r^*(\chi)$ if we already understand $\mathrm{Irr}(G)$. Since our purpose will often be to use $\mathrm{Irr}(H)$ to understand $\mathrm{Irr}(G)$, the formula (1) will typically prove more useful.

Example. $G = S_3$ and $H = A_3 = \{1, (123), (132)\}$.

If $f \in \mathcal{C}_H$ then

$$\begin{aligned} r^*(f)(e) &= \frac{6}{3}f(e) = 2f(e), \\ r^*(f)((12)) &= 0, \text{ and} \\ r^*(f)((123)) &= \frac{3}{3}f((123)) + \frac{3}{3}f((132)) = f((123)) + f((132)). \end{aligned}$$

Thus

A_3	1	(123)	(132)
χ_1	1	1	1
χ_2	1	w	w^2
χ_3	1	w^2	w

S_3	1	(12)	(123)
$r^*(\chi_1)$	2	0	2
$r^*(\chi_2)$	2	0	-1
$r^*(\chi_3)$	2	0	-1

So $r^*(\chi_1) = \mathbf{1} + \epsilon$ and $r^*(\chi_2) = r^*(\chi_3)$ is the 2-dimensional irreducible character of S_3 consistent with the formula (2).

Note that if χ is an irreducible character then $r^*(\chi)$ may be irreducible but need not be so.

If G is a finite group and W is a k-vector space we may define $\mathrm{Hom}(G, W)$ to be the vector space of all functions $G \to W$ under pointwise addition and scalar multiplication. This may be made into a representation of G by defining

$$(g \cdot f)(x) := f(g^{-1}x)$$

for each $g, x \in G$. If $w_1, \ldots, w_n$ is a basis for W then $\{\delta_g w_i \mid g \in G, 1 \leqslant i \leqslant n\}$ is a basis for $\mathrm{Hom}(G, W)$. So $\dim \mathrm{Hom}(G, W) = |G| \dim W$.

Lemma. $\mathrm{Hom}(G, W) \cong (\dim W)kG$ *as representations of* G.

Proof. Given a basis $w_1, \ldots, w_n$ for W, define the linear map

$$\Theta\colon \bigoplus_{i=1}^{n} kG \to \mathrm{Hom}(G, W)$$

by

$$\Theta((f_i)_{i=1}^n)(x) = \sum_{i=1}^{n} f_i(x)w_i.$$

It is easy to see that Θ is injective because the w_i are linearly independent so by comparing dimensions we see that Θ is a vector-space isomorphism.

It remains to prove that Θ is G-linear. If $g, x \in G$ then

$$g \cdot (\Theta((f_i)_{i=1}^n))(x) = \sum_{i=1}^n f_i(g^{-1}x)w_i = \Theta(g \cdot (f_i)_{i=1}^n)(x)$$

as required. $\square$

Exercise. Use the basis of $\mathrm{Hom}(G, W)$ given above to find a character-theoretic proof of the lemma.

Now, if H is a subgroup of G and W is a representation of H then we can define

$$\mathrm{Hom}_H(G, W) := \{f \in \mathrm{Hom}(G, W) \mid f(xh) = h^{-1}f(x)\ \forall x \in G, h \in H\},$$

a k-linear subspace of $\mathrm{Hom}(G, W)$.

Example. If $W = \mathbf{1}$ is the trivial representation of H and $f \in \mathrm{Hom}(G, \mathbf{1})$, then $f \in \mathrm{Hom}_H(G, \mathbf{1})$ if and only if $f(xh) = f(x)$ for $h \in H$ and $x \in G$. That is $\mathrm{Hom}_H(G, \mathbf{1})$ consists of the functions that are constant on each left coset in G/H. Thus $\mathrm{Hom}_H(G, \mathbf{1})$ can be identified with kG/H. One can check that this identification is G-linear.

Lemma. *$\mathrm{Hom}_H(G, W)$ is a G-invariant subspace of $\mathrm{Hom}(G, W)$.*

Proof. Let $f \in \mathrm{Hom}_H(G, W)$, $g, x \in G$ and $h \in H$ we must show that

$$(g \cdot f)(xh) = h^{-1}(g \cdot f)(x).$$

But $(g \cdot f)(xh) = f(g^{-1}xh) = h^{-1}f(g^{-1}x) = h^{-1}(g \cdot f)(x)$ as required. $\square$

Definition. Suppose that H is a subgroup of G of finite index and W is a representation of H. We define the *induced representation* to be $\mathrm{Ind}_H^G W := \mathrm{Hom}_H(G, W)$

Proposition. *Suppose W is a representation of H then for $g \in G$,*

$$\begin{aligned}
\chi_{\mathrm{Ind}_H^G W}(g) &= r^*(\chi_W) \\
&= \sum_{[h]_H \subseteq [g]_G} \frac{|C_G(g)|}{|C_H(h)|}\chi_W(h) \\
&= \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{x \in G} \chi_W^\circ(x^{-1}gx)
\end{aligned}$$

where $\chi_W^\circ \in \mathcal{C}_G$ is given by

$$\chi_W^\circ(g) = \begin{cases} \chi_W(g) & \text{when } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\dim \mathrm{Ind}_H^G W = \chi_{\mathrm{Ind}_H^G W}(e) = \frac{|G|}{|H|} \dim W$*;*

Lecture 15

Remark. $x^{-1}gx \in H$ if and only if $gxH = xH$ so if W is the trivial representation

$$\begin{aligned}\frac{1}{|H|}\sum_{x\in G}\chi_W^\circ(x^{-1}gx) &= \frac{1}{|H|}\{x\in G \mid gxH = xH\}| \\ &= |\{xH \in G/H \mid gxH = xH\}|\end{aligned}$$

and we get the permutation character of kG/H as required.

Proof of Proposition. First we observe that for $x,y,g\in G$, since $x^{-1}gx = y^{-1}gy$ if and only if $xy^{-1}\in C_G(g)$,

$$\begin{aligned}\sum_{[h]_H\subseteq[g]_G}\frac{|C_G(g)|}{|C_H(h)|}\chi_W(h) &= \sum_{h\in H\cap[g]_G}\frac{|C_G(g)|}{|H|}\chi_W(h) \\ &= \frac{1}{|H|}\sum_{x\in G}\chi_W^\circ(x^{-1}gx).\end{aligned}$$

So it suffices to show that

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|}\sum_{x\in G}\chi_W^\circ(x^{-1}gx).$$

Let $x_1,\ldots,x_r$ be left coset representatives in G/H. Then $f\in\operatorname{Hom}_H(G,W)$ is determined by $f(x_1),\ldots,f(x_r)\in W$ since $f(x_ih) = h^{-1}f(x_i)$ for all $i = 1,\ldots r$ and $h\in H$.

Moreover, given $w_1,\ldots,w_r\in W$ we can define $f\in\operatorname{Hom}_H(G,W)$ via

$$f(x_ih) = h^{-1}w_i \text{ for } i = 1,\ldots,r \text{ and } h\in H.$$

Thus

$$\Theta\colon \operatorname{Hom}_H(G,W)\to\bigoplus_{i=1}^r W$$

defined by $f\mapsto (f(x_i))_{i=1}^r$ is an isomorphism of vector spaces.

In particular given $w\in W$, and $1\leqslant j\leqslant r$, we can define $\varphi_{j,w}\in\operatorname{Hom}_H(G,W)$ by

$$\varphi_{j,w}(x_kh) = \delta_{jk}h^{-1}w$$

for each $h\in H$ and $1\leqslant k\leqslant r$ so that $\Theta\left(\{\phi_{j,w}|w\in W\}\right)$ is the jth copy of W in $\bigoplus_{i=1}^r W$

Now given $g\in G$, let's consider how g acts on a $\varphi_{i,w}$. For each coset representative x_i there is a unique $\sigma(i)$ and $h_i\in H$ such that $g^{-1}x_i = x_{\sigma(i)}h_i^{-1}\in x_{\sigma(i)}H$, and

$$(g\cdot\varphi_{i,w})(x_j) = \varphi_{i,w}(g^{-1}x_j) = \varphi_{i,w}(x_{\sigma(j)}h_j^{-1}) = \delta_{i\sigma(j)}h_jw.$$

Thus $g\cdot\varphi_{i,w} = \varphi_{\sigma^{-1}(i),h_{\sigma^{-1}(i)}w}$.

Thus g acts on $\bigoplus_{i=1}^r W$ via a block permutation matrix and we only get contributions to the trace from the non-zero diagonal blocks which correspond to the fixed points of σ. Moreover if $\sigma(i) = i$ then g acts on W_i via $h_i = x_i^{-1}gx_i$

Thus

$$\operatorname{tr} g_{\operatorname{Ind}_H^G W} = \sum_i \chi_W^\circ(x_i^{-1}gx_i).$$

Since $G = \{x_i h \mid h \in H\}$ and $\chi_W^\circ(h^{-1}yh) = \chi_W^\circ(y)$ for all $y \in G$ and $h \in H$ we may rewrite this as

$$\operatorname{tr} g_{\operatorname{Ind}_H^G W} = \frac{1}{|H|} \sum_{x \in G} \chi_W^\circ(xgx^{-1})$$

as required. $\square$

If V is a representation of G, we'll write $\operatorname{Res}_H^G V$ for the representation of H obtained by restriction.

Corollary (Frobenius reciprocity)**.** *Let V be a representation of G, and W a representation of H, then*

(i) $\langle \chi_V, \operatorname{Ind}_H^G \chi_W \rangle_G = \langle \operatorname{Res}_H^G \chi_V, \chi_W \rangle_H$*;*
(ii) $\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$*.*

Proof. (i) follows from $\operatorname{Ind}_H^G \chi_W = r^*(\chi_W)$ and the fact that $\operatorname{Res}_H^G \chi_V = r(\chi_V)$. We've already seen that (i) implies (ii) since the dimension of the LHS of (ii) is the LHS of (i) and the dimension of the RHS of (ii) is the dimension of the RHS of (i). $\square$

Exercise. Prove (ii) directly by considering

$$\Theta\colon \operatorname{Hom}_G(V, \operatorname{Hom}_H(G, W)) \leftrightarrow \operatorname{Hom}_H(V, W) \colon \Psi$$

defined by $\Theta(f)(v) = f(v)(e)$ and $\Psi(\beta)(v)(g) = \beta(g^{-1}v)$. This gives an alternative proof that $\chi_{\operatorname{Hom}_H(G,W)} = r^*(\chi_W)$.

6.1. **Mackey Theory.** This is the study of representations like $\operatorname{Res}_K^G \operatorname{Ind}_H^G W$ for H, K subgroups of G and W a representation of H. We can (and will) use it to characterise when $\operatorname{Ind}_H^G W$ is irreducible using that

$$\langle \operatorname{Ind}_H^G \chi_W, \operatorname{Ind}_H^G \chi_W \rangle_G = \langle \operatorname{Res}_H^G \operatorname{Ind}_H^G W, W \rangle_H.$$

If H, K are subgroups of G we can restrict the action of G on G/H to K

$$K \times G/H \to G/H; (k, gH) \mapsto kgH.$$

The the union of an orbit of this action is called a *double coset*. The union of the K-orbit of gH is written $KgH := \{kgh \mid k \in K, h \in H\}$.

