Homotopy Theory

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https://www.dpmms.cam.ac.uk/∼or257/teaching/notes/HomotopyTheory.pdf

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Chapter 1

Homotopy groups, CW complexes, and fibrations

Conventions. The term "map" means continuous function. We write $I = [0,1]$, I^n for the n-fold product, and

$$\partial I^n = \{(t_1, t_2, \dots, t_n) \in I^n \mid \text{ some } t_i \text{ is } 0 \text{ or } 1\}.$$

We write $D^d \subset \mathbb{R}^d$ for the unit disc, and $S^{d-1} \subset D^d \subset \mathbb{R}^d$ for the unit sphere. In particular

$$D^0 = \{0\}, \qquad S^0 = \{\pm 1\}, \qquad D^{-1} = \emptyset, \qquad S^{-1} = \emptyset.$$

A homotopy between maps $f_0, f_1 : X \to Y$ is a map $F : X \times [0,1] \to Y$ such that $F(-,0) = f_0(-)$ and $F(-,1) = f_1(-)$. If $A \subset X$ is a subspace we say that the homotopy F is relative to A if $F(a,t)$ is independent of t for all $a \in A$.

We write $[X,Y] := \{\text{maps } f : X \to Y\}/\text{homotopy}$ for the set of homotopy classes of maps from X to Y.

1.1 Homotopy groups

Lecture 1

Let (X, x_0) be a based space, i.e. a space with a distinguished point $x_0 \in X$.

Definition 1.1.1. Let the **nth homotopy group** $\pi_n(X, x_0)$ denote the quotient of the set of maps

$$f : I^n \longrightarrow X \text{ such that } f(\partial I^n) \subset \{x_0\},$$

by the equivalence relation $\simeq$, where $f_0 \simeq f_1$ if there is a homotopy

$$F : I^n \times [0,1] \longrightarrow X$$

such that

(i) $F(\partial I^n \times [0,1]) \subset \{x_0\}$,

(ii) $F(-,0) = f_0(-)$, and $F(-,1) = f_1(-)$.

Let us condense this definition a bit. For a pair (X, A) of a space X and a subspace $A \subset X$, and another pair (Y, B), a **map of pairs** $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subset B$. A **homotopy between maps of pairs**, $F : (X,A) \times [0,1] \to (Y,B)$ is a homotopy $F : X \times [0,1] \to Y$ such that $F(A \times [0,1]) \subset B$. In these terms we can just say that

$$\pi_n(X, x_0) = \{\text{homotopy classes of maps } f : (I^n, \partial I^n) \to (X, \{x_0\})\}.$$

Sometimes we will write elements of these sets as $[f]$, to emphasise that they are equivalence classes and that f is merely a representative, but often we will not.

Example 1.1.2. If $n = 1$ then $\pi_1(X, x_0)$ is the usual fundamental group (but so far we have only descried it as a set, not as a group).

If $n = 0$ then, as $I^0 = \{*\}$ and $\partial I^0 = \emptyset$, $\pi_0(X, \{x_0\})$ is the set of path components of X (and is independent of x_0). △

For $n \geq 1$ we define a composition law $\cdot$ on $\pi_n(X, x_0)$ by the formula

$$(f \cdot g)(t_1, t_2, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \ldots, t_n) & 1/2 \leq t_1 \leq 1 \end{cases}. \tag{1.1.1}$$

Theorem 1.1.3. *The operation $\cdot$ is well-defined on $\pi_n(X, x_0)$, and endows this set with the structure of a group with identity element given by the constant map:*

$$\begin{aligned} \mathrm{const}_{x_0} : I^n &\longrightarrow X \\ (t_1, t_2, \ldots, t_n) &\longmapsto x_0. \end{aligned}$$

Furthermore, if $n \geq 2$ then this group is abelian.

