Michaelmas 2019

Representation Theory — Examples Sheet 1

SJW

- 1. Let ρ be a representation of a group G. Show that det ρ is a representation of G. What is its degree?
- 2. Let θ be a one-dimensional representation of a group G and $\rho: G \to GL(V)$ another representation of G. Show that $\theta \otimes \rho: G \to GL(V)$ given by $\theta \otimes \rho(g) = \theta(g) \cdot \rho(g)$ defines a representation of G. If ρ is irreducible, must $\theta \otimes \rho$ also be irreducible?
- 3. Suppose that N is a normal subgroup of a group G. Given a representation of the quotient group G/N on a vector space V, explain how to construct an associated representation of G on V. Which representations of G arise in the way? Recall that G' is the normal subgroup of G generated by all elements of the form $ghg^{-1}h^{-1}$ with $g,h \in G$. Show that the 1-dimensional representations of G are precisely those that arise from 1-dimensional representations of G/G'.
- 4. Suppose that (ρ, V) and (σ, W) are representations of a group G. Show that $(\tau, \text{Hom}(V, W))$ is a representation of G where $\tau(g)(f)(v) := \sigma(g)f(\rho(g^{-1})v)$ for all $g \in G$, $f \in \text{Hom}(V, W)$ and $v \in V$.
- 5. Let $\rho: \mathbb{Z} \to GL_2(\mathbb{C})$ be the representation defined by $\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that ρ is not completely reducible. By a similar construction, show that if k is a field of characteristic p there is a two dimensional k-representation of C_p that is not completely reducible.
- 6. Let C_n be the cyclic group of order n. Explicitly decompose the complex regular representation $\mathbb{C}C_n$ as a direct sum of irreducible subrepresentations.
- 7. Let D_{10} be the dihedral group of order 10. Show that every irreducible \mathbb{C} -representation of D_{10} has degree 1 or 2. By describing them explicitly, show that there are precisely four such representations up to isomorphism. Show moreover that for each such representation it is possible to choose a basis so that all the representing matrices have real entries.
- 8. What are the irreducible real representations $\rho: C_n \to GL(V)$ of a cyclic group of order n? Compute $\operatorname{Hom}_G(V,V)$ in each case. How does the real regular representation $\mathbb{R}C_n$ of C_n break up as a direct sum of irreducible representations?
- 9. Show that (up to isomorphism) there is only one irreducible complex representation of Q_8 of dimension at least two. Show that this representation cannot be realised over \mathbb{R} and deduce that that Q_8 is not isomorphic to a subgroup of $GL_2(\mathbb{R})$. Find a four-dimensional irreducible real representation V of Q_8 . Compute $\operatorname{Hom}_G(V,V)$ in this case.
- 10. Suppose that k is algebraically closed. Using Schur's Lemma, show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible after restriction to H. What happens for $k = \mathbb{R}$?
- 11. Let (ρ, V) be an irreducible complex representation of a finite group G. For each $v \in V$, show that the \mathbb{C} -linear map $\mathbb{C}G \to V$ given by $\delta_g \mapsto \rho(g)(v)$ is G-linear and deduce that V is isomorphic to a subrepresentation of $\mathbb{C}G$. What is dim $\operatorname{Hom}_G(\mathbb{C}G, V)$?
- 12. Let G be the subgroup of the symmetric group S_6 generated by (123), (456) and (23)(56). Show that G has an index two subgroup of order 9 and four normal subgroups of order 3. By considering quotients show that G has two complex representations of degree 1, and four pairwise non-isomorphic irreducible complex representations of degree 2, none of which is faithful. Does G have a faithful irreducible complex representation?
- 13. Show that if $\rho: G \to GL(V)$ is a representation of a finite group G on a real vector space V then there is a basis for V with repect to which the matrix representing $\rho(g)$ is orthogonal for every $g \in G$. Which finite groups have a faithful two-dimensional real representation?

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