Definition. $K\backslash G/H := \{KgH \mid g \in G\}$ is the set of double cosets.

The double cosets partition G.

Given any representation (ρ, W) of H and $g \in G$, we can define $({}^g\rho, {}^gW)$ to be the representation of ${}^gH := gHg^{-1} \leqslant G$ on W given by $({}^g\rho)(ghg^{-1}) = \rho(h)$ for $h \in H$.

Theorem (Mackey's Restriction Formula)**.** *If G is a finite group with subgroups H and K, and W is a representation of H then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{g \in K\backslash G/H} \operatorname{Ind}_{K \cap {}^gH}^K \operatorname{Res}_{{}^gH \cap K}^{{}^gH} {}^gW.$$

Proof. For each double coset KgH we can define

$$V_{KgH} = \{f \in \operatorname{Ind}_H^G W \mid f(x) = 0 \text{ for all } x \notin KgH\}.$$

Then V_{KgH} is a K-invariant subspace of $\operatorname{Ind}_H^G W$ since we always have $(kf)(x) = f(k^{-1}x)$. Thus there is a decomposition

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{g \in K\backslash G/H} V_{KgH}$$

and it suffices to show that for each g,

$$V_{KgH} \cong \operatorname{Ind}_{K\cap {}^gH}^K \operatorname{Res}_{{}^gH\cap K}^{{}^gH} {}^gW$$

as representations of K.

It is easy to see that

$$V_{KgH} \cong \{f\colon KgH \to W \mid f(xh) = h^{-1}f(x) \text{ for all } x \in KgH\}$$

with K-action given by $kf(x) = f(k^{-1}x)$ and we must show

$$V_{KgH} \cong \operatorname{Hom}_{K\cap {}^gH}(K, {}^gW).$$

Define such a Θ by $\Theta(f)(k) = f(kg)$. If $ghg^{-1} \in K$ for some $h \in H$,

$$\begin{aligned}\Theta(f)(kghg^{-1}) &= f(kgh)\\ &= \rho(h^{-1})f(kg)\\ &= ({}^g\rho)(ghg^{-1})^{-1}\Theta(f)(k)\end{aligned}$$

Thus $\operatorname{Im}\Theta \leqslant \operatorname{Ind}_{K\cap {}^gH}^K \operatorname{Res}_{K\cap {}^gH}^{{}^gH} {}^gW$.

Also, if $k' \in K$ then

$$(k'\Theta(f))(k) = f(k'^{-1}kg) = (k'f)(kg) = \Theta(k'f)(k)$$

and so Θ is K-linear.

Θ is injective since, if $f \in V_{KgH}$ with $f(kg) = 0$ for all $k \in K$, then

$$f(kgh) = h^{-1}f(kg) = h^{-1}(0) = 0$$

for all $k \in K$ and $h \in H$ so $f = 0$. It thus remains to show that

$$\dim V_{KgH} = \dim \operatorname{Ind}_{K\cap {}^gH}^K {}^gW.$$

Since choosing $f \in V_{KgH}$ is equivalent to choosing an element of W for coset representatives of each element $\operatorname{Orb}_K(gH) \subseteq G/H$,

$$\dim V_{KgH} = \dim W|\operatorname{Orb}_K(gH)| = \dim W\frac{|K|}{|\operatorname{Stab}_K(gH)|} = \dim W\frac{|K|}{|K\cap gHg^{-1}|}$$

(since $kgH = gH$ if and only if $k \in gHg^{-1}$). Finally

$$\dim \operatorname{Ind}_{K\cap {}^gH}^K \operatorname{Res}_{{}^gH\cap K}^{{}^gH} {}^gW = \frac{|K|}{|K\cap {}^gH|}\dim W$$

by the Proposition stated at the end of the last lecture. □

Lecture 16

Corollary (Character version of Mackey's Restriction Formula)**.** *If χ is a character of a representation of H then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \chi = \sum_{KgH\in K\backslash G/H} \operatorname{Ind}_{{}^gH\cap K}^K {}^g\chi.$$

where ${}^g\chi$ is the class function on ${}^gH\cap K$ given by ${}^g\chi(x) = \chi(g^{-1}xg)$.

Exercise. Prove this corollary directly with characters

Corollary (Mackey's irreducibility criterion)**.** *If H is a subgroup of G and W is a representation of H, then $\operatorname{Ind}_H^G W$ is irreducible if and only if*

(i) W is irreducible and

(ii) for each $g \in G \backslash H$, the two representations $\operatorname{Res}^{{}^gH}_{H\cap {}^gH}\, {}^gW$ and $\operatorname{Res}^H_{{}^gH\cap H} W$ of $H \cap^g H$ have no irreducible factors in common.

Proof.

$$
\begin{aligned}
< \operatorname{Ind}_H^G \chi_W, \operatorname{Ind}_H^G \chi_W\rangle_G &\overset{\text{Frob. recip.}}{=} \langle \chi_W, \operatorname{Res}_H^G \operatorname{Ind}_H^G \chi_W\rangle_H \\
&\overset{\text{Mackey}}{=} \sum_{g\in H\backslash G/H} \langle \chi_W, \operatorname{Ind}^H_{H\cap^g H} \operatorname{Res}^{{}^gH}_{H\cap^g H} {}^g\chi_W\rangle_H \\
&\overset{\text{Frob. recip.}}{=} \sum_{g\in H\backslash G/H} \langle \operatorname{Res}^H_{H\cap^g H} \chi_W, \operatorname{Res}^{{}^gH}_{H\cap^g H} {}^g\chi_W\rangle_{H\cap^g H}
\end{aligned}
$$

So $\operatorname{Ind}_H^G W$ is irreducible precisely if

$$\sum_{g\in H\backslash G/H} \langle \operatorname{Res}^H_{H\cap^g H} \chi_W, \operatorname{Res}^{{}^gH}_{H\cap^g H} {}^g\chi_W\rangle_{H\cap^g H} = 1.$$

The term corresponding to the coset $HeH = H$ is $\langle \chi_W, \chi_W\rangle_H$ which is at least 1 and equal to 1 precisely if W is irreducible. The other terms are all $\geqslant 0$ and are zero precisely if condition (ii) of the statement holds. □

Corollary. *If H is a normal subgroup of G ,and W is an irreducible rep of H then $\operatorname{Ind}_H^G W$ is irreducible if and only if ${}^g\chi_W \neq \chi_W$ for all $g \in G\backslash H$.*

Proof. Since H is normal, $gHg^{-1} = H$ for all $g \in G$. Moreover ${}^gW$ is irreducible since W is irreducible.

So by Mackey's irreducibility criterion, $\operatorname{Ind}_H^G W$ irreducible precisely if $W \ncong {}^gW$ for all $g \in G\backslash H$. This last is equivalent to $\chi_W \neq {}^g\chi_W$ as required.

□

Examples.

(1) $H = \langle r\rangle \cong C_n$, the rotations in $G = D_{2n}$. The irreducible characters χ of H are all of the form $\chi(r^j) = e^{\frac{2\pi ijk}{n}}$. We see that $\operatorname{Ind}_H^G \chi$ is irreducible if and only if $\chi(r^j) \neq \chi(r^{-j})$ for some j. This is equivalent to χ not being real valued.

(2) $G = S_n$ and $H = A_n$. If $g \in S_n$ is a cycle type that splits into two conjugacy classes in A_n and χ is an irreducible character of A_n that takes different values of the two classes then $\operatorname{Ind}_H^G \chi$ is irreducible.

6.2. **Frobenius groups.**

Definition. A *Frobenius group* is a finite group G that has a transitive action on a set X with $|X| > 1$ such that each $g \in G\backslash\{e\}$ fixes at most one $x \in X$ and $\operatorname{Stab}_G(x) \neq \{e\}$ for some (all) $x \in X$.

Examples.

(a) $G = D_{2n}$ with n odd acting naturally on the vertices of an n-gon.

(b) $G = \left\{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} \mid a, b \in \mathbb{F}_p, a \neq 0\right\}$ acting on $X = \left\{\begin{pmatrix} x \\ 1\end{pmatrix} \mid x \in \mathbb{F}_p\right\}$ by matrix multiplication.

Lemma. *G is a Frobenius group if and only if G has a proper subgroup H such that $H \cap gHg^{-1} = \{e\}$ for all $g \in G \backslash H$.*

Proof. Suppose the action of G on X shows G to be Frobenius and pick $x \in X$.

Let $H := \mathrm{Stab}_G(x)$ for some fixed $x \in X$, a proper subgroup of G. Then $gHg^{-1} = \mathrm{Stab}_G(gx)$ for each $g \in G$. Since no element of $G\backslash\{e\}$ fixes more than one $x \in X$ it follows that $gHg^{-1} \cap H = \{e\}$ for each $g \in G\backslash H$.

For the converse let $X = G/H$ with the left regular action and reverse the argument. □

Theorem. *(Frobenius 1901) Let G be a finite group acting transitively on a set X. If each $g \in G\backslash\{e\}$ fixes at most one element of X then*

$$K = \{1\} \cup \{g \in G \mid gx \neq x \textit{ for all } x \in X\}$$

is a normal subgroup of G of order $|X|$.

It follows that no Frobenius group can be simple. The normal subgroup K is called the *Frobenius kernel* and the group H is called the *Frobenius complement.* No proof of the theorem is known that does not use representation theory.

Proof. For $x \in X$, let $H = \mathrm{Stab}_G(x)$ so $|G| = |X||H|$ by the orbit-stabiliser theorem.

By hypothesis if $g \in G\backslash H$ then

$$\mathrm{Stab}_G(gx) \cap \mathrm{Stab}_G(x) = gHg^{-1} \cap H = \{e\}.$$

Thus

(i) $|\bigcup_{x\in X} \mathrm{Stab}_G(x)| = |\bigcup_{g\in G} gHg^{-1}| = (|H|-1)|X| + 1$

(ii) h and h' in H are conjugate in G if and only if they are conjugate in H.

(iii) $|C_G(h)| = |C_H(h)|$ for $e \neq h \in H$

By (i) $|K| = |\{e\} \cup \left(|G| \backslash \bigcup_{x\in X} \mathrm{Stab}_G(x)\right)| = |H||X| - (|H|-1)|X| = |X|$ as required.

We must show that $K \lhd G$. Our strategy will be to prove that it is the kernel of some representation of G.