Proof. The first part is identical to the argument that the fundamental group is indeed a group. For the second part we use the homotopy shown below

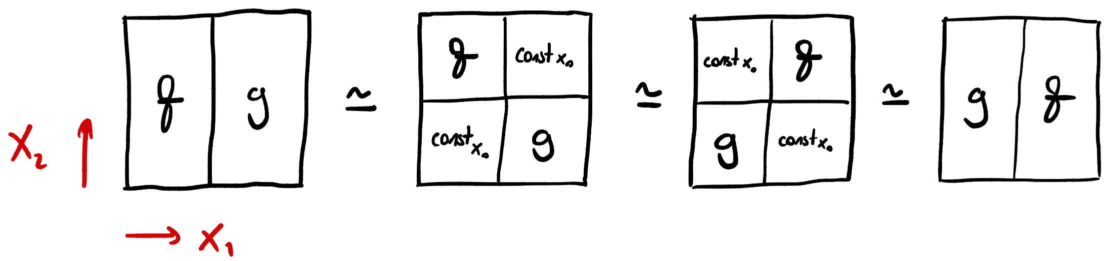


Figure 1.1

to swap the order of operations. □

If $\varphi : (X, x_0) \to (Y, y_0)$ is a based map then

$$\begin{aligned} \varphi_* : \pi_n(X, x_0) &\longrightarrow \pi_n(Y, y_0) \\ [f] &\longmapsto [\varphi \circ f] \end{aligned}$$

defines a group homomorphism. If $\psi : (Y, y_0) \to (Z, z_0)$ is another based map then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$, and $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_n(X, x_0)}$.

If two maps $\varphi_0, \varphi_1 : (X, x_0) \to (Y, y_0)$ are based homotopic[1] then the evident homotopy shows that $(\varphi_0)_* = (\varphi_1)_*$. From this it is immediate that if φ has a based homotopy inverse (i.e. a based map $\psi : (Y, y_0) \to (X, x_0)$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are both based homotopic to the identity) then φ_* is an isomorphism. In fact, if φ is *any* homotopy equivalence then $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism: this is proved in the same way as for the fundamental group; it is Example Sheet 1 Q1.

[1] i.e. homotopic as maps of pairs

1.2 Change of basepoint

If $u : I \to X$ is a path from x_0 to x_1, and $f : (I^n, \partial I^n) \to (X, x_1)$ is a map of pairs, then we get a map of pairs

$$u_\#(f) : (I^n, \partial I^n) \longrightarrow (X, x_0)$$

given graphically as shown in Figure 1.2 below: that is, apply f to a smaller cube, and interpolate between the smaller and larger cube using u on each radial ray.

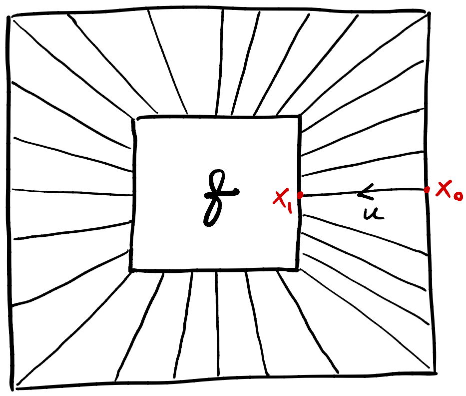


Figure 1.2

This construction is well-defined on homotopy classes, and yields a function

$$u_\# : \pi_n(X, x_1) \longrightarrow \pi_n(X, x_0)$$

satisfying

(i) $u_\# = u'_\#$ if u is homotopic to u' relative to their end-points,

(ii) $(\mathrm{const}_{x_0})_\#$ is the identity,

(iii) $u_\#$ is a group homomorphism,

(iv) if $v : I \to X$ is a path from x_1 to x_2 and $u \cdot v$ is the concatenation of these paths, then $(u \cdot v)_\# = u_\# \circ v_\#$.

These properties all follow from fairly obvious homotopies.

It follows that if x_0 and x_1 lie in the same path component, then the groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic, but a *choice* of path from x_0 to x_1 is necessary to obtain a specific isomorphism.

In particular, taking $x_1 = x_0$ gives a left action of the group $\pi_1(X, x_0)$ on each of the groups $\pi_n(X, x_0)$ for $n \geq 1$, which for $n = 1$ is simply the action by conjugation:

$u_\#(f) = u \cdot f \cdot u^{-1}$. Thus for $n \geq 2$ the abelian group $\pi_n(X, x_0)$ has the structure of a $\mathbb{Z}[\pi_1(X, x_0)]$-module[2].

1.3 Relative homotopy groups

Let (X, A) be a pair and $x_0 \in A$ be a basepoint. Let $\sqcap^{n-1} \subset \partial I^n$ denote the closure of the complement of $I^{n-1} \times \{0\} \subset \partial I^n$.

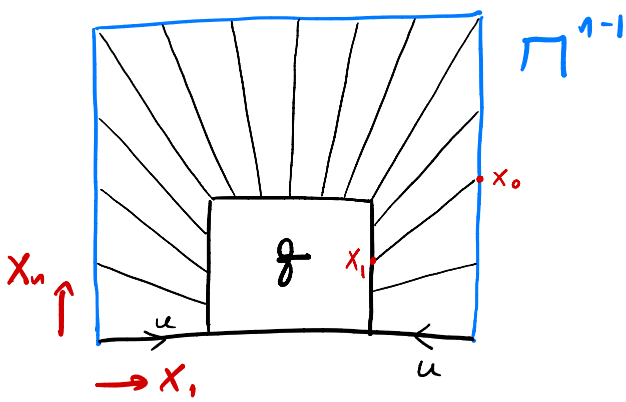
Definition 1.3.1. For $n \geq 1$ let $\pi_n(X, A, x_0)$ denote the quotient of the set of maps

$$f : I^n \longrightarrow X \text{ such that } f(\partial I^n) \subset A \text{ and } f(\sqcap^{n-1}) \subset \{x_0\}$$

by the equivalence relation given by homotopies through such maps. Extending the notation for maps of pairs, we can write such a map as a map of triples

$$f : (I^n, \partial I^n, \sqcap^{n-1}) \longrightarrow (X, A, \{x_0\}).$$

For $n \geq 2$ the formula (1.1.1) defines a group structure on $\pi_n(X, A, x_0)$, with identity element given by the constant map const_{x_0}, and for $n \geq 3$ the homotopy of Figure 1.1 shows that this group is abelian. For $n = 1$ the sets $\pi_1(X, A, x_0)$ do not have a group structure, but const_{x_0} still provides a canonical element of these sets. Analogously to Figure 1.2 above, if $u : I \to A$ is a path from x_0 to x_1 then the following figure describes the change-of-basepoint isomorphism in relative homotopy groups, so that in particular $\pi_1(A, x_0)$ acts on each $\pi_n(X, A, x_0)$.



Maps of triples $\varphi : (X, A, x_0) \to (Y, B, y_0)$ induce functions $\varphi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ with all the properties one expects:

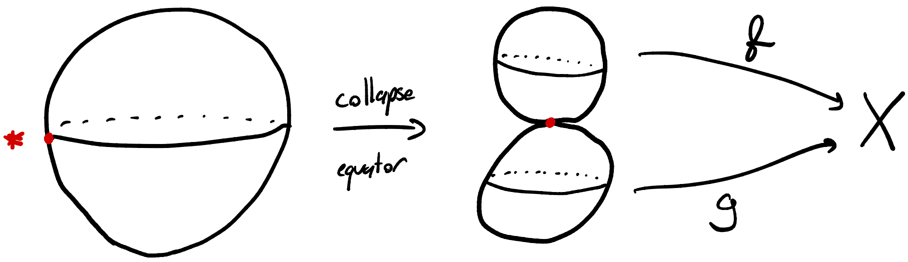
(i) If $\varphi_0 \simeq \varphi_1$ through such maps then $(\varphi_0)_* = (\varphi_1)_*$,