Now if χ is a character of H we can compute $\mathrm{Ind}_H^G \chi$:

$$\mathrm{Ind}_H^G \chi(g) = \begin{cases} |X|\chi(e) & \text{if } g = e \\ \chi(h) & \text{if } g = h \in H\backslash\{e\} \\ 0 & \text{if } g \in K\backslash\{e\} \end{cases}$$

Suppose now that $\chi_1, \ldots, \chi_r$ is a list of the irreducible characters of H and let $\theta_i = \mathrm{Ind}_H^G \chi_i + \chi_i(e)\mathbf{1}_G - \chi_i(e)\,\mathrm{Ind}_H^G \mathbf{1}_H \in R(G)$ for $i = 1, \ldots, r$ and so

$$\theta_i(g) = \begin{cases} \chi_i(e) & \text{if } g = e \\ \chi_i(h) & \text{if } g = h \in H \\ \chi_i(e) & \text{if } g \in K \end{cases}$$

If θ_i were a character then the corresponding representation would have kernel containing K. Since $\theta_i \in R(G)$ we can write it as a $\mathbb{Z}$-linear combination of irreducible characters $\theta_i = \sum n_i\psi_i$, say.

Now we can compute

$$\begin{aligned}\langle \theta_i, \theta_i \rangle_G &= \frac{1}{|G|} \sum_{g \in G} |\theta_i(g)|^2 \\ &= \frac{1}{|G|} \left(\sum_{h \in H \setminus \{e\}} |X||\chi_i(h)|^2 + \sum_{k \in K} \chi_i(e)^2 \right) \\ &= \frac{|X|}{|G|} \left(\sum_{h \in H} |\chi_i(h)|^2 \right) \\ &= \langle \chi_i, \chi_i \rangle_H = 1\end{aligned}$$

But on the other hand it must be $\sum n_i^2$. Thus θ_i is $\pm\psi$ for some character ψ of G. Since $\theta_i(e) > 0$ it must actually be an irreducible character.

To finish we write $\theta = \sum \chi_i(e)\theta_i$ and so $\theta(h) = \sum \chi_i(e)\chi_i(h) = 0$ for $h \in H \setminus \{e\}$ by column orthogonality, and $\theta(k) = \sum \chi_i(e)^2 = |H|$ for $k \in K$. Thus $K = \ker \theta$ is a normal subgroup of G. □

Lecture 17

7. Arithmetic properties of characters

In this section we'll investigate how arithmetic properties of characters produce a suprising interplay between the structure of the group and properties of the character table. The highlight of this will be the proof of Burnside's famous p^aq^b-theorem that says that the order of a simple group cannot have precisely two distinct prime factors.

7.1. **Arithmetic results.**

We'll need to quote some results about arithmetic without proof; proofs should be provided in the Number Fields course (or in one later case Galois Theory). We'll continue with our assumption that $k = \mathbb{C}$ and also assume that our groups are finite.

Definition. $x \in \mathbb{C}$ is an *algebraic integer* if it is a root of a monic polynomial with integer coefficients.

Facts.

Fact 1 The algebraic integers form a subring of $\mathbb{C}$. (cf Groups, Rings and Modules 2021 Examples Sheet 4 Q13)

Fact 2 Any subring of $\mathbb{C}$ that is finitely generated as an abelian group consists of algebraic integers. (cf Groups, Rings and Modules 2021 Examples Sheet 4 Q13)

Fact 3 If $x \in \mathbb{Q}$ is an algebraic integer then $x \in \mathbb{Z}$. (cf Numbers and Sets 2010 Example Sheet 3 Q12)

Lemma. *If χ is the character of a representation of a finite group G, then $\chi(g)$ is an algebraic integer for all $g \in G$.*

Proof. We know that $\chi(g)$ is a sum of n^{th} roots of unity for $n = |G|$. Since each n^{th} root of unity is by defintion a root of $X^n - 1$ and so an algebraic integer. The lemma follows from Fact 1. $\square$

7.2. **The group algebra.**

Before we go further we need to explain how to make the vector space kG into a ring. There are in fact two sensible ways to do this. The first of these is by pointwise multiplication: $f_1f_2(g) = f_1(g)f_2(g)$ for all $g \in G$ will make kG into a commutative ring. But more usefully for our immediate purposes we have the convolution product

$$f_1f_2(g) := \sum_{x\in G} f_1(gx)f_2(x^{-1}) = \sum_{\substack{x,y\in G\\ xy=g}} f_1(x)f_2(y)$$

that makes kG into a (possibly) non-commutative ring. Notice in particular that with this product $\delta_{g_1}\delta_{g_2} = \delta_{g_1g_2}$ and so we may rephrase the multiplication as

$$(\sum_{g\in G}\lambda_g\delta_g)(\sum_{h\in G}\mu_h\delta_h) = \sum_{k\in G}\left(\sum_{gh=k}\lambda_g\mu_h\right)\delta_k.$$

From now on this will be the product we have in mind when we think of kG as a ring.

We notice in passing that a (finitely generated) kG-module is the 'same' as a representation of G: given a representation (ρ, V) of G we can make it into a

kG-module via
$$fv = \sum_{g \in G} f(g)\rho(g)(v).$$
for $f \in kG$ and $v \in V$. Conversely, given a finitely generated kG-module M we can view M as a representation of G via $\rho(g)(m) = \delta_g m$.

Exercise. Suppose that kX is a permutation representation of G. Calculate the action of $f \in kG$ on kX under this correspondence.

It will prove useful understand the *centre* $Z(kG)$ of kG; that is the set of $f \in kG$ such that $fh = hf$ for all $h \in kG$.

Lemma. *Suppose that $f \in kG$. Then f is in $Z(kG)$ if and only if $f \in \mathcal{C}_G$, the set of class functions on G. In particular* $\dim_k Z(kG)$ *is the number of conjugacy classes in G.*

Proof. Suppose $f \in kG$. Notice that $fh = hf$ for all $h \in kG$ if and only if $f\delta_g = \delta_g f$ for all $g \in G$: the forward direction is clear and for the backward direction if $f\delta_g = \delta_g f$ for all $g \in G$ then
$$fh = \sum_{g \in G} fh(g)\delta_g = \sum_{g \in G} h(g)\delta_g f = hf.$$
But $\delta_g f = f\delta_g$ if and only if $\delta_g f \delta_{g^{-1}} = f$ and
$$(\delta_g f \delta_{g^{-1}})(x) = (\delta_g f)(xg) = f(g^{-1}xg).$$
So if $f \in Z(kG)$ if and only if $f \in \mathcal{C}_G$ as required. $\square$

Remark. The multiplication on $Z(kG)$ is not the same as the multiplication on $\mathcal{C}_G$ that we have seen before even though both have the same additive groups and both are commutative rings.

Definition. Suppose $\mathcal{O}_1 = \{e\}, \ldots, \mathcal{O}_r$ are the conjugacy classes of G, define the *class sums* $C_1, \ldots, C_r$ to be the class functions on G so that
$$C_i = \begin{cases} 1 & g \in \mathcal{O}_i \\ 0 & g \notin \mathcal{O}_i. \end{cases}$$
We called these $\mathbf{1}_{\mathcal{O}_i}$ before but have changed notation to remind ourselves that the multiplication is not pointwise. Also we'll fix $g_i \in \mathcal{O}_i$ for convenience.

We've seen that the class sums form a basis for $Z(kG)$.

Proposition. *There are non-negative integers a_{ijk} such that $C_iC_j = \sum_k a_{ijk}C_k$ for $i, j, k \in \{1, \ldots, r\}$. Indeed*
$$a_{ijk} = |(x, y) \in \mathcal{O}_i \times \mathcal{O}_j \mid xy = g_k\}|.$$

The a_{ijk} are called the *structure constants* for $Z(kG)$.

Proof. Since $Z(kG)$ is a ring, we can certainly write $C_iC_j = \sum a_{ijk}C_k$ for some $a_{ijk} \in k$.

However, we can explicitly compute for $g_k \in \mathcal{O}_k$,
$$a_{ijk} = (C_iC_j)(g_k) = \sum_{\substack{x,y \in G \\ xy = g_k}} C_i(x)C_j(y) = |\{(x, y) \in \mathcal{O}_i \times \mathcal{O}_j \mid xy = g_k\}|$$
as claimed. $\square$

Suppose now that (ρ, V) is an irreducible representation of G. Then if $z \in Z(kG)$ we see that $z\colon V \to V$ given by $zv = \sum_{g\in G} z(g)\rho(g)v \in \mathrm{Hom}_G(V,V)$.

By Schur's Lemma it follows that z acts by a scalar $\lambda_z \in k$ on V. In this way we get an algebra homomorphism $w_\rho\colon Z(kG) \to k; z \mapsto \lambda_z$.

Taking traces we see that

$$\dim V \cdot \lambda_z = \sum_{g\in G} z(g)\chi_V(g).$$

So

$$w_\rho(C_i) = \frac{\chi(g_i)}{\chi(e)}|\mathcal{O}_i| \text{ for } g_i \in \mathcal{O}_i.$$

We now see that w_ρ only depends on χ_ρ (and so on the isomorphism class of ρ) and we write $w_\chi = w_\rho$.

Lemma. *The values $w_\chi(C_i)$ are algebraic integers.*

Note this isn't *a priori* obvious since $\frac{1}{\chi(e)}$ will not be an algebraic integer for $\chi(e) \neq 1$.

Proof. Since w_χ is an algebra homomorphism $Z(kG) \to k$,

$$w_\chi(C_i)w_\chi(C_j) = \sum_k a_{ijk}w_\chi(C_k).$$

So the subring of $\mathbb{C}$ generated by $w_\chi(C_i)$ for $i = 1, \ldots, r$ is a finitely generated abelian group. The result follows from Fact 2 above. $\square$

Lemma.

$$a_{ijk} = \frac{|G|}{|C_G(g_i)||C_G(g_j)|}\sum_\chi \frac{\chi(g_i)\chi(g_j)\chi(g_k^{-1})}{\chi(e)}.$$

In particular the a_{ijk} are determined by the character table.

Proof. For each irreducible character χ,

$$\frac{\chi(g_i)}{\chi(e)}|\mathcal{O}_i|\frac{\chi(g_j)}{\chi(e)}|\mathcal{O}_j| = \sum_k a_{ijk}\frac{\chi(g_k)}{\chi(e)}|\mathcal{O}_k|$$

Multiplying both sides by $\frac{\chi(e)\chi(g_l^{-1})}{|G|}$, using $|\mathcal{O}_l| = \frac{|G|}{|C_G(g_l)|}$ for $l = i,j,k$ and summing over $\chi \in \mathrm{Irr}(G)$ we obtain

$$\frac{|G|}{|C_G(g_i)||C_G(g_j)|}\sum_\chi \frac{\chi(g_i)\chi(g_j)\chi(g_l^{-1})}{\chi(e)} = \sum_{k=1}^r \frac{1}{|C_G(g_k)|}\sum_{\chi\in\mathrm{Irr}(G)}\chi(g_k)\chi(g_l^{-1}) = a_{ijl}$$

by column orthogonality. $\square$

Lecture 18

7.3. **Degrees of irreducibles.**

Theorem. *If V is an irreducible representation of a group G then* $\dim V$ *divides* $|G|$.

Proof. Let χ be the character of V. We'll show that $\frac{|G|}{\chi(e)}$ is an algebraic integer and so, since it is rational, an actual integer by Fact 3 from §7.1.

$$\begin{aligned}\frac{|G|}{\chi(e)} &= \frac{1}{\chi(e)}\sum_{g\in G}\chi(g)\chi(g^{-1})\\ &= \sum_{i=1}^{r}\frac{1}{\chi(e)}|\mathcal{O}_i|\chi(g_i)\chi(g_i^{-1})\\ &= \sum_{i=1}^{r} w_\chi(C_i)\chi(g_i^{-1})\end{aligned}$$

But the set of algebraic integers form a ring (by Fact 1 in §7.1) and each $w_\chi(C_i)$ and each $\chi(g_i^{-1})$ is an algebraic integer so $\frac{|G|}{\chi(e)}$ is an algebraic integer as required. □

Examples.

(1) If G is a p-group and χ is an irreducible character then $\chi(e)$ is always a power of p. In particular if $|G| = p^2$ then, since $\sum_\chi \chi(e)^2 = p^2$, every irreducible rep is 1-dimensional and so G is abelian.
(2) If $G = A_n$ or S_n and $p > n$ is a prime, then p cannot divide the dimension of an irreducible rep.

In fact a stronger result is true:

Theorem (Burnside (1904))**.** *If (ρ, V) is an irreducible representation then* $\dim V$ *divides* $|G/Z(G)|$.

You could compare this with $|\mathcal{O}_i| = |G|/|C_G(g_i)|$ divides $|G/Z(G)|$.

Proof. If $z \in Z = Z(G)$ then by Schur's Lemma $\rho|_Z\colon Z \to GL(V)$ is of the form $\rho(z) = \lambda_z \mathrm{id}_V$ with $\lambda_z \in k$.

For each $m \geqslant 2$, consider the irreducible representation of G^m given by

$$\rho^{\otimes m}\colon G^m \to GL(V^{\otimes m}).$$

If $z = (z_1, \ldots, z_m) \in Z^m$ then z acts on $V^{\otimes m}$ via $\prod_{i=1}^m \lambda_{z_i}\mathrm{id}_V = \lambda_{\prod_1^m z_i}\mathrm{id}_V$. Thus if $\prod_1^m z_i = 1$ then $z \in \ker \rho^{\otimes m}$.

Let $Z' = \{(z_1, \ldots, z_m \in Z^m \mid \prod_{i=1}^m z_i = 1\}$ so $|Z'| = |Z|^{m-1}$. We may view $\rho^{\otimes m}$ as a degree $(\dim V)^m$ irreducible representation of G^m/Z'.

Since $|G^m/Z'| = |G|^m/|Z|^{m-1}$ we can use the previous theorem to deduce that $(\dim V)^m$ divides $|G|^m/|Z|^{m-1}$.

Suppose that p is a prime such that p^a divides $\dim V$. Then p^{am} divides $|G/Z|^m|Z|$. By taking m to be large, in particular so that p^m does not divide $|Z|$, we see that p^a divides $|G/Z|$. Thus $\dim V$ divides $|G/Z|$ as claimed. □

Proposition. *If G is a simple group then G has no irreducible representations of degree* 2.

Proof. If G is cyclic then G has no irreducible representations of degree bigger than 1, so we may assume G is non-abelian.

If $|G|$ is odd then we may apply the theorem above.

If $|G|$ is even then G has an element x of order 2. By example sheet 2 Q2, for every irreducible χ, $\chi(x) \equiv \chi(e) \mod 4$. So if $\chi(e) = 2$ then $\chi(x) = \pm 2$, and $\rho(x) = \pm I$. Thus $\rho(x) \in Z(\rho(G))$, a contradiction since G is non-abelian simple. □

Remark. In 1963 Feit and Thompson published a 255 page paper proving that there is no non-abelian simple group of odd order.

7.4. **Burnside's $p^a q^b$ Theorem.**

Lemma. *Suppose α is an non-zero algebraic integer of the form $\alpha = \frac{1}{m}\sum_{i=1}^m \lambda_i$ with $\lambda_i^n = 1$ for all i. Then $|\alpha| = 1$.*[16]

Sketch proof (non-examinable). By assumption $\alpha \in \mathbb{Q}(\epsilon)$ where $\epsilon = e^{2\pi i/n}$.

Let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$. It is known that $\{\beta \in \mathbb{Q}(\epsilon) \mid \sigma(\beta) = \beta \text{ for all } \sigma \in \mathcal{G}\} = \mathbb{Q}$.

Consider $N(\alpha) := \prod_{\sigma\in\mathcal{G}} \sigma(\alpha)$. Since $N(\alpha)$ is fixed by every element of $\mathcal{G}$, $N(\alpha) \in \mathbb{Q}$. Moreover $N(\alpha)$ is an algebraic integer since Galois conjugates of algebraic integers are algebraic integers — they satisfy the same integer polynomials. Thus $N(\alpha) \in \mathbb{Z}$.

But for each $\sigma \in \mathcal{G}$, $|\sigma(\alpha)| = |\frac{1}{m}\sum \sigma(\lambda_i)| \leqslant 1$. Thus $N(\alpha) = \pm 1$, and $|\alpha| = 1$ as required. □

Lemma. *Suppose χ is an irreducible character of G, and $\mathcal{O}$ is a conjugacy class in G such that $\chi(e)$ and $|\mathcal{O}|$ are coprime. For $g \in \mathcal{O}$, $|\chi(g)| = \chi(e)$ or 0.*

Note if $\chi = \chi_V$ this is saying that either g acts as a scalar on V or $\chi(g) = 0$.

Proof. By Bezout, we can find $a, b \in \mathbb{Z}$ such that $a\chi(e) + b|\mathcal{O}| = 1$. Define

$$\alpha := \frac{\chi(g)}{\chi(e)} = a\chi(g) + b\frac{\chi(g)}{\chi(e)}|\mathcal{O}|$$

Then, since $\chi(g)$ is a sum of $|G|$th roots of unity, α satisfies the conditions of the previous lemma (or is zero) and so this lemma follows. □

Proposition. *If G is a non-abelian finite group with a conjugacy class $\mathcal{O}_i \neq \{e\}$ such that $|\mathcal{O}_i|$ has prime power order then G is not simple.*

Proof. Suppose for contradiction that G is simple and has an element $g \in G\backslash\{e\}$ that lives in a conjugacy class $\mathcal{O}$ of order p^r.

If χ is a non-trivial irreducible character of G then $|\chi(g)| < \chi(1)$ since otherwise $\rho(g)$ is a scalar matrix and so lies in $Z(\rho(G)) \cong Z(G)$.

Thus by the last lemma, for every non-trivial irreducible character, either p divides $\chi(e)$ or $|\chi(g)| = 0$. By column orthogonality,

$$0 = \sum_{\chi} \chi(e)\chi(g).$$

Thus $\frac{-1}{p} = \sum_{\chi\neq\mathbf{1}} \frac{\chi(e)}{p}\chi(g)$ is an algebraic integer in $\mathbb{Q}$. Thus $\frac{1}{p}$ in $\mathbb{Z}$ the desired contradiction. □

Theorem (Burnside (1904))**.** *Let p, q be primes and G a group of order $p^a q^b$ with a, b non-negative integers such that $a + b \geqslant 2$, then G is not simple.*

[16] i.e. all the λ_i are equal.

Proof. If $a, b > 0$, then let Q be a Sylow-q-subgroup of G. Since $Z(Q) \neq 1$ we can find $e \neq g \in Z(Q)$. Then q^b divides $|C_G(g)|$, so the conjugacy class containing g has order p^r for some $0 \leqslant r \leqslant a$. The theorem now follows immediately from the Proposition. □

Remarks.

(1) It follows that every group of order $p^a q^b$ is soluble. That is, there is a chain of subgroups $G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_r = \{e\}$ with G_{i+1} normal in G_i and G_i/G_{i+1} abelian for all i.

(2) Note that $|A_5| = 2^2 \cdot 3 \cdot 5$ so the order of a simple group can have precisely 3 prime factors.

(3) The first purely group theoretic proof of the $p^a q^b$-theorem appeared in 1972.

8. Topological groups

Consider $S^1 = U_1(\mathbb{C}) = \{g \in \mathbb{C}^\times \mid |g| = 1\} \cong \mathbb{R}/\mathbb{Z}$.

By considering $\mathbb{R}$ as a $\mathbb{Q}$-vector space we see that as a group

$$S^1 \cong \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_{x \in X} \mathbb{Q}$$

for an an uncountable set X.

Thus we see that as an abstract group S^1 has uncountably many irreducible representations: for each $x \in X$ we can define a one-dimensional representation by

$$\rho_x(e^{2\pi i\mu}) = \begin{cases} e^{2\pi i \mu} & \mu \in \mathbb{Q}x \\ 1 & \mu \in \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_{y \in X\setminus\{x\}} \mathbb{Q}y \end{cases}.$$

These ρ_x are non-isomorphic so in this way we get uncountably many irreducible representations of S^1 (we haven't listed them all). We don't really have any control over the situation.

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However, S^1 is not just a group; it comes with a topology as a subset of $\mathbb{C}$. Moreover S^1 acts naturally on complex vector spaces in a continuous way.

Definition. A *topological group* G is a group G which is also a topological space such that the multiplication map $G \times G \to G; (g,h) \mapsto gh$ and the inverse map $G \to G; g \mapsto g^{-1}$ are continuous maps.

Examples.

(1) $GL_n(\mathbb{C})$ with topology from $\mathbb{C}^{n^2}$.
(2) G finite — with the discrete topology.
(3) $O(n) = \{A \in GL_n(\mathbb{R}) \mid A^TA = I\}$; $SO(n) = \{A \in O(n) \mid \det A = 1\}$.
(4) $U(n) = \{A \in GL_n(\mathbb{C}) \mid \overline{A^T}A = I\}$; $SU(n) = \{A \in U(n) \mid \det A = 1\}$.
(5) *G profinite such as $\mathbb{Z}_p$, the completion of $\mathbb{Z}$ with respect to the p-adic metric.

Definition. A *representation* of a topological group G on a vector space V is a continuous group homomorphism $G \to GL(V)$.

Remarks.

(1) If X is a topological space then $\alpha\colon X \to GL_n(\mathbb{C})$ is continuous if and only if the maps $x \mapsto \alpha_{ij}(x) = \alpha(x)_{ij}$ are continuous for all i, j.
(2) If G is a finite group with the discrete topology. Then continous function $G \to X$ just means function $G \to X$.

8.1. **Compact Groups.** Our most powerful idea for studying representations of finite groups has been averaging over the group; that is the operation $\frac{1}{|G|}\sum_{g \in G}$. When considering more general topological groups we should replace $\sum$ by $\int$.

Definition. For G a topological group and $C(G,\mathbb{R}) = \{f\colon G \to \mathbb{R} \mid f \text{ is continuous}\}$, a linear map $\int_G\colon C(G,\mathbb{R}) \to \mathbb{R}$ (we write $\int_G f = \int_G f(g)\,\mathrm{d}g$) is called a *Haar integral* if

(i) $\int_G 1 = 1$ (so $\int_G$ is normalised so total volume is 1);
(ii) $\int_G f(xg)\,\mathrm{d}g = \int_G f(g)\,\mathrm{d}g = \int_G f(gx)\,\mathrm{d}g$ for all $x \in G$ (so $\int_G$ is translation invariant).