(ii) φ_* is a homomorphism for $n \geq 2$,

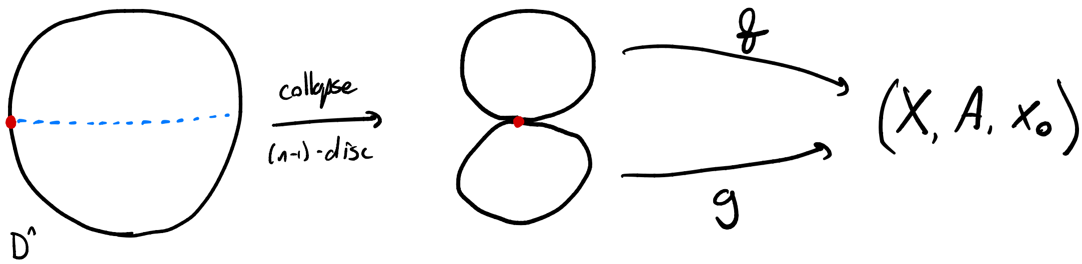
[2]A group G has a "group ring" $\mathbb{Z}[G]$ whose elements are finite $\mathbb{Z}$-linear sums of elements of G, and whose multiplication is determined by $\mathbb{Z}$-linearity and the group structure of G. A left $\mathbb{Z}[G]$-module is precisely the same as an abelian group with a left G-action by group homomorphisms.

(iii) $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

Remark 1.3.2 (Cubes and discs). As $I^n/\partial I^n \cong S^n$, with $\partial I^n/\partial I^n$ giving a basepoint $* \in S^n$, we may equally well take $\pi_n(X, x_0)$ to be given by the homotopy classes of based maps $f : (S^n, *) \to (X, x_0)$. The group operation is then given as follows:



Similarly, as $(I^n/\sqcap^{n-1}, \partial I^n/\sqcap^{n-1}) \cong (D^n, S^{n-1})$, we may take $\pi_n(X, A, x_0)$ to be given by homotopy classes of maps of triples $f : (D^n, S^{n-1}, *) \to (X, A, x_0)$. The group operation is given as follows:



We will use these points of view interchangeably.

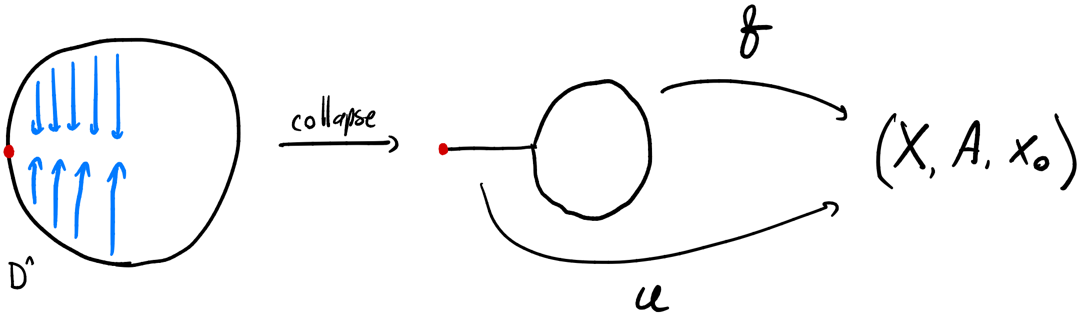
1.4 The Hurewicz homomorphism

We know that $H_n(D^n, S^{n-1}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, with generator u_n. As homotopic maps induce equal maps on homology, we may therefore define a function

$$\begin{aligned} h : \pi_n(X, A, x_0) &\longrightarrow H_n(X, A; \mathbb{Z}) \\ [f] &\longmapsto f_*(u_n). \end{aligned}$$

By contemplating the previous figure, this is a homomorphism: it is called the **Hurewicz homomorphism**. For a while this will be the only tool we have to show that elements of $\pi_n(X, A, x_0)$ are nonzero, but we will develop more tools.