Note that for any $\mathbb{R}$-vector space $\int_G$ induces a linear map $C(G,V) \to V$; if $v_1, \dots, v_n$ is a basis for V then $f \in C(G,V)$ is uniquely of the form $f = \sum f_i v_i$ with $f_1, \dots, f_n \in C(G,\mathbb{R})$ and $\int_G f = \sum_{i=1}^n \left(\int_G f_i\right) v_i$. Moreover this map is also translation invariant and sends a constant function to its unique value.

Moreover if $\alpha\colon V \to W$ is a linear map and $f \in C(G,V)$ then $\alpha\left(\int_G f\right) = \int_G (\alpha f)$. In particular if V is a $\mathbb{C}$-vector space then $\int_G$ is $\mathbb{C}$-linear.

Examples.
(1) If G finite, then $\int_G f = \frac{1}{|G|} \sum_{g \in G} f(g)$.
(2) If $G = S^1$, $\int_G f = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\,\mathrm{d}\theta$.

Theorem. *If G is a compact Hausdorff group, then there is a unique Haar integral on G.*

Proof. Omitted □

All the examples of topological groups from last time are compact Hausdorff except $GL_n(\mathbb{C})$ which is not compact. We'll follow standard practice and write 'compact group' instead of 'compact Hausdorff group'.

Corollary (Weyl's Unitary Trick)**.** *If G is a compact group then every representation (ρ, V) has a G-invariant invariant Hermitian inner product.*

Proof. Same as for finite groups: let $\langle -, - \rangle$ be any inner product on V, then

$$(v, w) = \int_G \langle \rho(g)v, \rho(g)w \rangle \,\mathrm{d}g$$

is the required G-invariant inner product since

$$(\rho(h)v, \rho(h)w) = \int_G \langle \rho(gh)v, \rho(gh)w \rangle \,\mathrm{d}g = (v, w)$$

for $v, w \in V$. Checking that $(-,-)$ is an inner product is straightforward. □

Thus every representation of a compact group is equivalent to a unitary representation.

Corollary (Maschke's Theorem)**.** *If G is a compact group and V is a representation of G then every subrepresentation of V has a G-invariant complement. Thus G is completely reducible.*

We can use the Haar integral to put an inner product on the space $\mathcal{C}_G$ of (continuous) class functions:

$$\langle f, f' \rangle := \int_G \overline{f(g)} f'(g) \,\mathrm{d}g.$$

If $\rho\colon G \to GL(V)$ is a representation then $\chi_\rho := \operatorname{tr} \rho$ is a continuous class function since each $\rho(g)_{ii}$ is continuous.

Corollary (Orthogonality of Characters)**.** *If G is a compact group and V and W are irreducible reps of G then*

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Proof. Same as for finite groups:

$$\begin{aligned}\langle \chi_V, \chi_W\rangle &= \int_G \overline{\chi_V(g)}\chi_W(g)\,\mathrm{d}g \\ &= \dim \mathrm{Hom}_G(\mathbf{1}, \mathrm{Hom}(V, W)) \\ &= \dim \mathrm{Hom}_G(V, W).\end{aligned}$$

Then apply Schur's Lemma.

Note along the way we require that $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ which follows from the fact that we may assume that $\rho_V(G) \subset U(V)$ and so the eigenvalues of $\rho_V(g)$ are contained in S^1 for all $g \in G$.

We also need to define a projection map $\pi\colon U \to U^G$ for $U = \mathrm{Hom}_k(V, W)$. For this we define $\pi = \int_G \rho \in \mathrm{Hom}_k(U, U)$: if $u \in U^G$ then

$$\left(\int_G \rho_U\right)(u) = \int_G \rho_U(g)u\,\mathrm{d}g = u$$

and

$$\rho_U(h)\pi = \int_G \rho_U(h)\rho_U(g)\,\mathrm{d}g = \int_G \rho_U(hg)\,\mathrm{d}g = \pi.$$

Moreover $\mathrm{tr}\,\pi = \chi_U$. □

It is also possible to make sense of 'the characters are a basis for the space of class functions' but this requires a little knowledge of Hilbert spaces.

8.2. **A worked example: S^1.**

Theorem. *Every one dimensional (cts) representation of S^1 is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.*

First we need to prove a couple of Lemmas.

Lemma. *If $\psi\colon (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a continous group homomorphism then there is some $\lambda \in \mathbb{R}$ such that $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$.*

Proof. Let $\lambda = \psi(1)$. Since ψ is a group homomorphism, $\psi(n) = \lambda n$ for all $n \in \mathbb{Z}$. Then m$\psi(n/m) = \psi(n) = \lambda n$ and so $\psi(n/m) = \lambda n/m$. That is $\psi(x) = \lambda x$ for all $x \in \mathbb{Q}$. But $\mathbb{Q}$ is dense in $\mathbb{R}$ and ψ is continuous so $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$. □

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Lemma. *If $\psi\colon (\mathbb{R}, +) \to S^1$ is a continuous group homomorphism then $\psi(x) = e^{2\pi i\lambda x}$ for some $\lambda \in \mathbb{R}$.*

Proof. **Claim:** if $\psi\colon \mathbb{R} \to S^1$ is any continuous function with $\psi(0) = 1$ then there is a unique continuous function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\psi(x) = e^{2\pi i\alpha(x)}$.[17] (Sketch proof of claim: locally $\alpha(x) = \frac{1}{2\pi i}\log\psi(x)$ and we can choose the branches of log to make the pieces glue together continuously).

Now given the claim, if ψ is a group homomorphism and α is the map defined by the claim we can define a continuous function $\mathbb{R}^2 \to \mathbb{R}$ by

$$\Delta(a, b) := \alpha(a + b) - \alpha(a) - \alpha(b).$$

[17] In the language of algebraic topology $\mathbb{R} \to S^1$; $x \mapsto e^{2\pi ix}$ is a covering map and so paths in S^1 lift uniquely to paths in $\mathbb{R}$ after choosing the lift of the starting point. In fact $\mathbb{R}$ is the universal cover of S^1 via this map.

Since $e^{2\pi i\Delta(a,b)} = \psi(a+b)\psi(a)^{-1}\psi(b)^{-1} = 1$, Δ only takes values in $\mathbb{Z}$. Thus as $\mathbb{R}^2$ is connected, Δ is constant. Since $\Delta(0,0) = 0$ we see that $\Delta \equiv 0$ and so α is a group homomorphism. By the previous lemma we can deduce that there is $\lambda \in \mathbb{R}$ such that $\alpha(x) = \lambda x$ for all $x \in \mathbb{R}$ and so $\psi(x) = e^{2\pi i\lambda x}$ as required. □

Theorem. *Every irreducible representation of S^1 has degree 1 and is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.*

Proof. Since S^1 is abelian, by Schur's Lemma every simple representation (ρ, V) of S^1 has degree 1 — each $\rho(g) \in \mathrm{Hom}_{S^1}(V,V)$ and so is a scalar endomorphism.

Let $\rho\colon S^1 \to GL_1(\mathbb{C})$ be a continuous homomorphism. Since S^1 is compact, $\rho(S^1)$ has closed and bounded image. Since $\rho(z^n) = \rho(z)^n$ for $n \in \mathbb{Z}$, it follows that $\rho(S^1) \subset S^1$.

Now let $\psi\colon \mathbb{R} \to S^1$ be defined by $\psi(x) = \rho(e^{2\pi ix})$, a continuous homomorphism. By the most recent Lemma, $\rho(e^{2\pi ix}) = \psi(x) = e^{2\pi i\lambda x}$ for some $\lambda \in \mathbb{R}$.

Since also $\rho(e^{2\pi i}) = 1$ we see $\lambda \in \mathbb{Z}$. □

The theorem tell us that the 'character table' of S^1 has rows χ_n indexed by $\mathbb{Z}$ with $\chi_n(e^{i\theta}) = e^{in\theta}$.

Notation. *Let*

$$\mathbb{N}_0[z,z^{-1}] := \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathbb{N}_0 \textit{ with } \sum_{n\in\mathbb{N}_0} a_n < \infty \right\}$$

Now if V is any rep of S^1 then by Machke's Theorem V breaks up as a direct sum of one dimensional subreps and so its character $\chi_V = \sum a_n z^n$ lies in $\mathbb{N}_0[z,z^{-1}]$ with $\sum a_n = \dim V$. As usual a_n is the number of copies of $\rho_n\colon z \mapsto z^n$ in the decomposition of V as a direct sum of simple subrepresentations. Thus we can compute

$$a_n = \langle \chi_n, \chi_V \rangle_{S^1} = \frac{1}{2\pi}\int_0^{2\pi} \chi_V(e^{i\theta})e^{-in\theta}\,\mathrm{d}\theta.$$

Thus

$$\chi_V(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi}\int_0^{2\pi} \chi_V(e^{i\phi})e^{-in\phi}\,\mathrm{d}\phi \right) e^{in\theta}.$$

So Fourier decomposition gives the decomposition of χ_V into irreducible characters and the Fourier mode is the multiplicity of an irreducible character.

Remark. In fact by the theory of Fourier series any continuous function on S^1 can be uniformly approximated by a finite $\mathbb{C}$-linear combination of the χ_n.

Moreover the χ_n form a complete orthonormal set in the Hilbert space

$$L^2(S^1) = \left\{ f\colon S^1 \to \mathbb{C} \mid \int_0^{2\pi} |f(e^{i\theta})|^2\,\mathrm{d}\theta \text{ exists and is finite} \right\}$$

of square-integrable complex-valued functions on S^1. That is every function $f \in L^2(S^1)$ has a unique series expansion

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta'})e^{-in\theta'}\,\mathrm{d}\theta' \right) e^{in\theta}$$

converging with respect to the norm $||f|| = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\,\mathrm{d}\theta$.

We can phrase this as

$$L^2(S^1) = \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}\chi_n{}^{18}$$

which is an analogue of

$$kG = \bigoplus_{V\in \mathrm{Irr}(G)} V^{\dim V}$$

for finite groups.

8.3. **Second worked example:** $SU(2)$**.**

Recall that $SU(2) = \{A \in GL_2(\mathbb{C}) \mid \overline{A^T}A = I, \det A = 1\}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ then since $\det A = 1$, $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Thus $d = \overline{a}$ and $c = -\overline{b}$. Moreover $a\overline{a} + b\overline{b} = 1$. In this way we see that

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}$$

which may be viewed topologically as $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$.

More precisely if

$$\mathbb{H} := \mathbb{R}\cdot SU(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid w, z \in \mathbb{C} \right\} \subset M_2(\mathbb{C}).$$

Then $||A||^2 = \det A$ defines a norm on $\mathbb{H} \cong \mathbb{R}^4$ and $SU(2)$ is the unit sphere in $\mathbb{H}$. If $A \in SU(2)$ and $X \in \mathbb{H}$ then $||AX|| = ||X||$ since $||A|| = 1$. So, after normalisation, usual integration of functions on S^3 defines a Haar integral on $SU(2)$. i.e.

$$\int_{SU(2)} f = \frac{1}{2\pi^2}\int_{S^3} f.$$

Here $2\pi^2$ is the volume of S^3 in $\mathbb{R}^4$ with respect to the usual measure.