The action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$ is shown in the following figure, which makes clear that the Hurewicz homomorphism is insensitive to this action, i.e. $h(u_\#([f])) = h([f])$.



1.5 Compression criterion

The constant map $\mathrm{const}_{x_0} : (I^n, \partial I^n, \sqcap^{n-1}) \to (X, A, x_0)$ represents an element of $\pi_n(X, A, x_0)$. When is $[f] = [\mathrm{const}_{x_0}]$ in $\pi_n(X, A, x_0)$? To answer this we adopt the "disc" perspective.

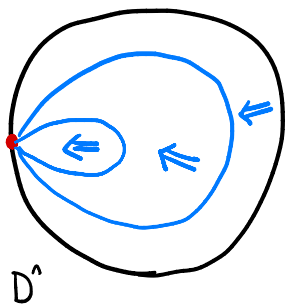
Lemma 1.5.1. *A map $f : (D^n, S^{n-1}, *) \to (X, A, x_0)$ represents $[\mathrm{const}_{x_0}] \in \pi_n(X, A, x_0)$ if and only if $f : D^n \to X$ is homotopic relative to S^{n-1} to a map with image in A.*

Lecture 2

Proof. Suppose first that f is homotopic relative to S^{n-1} to a map g which has image in A. Such a homotopy shows that $[f] = [g] \in \pi_n(X, A, x_0)$. Now choose a deformation retraction[3]

$$r : D^n \times [0, 1] \to D^n$$

of D^n to $* \in S^{n-1} \subset D^n$, and consider $g \circ r : D^n \times [0, 1] \to X$. At time 0 this is the map g, and at time 1 it is const_{x_0}: furthermore, for all times it sends S^{n-1} into A (as the map g has image in A) and sends $*$ to x_0 (as r fixes $*$ throughout). Thus this homotopy shows that $[g] = [\mathrm{const}_{x_0}]$.



Now suppose that $[f] = [\mathrm{const}_{x_0}]$, and let $H : D^n \times [0, 1] \to X$ be a homotopy giving this identity, so $H(D^n \times \{1\}) \subset \{x_0\} \subset A$ and $H(S^{n-1} \times [0, 1]) \subset A$. As $D^n \times [0, 1]$

[3] In this course a deformation retraction of X to a subspace A is a homotopy from the identity map of X to a map into A, and this homotopy should be constant on A.

deformation retracts to $D^n \times \{1\} \cup S^{n-1} \times [0,1]$, by radial projection from $(0,-1)$ as in Figure 1.3, it follows that $f(-) = H(-,0)$ is homotopic relative to S^{n-1} to a map into A. $\square$

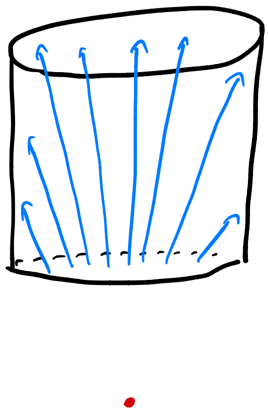


Figure 1.3

1.6 The long exact sequence of a pair

The map of pairs $i : (A, x_0) \to (X, x_0)$ and map of triples $j : (X, x_0, x_0) \to (X, A, x_0)$ induce maps

$$i_* : \pi_n(A, x_0) \to \pi_n(X, x_0) \quad \text{and} \quad j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0).$$

In addition, given a map $f : (I^n, \partial I^n, \sqcap^{n-1}) \to (X, A, x_0)$ its restriction to $I^{n-1} \times \{0\} \subset \partial I^n$ gives a map $\partial f : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$, inducing a function

$$\partial : \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0).$$

If $n \geq 2$ then $\partial(f \cdot g) = \partial f \cdot \partial g$, so this function is a homomorphism in this case.