We now try to compute the conjugacy classes in $SU(2)$.

Definition. Let $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}, |a| = 1 \right\} \cong S^1$, a *maximal torus* in $SU(2)$.

Also define $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$

Lemma.
(i) if $t \in T$ then $sts^{-1} = t^{-1}$;
(ii) $s^2 = -I \in Z(SU(2))$
(iii) $N_{SU(2)}(T) = T \cup sT = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \mid a \in \mathbb{C}, |a| = 1 \right\}$

Proof. All three parts follow from direct computation (exercise). □

Proposition.
(i) Every conjugacy class $\mathcal{O}$ in SU_2 contains an element of T.
(ii) More precisely. if $\mathcal{O}$ is a conjugacy class then $\mathcal{O}\cap T = \{t, t^{-1}\}$ for some $t \in T$ — $t = t^{-1}$ if and only if $t = \pm I$ when $\mathcal{O} = \{t\}$.

[18] $\widehat{\bigoplus}$ is supposed to mean a completed direct sum or more precisely a direct sum in the category of Hilbert spaces.

(iii) There is a bijection

$$\{\textit{conjugacy classes in } SU(2)\} \to [-1,1]$$

given by $A \mapsto \frac{1}{2}\operatorname{tr} A$.

Proof. (i) For every unitary matrix has an orthonormal basis of eigenvectors. So if $A \in \mathcal{O}$ there is a unitary matrix P such that PAP^{-1} is diagonal. We want to arrange that $\det P = 1$. But we can replace P by $Q = \frac{1}{\sqrt{\det P}}P$. Thus every conjugacy class $\mathcal{O}$ in $SU(2)$ contains a diagonal matrix t. Since additionally $t \in SU(2)$, $t \in T$.

(ii) If $\pm I \in \mathcal{O}$ the result is clear.

Suppose $t \in \mathcal{O} \cap T$ for some $t \neq \pm I$. Then

$$\mathcal{O} = \{gtg^{-1} \mid g \in SU(2)\}.$$

We've seen before that $sts^{-1} = t^{-1}$ so $\mathcal{O} \cap T \supset \{t, t^{-1}\}$.

Conversely, if $t' \in \mathcal{O} \cap T$ then t' and t must have the same eigenvalues since they are conjugate. This suffices to see that $t' \in \{t^{\pm 1}\}$.

(iii) To see the given function is injective, suppose that $\frac{1}{2}\operatorname{tr} A = \frac{1}{2}\operatorname{tr} B$. Then since $\det A = \det B = 1$, A and B must have the same eigenvalues. By part (i) they are both diagonalisable and by the proof of part (ii) this suffices to see that they are conjugate.

To see that it is surjective notice that $\frac{1}{2}\operatorname{tr}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} = \cos\theta$. Since $\cos\colon \mathbb{R} \to \mathbb{R}$ has image $[-1,1]$ the given function is surjective. □

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Let's write $\mathcal{O}_x = \{A \in SU(2) \mid \frac{1}{2}\operatorname{tr} A = x\}$ for $x \in [-1,1]$. We've proven that the $\mathcal{O}_x$ are the conjugacy classes in $SU(2)$. Clearly $\mathcal{O}_1 = \{I\}$ and $\mathcal{O}_{-1} = \{-I\}$. For $-1 < x < 1$ there is some $\theta \in (0,\pi)$ such that $\cos\theta = x$ then

$$\mathcal{O}_x = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid (\operatorname{Im} a)^2 + |b|^2 = \sin^2\theta \right\}$$

since $Re\, a = x = \cos\theta$. That is $\mathcal{O}_x$ is a 2-sphere of radius $|\sin\theta|$.

8.4. Representations of $SU(2)$.

Let V_n be the complex vector space of homogeneous polynomials in two variables x, y. So $\dim V_n = n+1$. Then $GL(\mathbb{C}^2)$ acts on V_n via

$$\rho_n \colon GL(\mathbb{C}^2) \to GL(V_n)$$

given by

$$\rho_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(x,y) = f(ax+cy, bx+dy).$$

Examples.

$V_0 = \mathbb{C}$ has the trivial action.

$V_1 = \mathbb{C}^2$ is the standard representation of $GL(\mathbb{C}^2)$ on $\mathbb{C}^2$ with basis x, y.

$V_2 = \mathbb{C}^3$ has basis x^2, xy, y^2 then

$$\rho_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

In general $V_n \cong S^n V_1$ as representations of $GL_2(\mathbb{C})$.

Since $SU(2)$ is a subgroup of $GL_2(\mathbb{C})$ we can view V_n as a representation of $SU(2)$ by restriction. In fact as we'll see, the V_n are all irreducible reps of $SU(2)$ and every irreducible rep of $SU(2)$ is isomorphic to one of these.

Lemma. *A (continuous) class function $f\colon SU(2) \to \mathbb{C}$ is determined by its restriction to T and $f|_T$ is even ie* $f\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right) = f\left(\begin{pmatrix} z^{-1} & 0 \\ 0 & z\end{pmatrix}\right)$.

Proof. We've seen that each conjugacy class in $SU(2)$ meets T and so a class fucntion is determined by its restriction to T. Then evenness follows from the additional fact that $T \cap \mathcal{O} = \{t^{\pm 1}\}$ for some $t \in T$. $\square$

Let $\mathbb{N}[z, z^{-1}]^{ev} = \{f \in \mathbb{N}[z, z^{-1}] \mid f(z) = f(z^{-1})\}$.

Lemma. *If χ is a character of a representation of $SU(2)$ then $\chi|_T \in \mathbb{N}_0[z, z^{-1}]^{ev}$.*

Proof. If V is a representation of $SU(2)$ then $\operatorname{Res}_T^{SU(2)} V$ is a representation of T and $\chi_{\operatorname{Res}_T V}$ is the restriction of χ_V to T. Since every character of T is in $\mathbb{N}_0[z, z^{-1}]$ and $\chi|_T$ is even we're done. $\square$

The next thing to do is compute the character $\chi_{V_n}|_T$ of (ρ_n, V_n), the representation consisting of degree n homogeneous polynomials in x and y.

$$\rho_n\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right)(x^i y^j) = (zx)^i (z^{-1}y)^j = z^{i-j} x^i y^j.$$

So $\{x^j y^{n-j} | 0 \leqslant j \leqslant n\}$ is an eigenbasis for V_n with respect to the T-action and

$$\chi_{V_n}\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \in \mathbb{N}[z, z^{-1}]^{ev}.$$

Theorem. *V_n is irreducible as a reperesentation of $SU(2)$.*

Proof. Let $0 \neq W \leqslant V_n$ be a $SU(2)$-invariant subspace. We want to show that $W = V_n$.

W is T-invariant so as $\operatorname{Res}_T^{SU(2)} V_n = \bigoplus_{j=0}^n \mathbb{C} x^j y^{n-j}$ is a direct sum of non-isomorphic representations of T,

(3) $\qquad W$ has as a basis a subset of $\{x^j y^{n-j} \mid 0 \leqslant j \leqslant n\}$.

Thus $x^j y^{n-j} \in W$ for some $0 \leqslant j \leqslant n$. Since

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix} x^j y^{n-j} = \frac{1}{\sqrt{2}}((x-y)^j (x+y)^{n-j}) \in W$$

so by (3) we can deduce that $x^n \in W$. Repeating the same calculation for $i = n$, we see that $(x+y)^n \in W$ and so, by (3) again, $x^i y^{n-i} \in W$ for all i.

Thus $W = V_n$. $\square$

Alternative proof:

We've seen that $\mathcal{O}_{\cos\theta} = \{A \in SU(2) \mid \frac{1}{2}\operatorname{tr} A = \cos\theta\}$ with the two-sphere $\{(\operatorname{Im}(a))^2 + |b|^2 = \sin^2\theta\}$ of radius $|\sin\theta|$. Thus if f is a class-function on $SU(2)$, since f is constant on each $\mathcal{O}_{\cos\theta}$,

$$\int_{SU(2)} f(g)\,\mathrm{d}g = \frac{1}{2\pi^2}\int_0^\pi f\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\right) 4\pi \sin^2\theta\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\sin^2\theta\,\mathrm{d}\theta.$$

Note this is normalised correctly, since $\frac{1}{\pi}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = 1$. So it suffices to prove that $\frac{1}{\pi}\int_0^{2\pi}|\chi_{V_n}(e^{i\theta})|^2\sin^2\theta\,\mathrm{d}\theta = 1$ for $z = e^{i\theta}$. (exercise: verify this).

Theorem. *Every irreducible representation of $SU(2)$ is isomorphic to V_n for some $n \geqslant 0$.*

Proof. Let V be an irreducible representation of $SU(2)$ so $\chi_V \in \mathbb{N}[z, z^{-1}]^{ev}$. Now $\chi_0 = 1, \chi_1 = z + z^{-1}, \chi_2 = z^2 + 1 + z^{-2}, \ldots$ form a basis of $\mathbb{Q}[z, z^{-1}]^{ev}$ as (non-f.d.) $\mathbb{Q}$-vector spaces. Thus $\chi_V = \sum a_i\chi_i$ for some $a_i \in \mathbb{Q}$, only finitely many non-zero.

Clearing denominators and moving negative terms to the left-hand-side, we get a formula

$$m\chi_V + \sum_{i\in I} m_i\chi_i = \sum_{j\in J} m_j\chi_j$$

for some disjoint finite subsets $I, J \subset \mathbb{N}$ and $m, m_i \in \mathbb{N}$. By orthogonality of characters and complete reducibility we obtain

$$mV \oplus \bigoplus_{i\in I} m_iV_i \cong \bigoplus_{j\in J} m_jV_j$$

since V is irreducible, $V \cong V_j$ some $j \in J$. □

8.5. **Tensor products of representations of $SU(2)$.** We've seen that if V, W are representations of $SU(2)$ such that $\mathrm{Res}_T^{SU(2)} V \cong \mathrm{Res}_T^{SU(2)} W$ then $V \cong W$. We want to understand $\otimes$ for representations of $SU(2)$.

Recall that if G is a group and V, W are representations of G then $\chi_{V\otimes W} = \chi_V\chi_W$.