Theorem 1.6.1. *The sequence*

$$\cdots \longrightarrow \pi_{n+1}(X,A,x_0) \xrightarrow{\partial} \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \xrightarrow{i_*} \pi_{n-1}(X,x_0) \xrightarrow{j_*} \cdots$$

$$\cdots \xrightarrow{i_*} \pi_1(X,x_0) \xrightarrow{j_*} \pi_1(X,A,x_0) \xrightarrow{\partial} \pi_0(A,x_0) \xrightarrow{i_*} \pi_0(X,x_0)$$

is exact. Exactness at the last three positions must be interpreted carefully: it means that $\mathrm{Im}(j^*) = \partial^{-1}([\mathrm{const}_{x_0}])$ *and* $\mathrm{Im}(\partial) = (i_*)^{-1}([\mathrm{const}_{x_0}])$.

This long exact sequence is natural with respect to maps $\varphi : (X,A,x_0) \to (Y,B,y_0)$.

Proof.
Exactness at $\pi_n(X,x_0)$: Let $f : (I^n, \partial I^n) \to (X,x_0)$ be such that $j_*([f]) = [\mathrm{const}_{x_0}]$. By the compression criterion this means that f is homotopic relative to ∂I^n to a map with image in A, so $[f] \in \mathrm{Im}(i_*)$. The compression criterion also shows that $j_* \circ i_* = 0$.

Exactness at $\pi_n(X,A,x_0)$: We have $\partial j_*([f]) = [f|_{I^n \times \{0\}}] = [\mathrm{const}_{x_0}]$ because f sends ∂I^n to x_0. Now let $f : (I^n, \partial I^n, \sqcap^{n-1}) \to (X,A,x_0)$ be such that $\partial f : (I^{n-1}, \partial I^{n-1}) \to (A,x_0)$ is homotopic to const_{x_0}, say via a homotopy $H : I^{n-1} \times [0,1] \to A$. Form the map

$$\begin{aligned} g : (I^n, \partial I^n) &\longrightarrow (X,x_0) \\ (t_1,t_2,\dots,t_n) &\longmapsto \begin{cases} H(t_2,\dots,t_n; 1-2t_1) & 0 \le t_1 \le 1/2 \\ f(2t_1-1,t_2,\dots,t_n) & 1/2 \le t_1 \le 1. \end{cases} \end{aligned}$$

These indeed glue as $H(t_2,\dots,t_n;0) = (\partial f)(t_2,\dots,t_n) = f(0,t_2,\dots t_n)$, and g indeed takes the value x_0 if any t_i is 0 or 1. See Figure 1.4 (a).

Thus $[g] \in \pi_n(X,x_0)$, and we can form $j_*([g]) \in \pi_n(X,A,x_0)$. This is equal to $[f] \in \pi_n(X,A,x_0)$ via the homotopy of triples

$$\begin{aligned} G : (I^n, \partial I^n, \sqcap^{n-1}) \times [0,1] &\longrightarrow (X,A,x_0) \\ (t_1,t_2,\dots,t_n,s) &\longmapsto \begin{cases} H(t_2,\dots,t_n; 1-2t_1) & 0 \le t_1 \le 1/2 \cdot s \\ f(\frac{2t_1-s}{2-s},t_2,\dots,t_n) & 1/2 \cdot s \le t_1 \le 1, \end{cases} \end{aligned}$$

as the homotopy H has image in A. See Figure 1.4 (b).

Exactness at $\pi_{n-1}(A,x_0)$: For $f : (I^n, \partial I^n, \sqcap^{n-1}) \to (X,A,x_0)$, the element $i_*\partial([f])$ is zero because

$$(I^{n-1} \times \{0\}, \partial I^{n-1} \times \{0\}) \xrightarrow{\partial f} (A,x_0) \xrightarrow{i} (X,x_0)$$

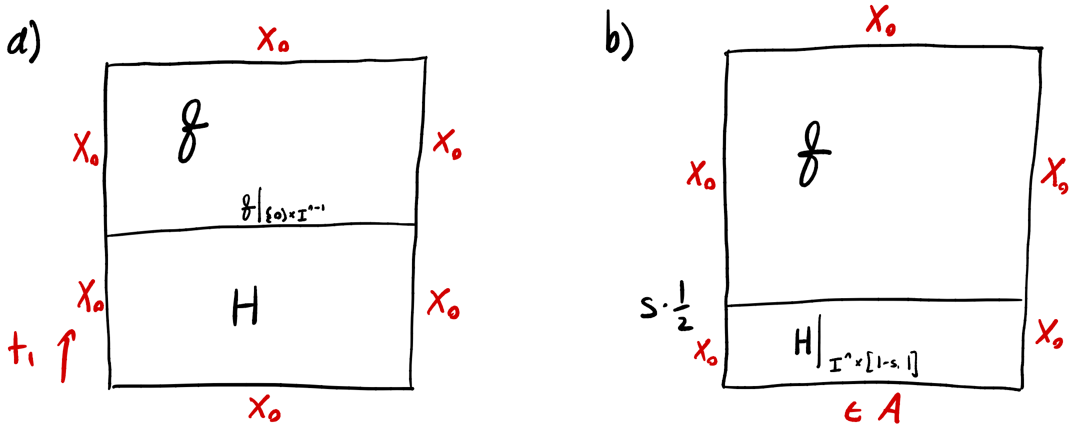


Figure 1.4

is homotopic to const_{x_0} via the homotopy f! Conversely, if $g : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$ has $i \circ g$ homotopic to const_{x_0} via a homotopy $H : (I^{n-1}, \partial I^{n-1}) \times [0,1] \to (X, x_0)$, then the map $H : I^n \to X$ satisfies

$$H|_{I^{n-1}\times\{0\}} = g, \quad H(\partial I^n) \subset A, \quad H(\sqcap^{n-1}) \subset \{x_0\},$$

and so $[H] \in \pi_n(X, A, x_0)$ satisfies $\partial([H]) = [g]$. $\square$

For a path $u : [0,1] \to A$, the change-of-basepoint bijections

$$u_\# : \pi_n(X, A, x_0) \longrightarrow \pi_n(X, A, x_1),$$

$$u_\# : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1) \quad \text{and} \quad u_\# : \pi_n(A, x_0) \longrightarrow \pi_n(A, x_1)$$

give a map of long exact sequences from that based at x_0 to that based at x_1. In particular $\pi_1(A, x_0)$ acts on the terms of the long exact sequence of Theorem 1.6.1, making it a long exact sequence of $\mathbb{Z}[\pi_1(A, x_0)]$-modules (and of sets-with-a-$\pi_1(A, x_0)$-action near the bottom).

1.7 Maps are pairs (up to homotopy)

A pair of spaces (Y, B) in particular gives an (inclusion) map $i : B \to Y$.

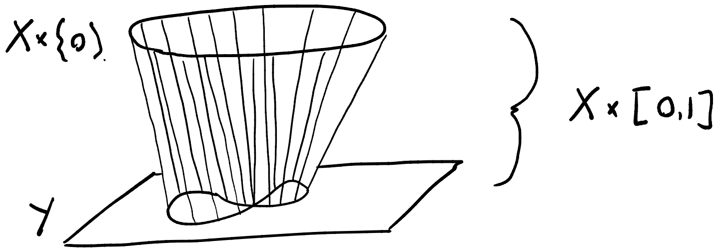
If $f : X \to Y$ is a map, let

$$M_f := (X \times [0,1] \sqcup Y)/(x,1) \in X \times [0,1] \sim f(x) \in Y,$$

called the **mapping cylinder of** f.