Let's compute some examples for $SU(2)$:

$$\chi_{V_1\otimes V_1}(z) = (z + z^{-1})^2 = z^2 + 1 + z^{-2} + 1 = \chi_{V_2} + \chi_{V_0}$$

and

$$\chi_{V_2\otimes V_1}(z) = (z^2 + 1 + z^{-2})(z + z^{-1}) = z^3 + 2z + 2z^{-1} + z^{-3} = \chi_{V_3} + \chi_{V_1}.$$

Proposition (Clebsch–Gordan rule)**.** *For $n, m \in \mathbb{N}$,*

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|+2} \oplus V_{|n-m|}.$$

Proof. Without loss of generality, $n \geqslant m$. Then

$$\begin{aligned}(\chi_n \cdot \chi_m)(z) &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \cdot (z^m + z^{m-2} + \cdots + z^{-m}) \\ &= \sum_{j=0}^{m} \frac{z^{n+m+1-2j} - z^{-(n+m+1-2j)}}{z - z^{-1}} \\ &= \sum_{j=0}^{m} \chi_{n+m-2j}(z)\end{aligned}$$

as required. □

Lecture 22

8.6. **Representations of $SO(3)$.**

Proposition. *There is an isomorphism of topological groups $SU(2)/\{\pm I\} \cong SO(3)$.*

Proof. See Example Sheet 4 Q4.[19] □

[19] If you get stuck then consult my notes from 2012 for some hints.

Corollary. *Every irreducible representation of $SO(3)$ is of the form V_{2n} for some $n \geqslant 0$.*

Proof. It follows from the Proposition that irreducible representations of $SO(3)$ correspond to irreducible representations of $SU(2)$ such that $-I$ acts trivially. But it is easy to verify that $-I$ acts on V_n as $(-1)^n$ □

9. Character table of $GL_2(\mathbb{F}_q)$

9.1. $\mathbb{F}_q$**.** Let $p > 2$ be a prime, $q = p^a$ a power of p for some $a > 0$, and $\mathbb{F}_q$ be the field with q elements. We know that $\mathbb{F}_q^\times \cong C_{q-1}$.

Notice that $\mathbb{F}_q^\times \to \mathbb{F}_q^\times; x \mapsto x^2$ is a group homomorphism with kernel ± 1. Thus half the elements of $\mathbb{F}_q^\times$ are squares and half are not. Let $\epsilon \in \mathbb{F}_q^\times$ be a fixed non-square and let $\mathbb{F}_{q^2} := \{a+b\sqrt{\epsilon} \mid a, b \in \mathbb{F}_q\}$, the field extension of $\mathbb{F}_q$ with q^2 elements under the obvious operations.

Every element of $\mathbb{F}_q$ has a square root in $\mathbb{F}_{q^2}$ since if λ is non-square then $\lambda/\epsilon = \mu^2$ is a square, and $(\sqrt{\epsilon}\mu)^2 = \lambda$. It follows by completing the square that every quadratic polynomial in $\mathbb{F}_q$ factorizes in $\mathbb{F}_{q^2}$.[20]

Notice that $(a + b\sqrt{\epsilon})^q = a^q + b^q\epsilon^{\frac{q-1}{2}}\sqrt{\epsilon} = (a - b\sqrt{\epsilon})$. Thus the roots of an irreducible quadratic over $\mathbb{F}_q$ are of the form λ, λ^q.

9.2. $GL_2(\mathbb{F}_q)$ **and its conjugacy classes.** We want to compute the character table of the group

$$G := GL_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}_q \text{ and } ad - bc \neq 0 \right\}.$$

The order of G is the number of bases for $\mathbb{F}_q^2$ over $\mathbb{F}_q$. This is $(q^2 - 1)(q^2 - q)$.

First, we compute the conjugacy classes in G. We know from linear algebra that 2×2-matrices are determined by their minimal polynomials up to conjugation. By Cayley–Hamilton each element A of $GL_2(\mathbb{F}_q)$ has minimal polynomial $m_A(X)$ of degree at most 2 and $m_A(0) \neq 0$.

There are four cases.

Case 1: $m_A = X - \lambda$ for some $\lambda \in {\mathbb{F}_q}^\times$. Then $A = \lambda I$. So $C_G(A) = G$, and $|[A]|_G = 1$. There are $q - 1$ such classes corresponding the possible choices of λ.

Case 2: $m_A = (X - \lambda)^2$ for some $\lambda \in {\mathbb{F}_q}^\times$ so A is conjugate to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Now

$$C_G\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{F}_q, a \neq 0 \right\}$$

so

$$|[A]|_G = \frac{(q-1)^2(q+1)q}{(q-1)q} = (q-1)(q+1).$$

There are $q - 1$ such classes.

Case 3: $m_A = (X - \lambda)(X - \mu)$ for some distinct $\lambda, \mu \in {\mathbb{F}_q}^\times$. Then A is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and to $\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}$. Moreover

$$C_G\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{F}_q^\times \right\} =: T.$$

So

$$|[A]_G| = \frac{q(q-1)(q^2-1)}{(q-1)^2} = q(q+1).$$

There are $\binom{q-1}{2}$ corresponding to each possible choice of the pair $\{\lambda, \mu\}$.

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[20] $\lambda \mapsto \lambda^q$ should be viewed as an analogue of complex conjugation.

Case 4: $m_A(X)$ is irreducible over $\mathbb{F}_q$ of degree 2 so $(X-\alpha)(X-\alpha^q) \in \mathbb{F}_{q^2}[X]$, $\alpha = \lambda + \mu\sqrt{\epsilon}$ with $\lambda, \mu \in \mathbb{F}_q$, $\mu \neq 0$. Then A is conjugate to $\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}$ and $\begin{pmatrix} \lambda & -\epsilon\mu \\ -\mu & \lambda \end{pmatrix}$. Now

$$C_G\left(\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}\right) = \left\{\begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} \mid a^2 - \epsilon b^2 \neq 0\right\} =: K.$$

If $a^2 = \epsilon b^2$ then ϵ is a square or $a = b = 0$. So $|K| = q^2 - 1$ and $\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}$ so

$$|[A]_G| = \frac{q(q-1)(q^2-1)}{q^2-1} = q(q-1).$$

There are $q(q-1)/2$ such classes corresponding to the choices of the pair $\{\alpha, \alpha^q\}$.

In summary

Representative A	C_G	$\lvert[A]_G\rvert$	No of such classes
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	G	1	$q-1$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$	$(q-1)(q+1)$	$q-1$
$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	T	$q(q+1)$	$\binom{q-1}{2}$
$\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}$	K	$q(q-1)$	$\binom{q}{2}$

The groups T and K are both *maximal tori.* That is they are maximal subgroups of G subject to the fact that they are conjugate to a subgroup of the group of diagonal matrices over some field extension of $\mathbb{F}_q$. T is called *split* and K is called *non-split.*

Some other important subgroups of G are Z which is the subgroup of scalar matrices (the centre); $N := \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q\right\}$ a Sylow p-subgroup of G; and $B := \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid b \in \mathbb{F}_q, a, d \in \mathbb{F}_q^\times\right\}$ a *Borel subgroup* of G. Then N is normal in B and $B/N \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times \cong C_{q-1} \times C_{q-1}$.

G acts transitively on $\mathbb{F}_q \cup \{\infty\}$ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \begin{cases} \frac{az+b}{cz+d} & \text{for } z \in \mathbb{F}_q \text{ and } cd + z \neq 0 \\ a/c & \text{for } z = \infty \text{ and } c \neq 0 \\ \infty & \text{for } (c = 0 \text{ and } z = \infty)1 \text{ or } (c \neq 0 \text{ and } z = -d/c). \end{cases}$$

so $B = \mathrm{Stab}_G(\infty)$.[21]

Writing $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we see that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} s \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a + b\beta \\ d & \beta d \end{pmatrix}$$

[21]Thus $|G| = |B|(q+1) = q(q-1)^2(q+1)$.

and these elements are all distinct. Hence BsN contains $q|B|$ elements so must be $G\backslash B$. Thus $BsN = BsB$ and $B\backslash G/B$ has two double cosets B and BsB (this is called Bruhat decomposition).

By Mackey's irreduciblity criterion it follows that if W is an irreducible representation of B, then $\mathrm{Ind}_B^G W$ is an irreducible representation of G precisely if $\mathrm{Res}^B_{B\cap {}^sB} W$ and $\mathrm{Res}^{{}^sB}_{B\cap {}^sB} {}^sW$ have no irreducible factors in common. Since s swaps $0, \infty \in \mathbb{F}_q \cup \{\infty\}$,

$$ {}^sB = \mathrm{Stab}_G(0) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a, d \in \mathbb{F}_q{}^\times, c \in \mathbb{F}_q \right\} $$

and $B \cap {}^sB = T$.

Lecture 23

9.3. The character table of B. Let's warm ourselves up by computing the character table of B.

Recall

$$ B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid b \in \mathbb{F}_q, a, d \in \mathbb{F}_q{}^\times \right\} $$

and

$$ N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q \right\} \lhd B \leqslant G = GL_2(\mathbb{F}_q). $$

$G = B \coprod BsB$ where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B/N \cong \mathbb{F}_q{}^\times \times \mathbb{F}_q{}^\times \cong C_{q-1} \times C_{q-1}$.

If $x, y \in B$ are conjugate in G then because $G = B \cup BsB$ either x is conjugate to y in B or $x = (b_2sb_1)y(b_2sb_1)^{-1}$ for some $b_1, b_2 \in B$ (or both). Thus $[x]_G \cap B$ splits into at most two conjugacy classes in B.

The conjugacy classes in B are

Representative	C_B	No of elts	No of such classes
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	B	1	$q-1$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	ZN	$q-1$	$q-1$
$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	T	q	$(q-1)(q-2)$

Now $B/N \cong T \cong \mathbb{F}_q{}^\times \times \mathbb{F}_q{}^\times$. So if $\Theta_q := \{\text{reps } \theta \colon \mathbb{F}_q^\times \to \mathbb{C}^\times\}$, then Θ_q is a cyclic group of order $q-1$ under pointwise operations. Moreover, for each pair $\theta, \phi \in \Theta_q$, we have a 1-dimensional representation of B given by

$$ \chi_{\theta,\phi}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \theta(a)\phi(d) $$

giving $(q-1)^2$ linear reps.

Suppose $\gamma \colon (\mathbb{F}_q, +) \to \mathbb{C}^\times$ is a degree 1 representation and $\theta \in \Theta_q$, we can define a 1-dimensional representation of ZN by

$$ \rho_{\theta,\gamma}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \theta(a)\gamma(a^{-1}b). $$

We see that

$$\begin{aligned}
\operatorname{Ind}_{ZN}^{B} \rho_{\theta,\gamma}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) &= (q-1)\theta(\lambda),\\
\operatorname{Ind}_{ZN}^{B} \rho_{\theta,\gamma}\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) &= \sum_{b\in\mathbb{F}_q{}^\times}\theta(\lambda)\gamma(b)\\
&= \theta(\lambda)\left(\sum_{b\in\mathbb{F}_q}\gamma(b)\right)-\theta(\lambda)\\
&= \theta(\lambda)(q\langle\mathbf{1},\gamma\rangle_{\mathbb{F}_q}-1)\\
&= \begin{cases}-\theta(\lambda) & \text{if } \gamma\neq\mathbf{1}\\ (q-1)\theta(\lambda) & \text{if } \gamma=\mathbf{1}\end{cases}\\
\operatorname{Ind}_{ZN}^{B} \rho_{\theta,\gamma}\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) &= 0
\end{aligned}$$

Let $\mu_\theta := \operatorname{Ind}_{ZN}^{B}\rho_{\theta,\gamma}$ for $\gamma\neq\mathbf{1}$ noting that this does not then depend on γ. Now

$$\langle\mu_\theta,\mu_\theta\rangle = \frac{1}{q(q-1)^2}\big((q-1)(q-1)^2+(q-1)(q-1)1+0\big)=1$$

so each μ_θ is irreducible and we have $(q-1)$ irreducible representations of degree $q-1$. Thus the character table of B is

	$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$
$\chi_{\theta,\phi}$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu)$
μ_θ	$(q-1)\theta(\lambda)$	$-\theta(\lambda)$	0

We note in passing that the 0 in the bottom right corner appears in $q-1$ rows and $(q-1)(q-2)$ columns. But they are forced to be 0 by a Lemma in §7.4 since the order of these conjugacy classes are all q, the degree of the irreducible representations are all $(q-1)$ which is coprime to q, and these elements don't act by scalars.

We also note that $B = Z\times\left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\mid a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q\right\}$ and the second factor is a Frobenius group. So Example Sheet 3 Q10, together with our construction of irreducible representations of a direct product as the tensor product of the irreducible representations of the factors, tells us that we should expect to be able to construct all the irreducible representation of B in the manner that we have done so.

9.4. **The character table of G.** Let's start computing some representations of G.

As $\det\colon G\to\mathbb{F}_q{}^\times$ is a surjective group homomorphism, for each $\theta\in\Theta_q$ gives a 1-dimensional representation of G via $\chi_\theta := \theta\circ\det$.

Let's continue by inducing $\chi_{\theta,\phi}$ from B to G. Notice that

$$^{s}\chi_{\theta,\phi}\left(\begin{pmatrix}a & 0\\ c & d\end{pmatrix}\right) = \chi_{\theta,\phi}\left(\begin{pmatrix}d & c\\ 0 & a\end{pmatrix}\right) = \theta(d)\phi(a)$$

and so $\operatorname{Res}_T^{{}^sB} {}^s\chi_{\theta,\phi} = \operatorname{Res}_T^B \chi_{\theta,\phi}$ if and only if $\theta = \phi$. So $W_{\theta,\phi} := \operatorname{Ind}_B^G \chi_{\theta,\phi}$ is irreducible precisely if $\theta \neq \phi$.

Now

$$\begin{aligned}
\chi_{W_{\theta,\phi}}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) &= (q+1)\theta(\lambda)\phi(\lambda),\\
\chi_{W_{\theta,\phi}}\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) &= \theta(\lambda)\phi(\lambda),\\
\chi_{W_{\theta,\phi}}\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) &= \theta(\lambda)\phi(\mu)+\phi(\lambda)\theta(\mu) \text{ and}\\
\chi_{W_{\theta,\phi}}\left(\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\right) &= 0.
\end{aligned}$$

Notice that $W_{\theta,\phi} \cong W_{\phi,\theta}$ so we get $\binom{q-1}{2}$ irreducible representations in this way. They are known as *principal series representations.*

We consider also $W_{\mathbf{1},\mathbf{1}} \cong \operatorname{Ind}_B^G \mathbf{1} = \mathbb{C}(\mathbb{F}_q \cup \{\infty\})$. Since G acts 2-transitively on $\mathbb{F}_q \cup \infty$, $W_{\mathbf{1},\mathbf{1}}$ decomposes as $\mathbf{1} \oplus V_{\mathbf{1}}$, with $V_{\mathbf{1}}$ irreducible of degree q. This representation is known as the *Steinberg representation.*

By tensoring $W_{\mathbf{1},\mathbf{1}}$ by χ_θ we also obtain $W_{\theta,\theta} \cong \chi_\theta \oplus V_\theta$ with V_θ irreducible of degree q.

So far we have

# classes	$q-1$ $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$	$q-1$ $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$	$\binom{q-1}{2}$ $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$	$\binom{q}{2}$ $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$	# of reps
χ_θ	$\theta(\lambda)^2$	$\theta(\lambda)^2$	$\theta(\lambda)\theta(\mu)$	$\theta(\lambda^2-\epsilon\mu^2)$	$q-1$
V_θ	$q\theta(\lambda)^2$	0	$\theta(\lambda)\theta(\mu)$	$-\theta(\lambda^2-\epsilon\mu^2)$	$q-1$
$W_{\theta,\phi}$	$(q+1)\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu)+\phi(\lambda)\theta(\mu)$	0	$\frac{(q-1)(q-2)}{2}$

We have explicitly constructed all these representations i.e. not just their characters. We have $\binom{q}{2}$ characters to go. It will turn out that they are indexes by irreducible representations φ of K such that $\varphi \neq \varphi^q$ but we won't we able to explicitly construct the representation.

Lecture 24

The next natural thing to do is compute $\operatorname{Ind}_B^G \mu_\theta$. It has character given by

$$\begin{aligned}
\operatorname{Ind}_B^G \mu_\theta\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) &= (q+1)(q-1)\theta(\lambda),\\
\operatorname{Ind}_B^G \mu_\theta\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) &= -\theta(\lambda),\\
\operatorname{Ind}_B^G \mu_\theta\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) &= 0 \text{ and}\\
\operatorname{Ind}_B^G \mu_\theta\left(\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\right) &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}\langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_B^G \mu_\theta \rangle =& \frac{1}{|G|}\left((q+1)^2(q-1)^2(q-1) + (q-1)(q^2-1)\right) \\ =& \frac{1}{q}\left(q^2-1)+1\right) = q\end{aligned}$$

so $\operatorname{Ind}_B^G \mu_\theta$ has many irreducible factors.

Our next strategy is to induce characters from K. We write $\alpha = \lambda + \mu\sqrt{\epsilon}$ for the matrix $\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}$. Notice that $Z \leqslant K$ with $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda$ in our new notation.

Suppose that $\varphi\colon K \to \mathbb{C}^\times$ is a 1-dimensional character of K. Then $\Phi := \operatorname{Ind}_K^G \varphi$ has character given by $\Phi(\lambda) = q(q-1)\varphi(\lambda)$, $\Phi(\alpha) = \varphi(\alpha) + \varphi(\alpha^q)$ for $\alpha \in \mathbb{F}_{q^2}^\times$ and $\Phi = 0$ away from these conjugacy classes.

Let's compute

$$\langle \Phi, \Phi \rangle = \frac{1}{|G|}\left((q-1)q^2(q-1)^2 + q(q-1)\sum_{\{\alpha,\alpha^q\}\subset K\backslash Z} |\varphi(\alpha)+\varphi(\alpha^q)|^2\right)$$

But

$$\begin{aligned}\sum_{\{\alpha,\alpha^q\}\subset K\backslash Z} |\varphi(\alpha)+\varphi(\alpha^q)|^2 &= \sum_{\{\alpha,\alpha^q\}\subset K\backslash Z} (\varphi(\alpha)+\varphi(\alpha^q))\left(\varphi(\alpha^{-1})+\varphi(\alpha^{-q})\right) \\ &= \sum_{\{\alpha,\alpha^q\}\subset K\backslash Z} \left(2+\varphi(\alpha^{q-1})+\varphi(\alpha^{1-q})\right) \\ &= (q^2-q) + \sum_{\alpha\in K}\varphi^{q-1}(\alpha) - \sum_{\lambda\in Z}\varphi(\lambda^{q-1})\end{aligned}$$

If $\varphi^{q-1} \neq \mathbf{1}$ then the middle term in the last sum is 0 since $\langle \varphi^{q-1}, \mathbf{1}\rangle = 0$. Since $\lambda^{q-1} = 1$ for $\lambda \in \mathbb{F}_q^\times$ the third term is also easy to compute. Putting this together we get $\langle \Phi, \Phi\rangle = q-1$ when $\varphi^{q-1} \neq \mathbf{1}$.

We similarly compute

$$\begin{aligned}\langle \operatorname{Ind}_B^G \mu_\theta, \Phi \rangle &= \frac{1}{|G|}\sum_{\lambda\in Z}(q^2-1)\overline{\theta(\lambda)}q(q-1)\varphi(\lambda) \\ &= (q-1)\langle \theta, \operatorname{Res}_Z^K \varphi\rangle_Z\end{aligned}$$

Thus $\operatorname{Ind}_B^G \mu_\theta$ and Φ have many factors in common when $\phi|_Z = \theta$.

Now, for each φ such that $\varphi^{q-1} \neq \mathbf{1}$ (there are q^2-q such choices) let $\theta := \operatorname{Res}_Z^K \varphi$ then our calculations tell us that if $\beta_\varphi = \operatorname{Ind}_B^G \mu_\theta - \Phi \in R(G)$ then

$$\langle \beta_\varphi, \beta_\varphi \rangle = q - 2(q-1) + (q-1) = 1.$$

Since also $\beta_\varphi(1) = q-1 > 0$ it follows that β_φ is an irreducible character. Since $\beta_\varphi = \beta_{\varphi^q}$ (and $\varphi^{q^2} = \varphi$) we get $\binom{q}{2}$ characters in this way and the character table of $GL_2(\mathbb{F}_q)$ is complete.

# classes	$q-1$	$q-1$	$\binom{q-1}{2}$	$\binom{q}{2}$	
rep	$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$	α, α^q	# of reps
χ_θ	$\theta(\lambda)^2$	$\theta(\lambda)^2$	$\theta(\lambda)\theta(\mu)$	$\theta(\alpha^{q+1})$	$q-1$
V_θ	$q\theta(\lambda)^2$	0	$\theta(\lambda)\theta(\mu)$	$-\theta(\alpha^{q+1})$	$q-1$
$W_{\theta,\phi}$	$(q+1)\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu)+\theta(\lambda)\phi(\mu)$	0	$\binom{q-1}{2}$
β_φ	$(q-1)\varphi(\lambda)$	$-\varphi(\lambda)$	0	$-(\varphi+\varphi^q)(\alpha)$	$\binom{q}{2}$

The representations corresponding to the β_φ known as *discrete series representations* have not been computed explicitly. Drinfeld found these representations in l-adic étale cohomology groups of an algebraic curve X over $\mathbb{F}_q$. These cohomology groups should be viewed as generalisations of 'functions on X'. This work was generalised by Deligne and Lusztig for all finite groups of Lie type.

This construction also enables us to compute the character table of $PGL_2(\mathbb{F}_q) := GL_2(\mathbb{F}_q)/Z(GL_2(\mathbb{F}_q))$ as its irreducible representations are the irreducible representations of $GL_2(\mathbb{F}_q)$ such that the scalar matrices act trivially. i.e. the χ_θ and V_θ such that $\theta^2 = 1$, the $W_{\theta,\theta^{-1}}$ such that $\theta^2 \neq 1$ and the β_φ such that $\varphi|_Z = \mathbf{1}_Z$ i.e. $\varphi^{q+1} = 1$ as well as $\varphi^{q-1} \neq 1$.

We can also then compute the character table of $PSL_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)/Z(SL_2(\mathbb{F}_q))$ which has index 2 in $PGL_2(\mathbb{F}_q)$ by restriction. These groups are all simple when $q \geqslant 5$ and this can be seen from the character table.