There is an inclusion

$$\begin{aligned} i : X &\longrightarrow M_f \\ x &\longmapsto [(x,0)] \end{aligned}$$



and linear interpolation gives a deformation retraction

$$\begin{aligned} r : M_f \times [0,1] &\longrightarrow M_f \\ ([y], t) &\longmapsto y \\ ([x,s], t) &\longmapsto [(x, s(1-t)+t)] \end{aligned}$$

to the subspace $Y \subset M_f$. Thus the pair (M_f, X) has inclusion map satisfying

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & M_f \arrow[d, "r(-{,}0)", "\simeq"'] \\ & Y \end{tikzcd}$$

In other words, M_f is a replacement of Y, up to homotopy equivalence, for which the map $f : X \to Y$ is represented by the inclusion of a subspace.

Using this device we may consider any map $f : X \to Y$ as being, morally, the inclusion of a subspace, and so can pretend that (Y, X) is a pair: we really mean the pair (M_f, X). For example, for $x_0 \in X$ we can define

$$\pi_n(Y, X, x_0) := \pi_n(M_f, X, x_0),$$

whereupon, using $\pi_n(Y) \cong \pi_n(M_f)$, we get a long exact sequence

$$\begin{tikzcd} & \cdots \arrow[r] & \pi_{n+1}(Y, X, x_0) \arrow[dll, "\partial"] \\ \pi_n(X, x_0) \arrow[r, "f_*"] & \pi_n(Y, f(x_0)) \arrow[r, "j_*"] & \pi_n(Y, X, x_0) \arrow[dll, "\partial"] \\ \pi_{n-1}(X, x_0) \arrow[r, "f_*"] & \pi_{n-1}(Y, f(x_0)) \arrow[r, "j_*"] & \cdots \end{tikzcd}$$

1.8 CW complexes

Definition 1.8.1. A **CW complex** is a space X obtained by the following process.

(i) Let $X^{-1} := \emptyset$.

(ii) Supposing the space X^{n-1} has been defined, let $\{\varphi_\alpha : S^{n-1} \to X^{n-1}\}_{\alpha \in I_n}$ be a set of maps, and let

$$X^n := (X^{n-1} \sqcup (I_n \times D^n))/(\alpha, x) \in I_n \times S^{n-1} \sim \varphi_\alpha(x) \in X^{n-1},$$

with the quotient topology.

We let e^n_α be the image of $\{\alpha\} \times \mathrm{int}(D^n)$ under the quotient map, called an **open cell**.

Let $X := \cup_{n \geq 0} X^n$ with the following topology: a set $S \subset X$ is open (resp. closed) if and only if each $S \cap X^n$ is open (resp. closed)[4]. We call X^n the n-**skeleton** of X.

A pair of spaces (X, A) is a **relative CW complex** if X is obtained as above but starting from $X^{-1} := A$.

If X is a CW complex then a **subcomplex** is a closed subspace $A \subset X$ which is a union of open cells. In this case A is a CW complex in its own right, and (X, A) is called a **CW pair**.

Definition 1.8.2. A pair (X, A) has the **homotopy extension property** if given a map $f : X \to Y$ and a homotopy $H : A \times [0,1] \to Y$ such that $H(-,0) = f|_A(-)$, there exists a homotopy $H' : X \times [0,1] \to Y$ such that $H'|_{A \times [0,1]} = H$ and $H'(-,0) = f(-)$.

Equivalently[5], we start with the data of a map

$$f \cup H : (X \times \{0\}) \cup (A \times [0,1]) \longrightarrow Y$$

and ask for an extension to a map $H' : X \times [0,1] \to Y$.

Theorem 1.8.3. *A relative CW complex (X, A) has the homotopy extension property.*

Proof. As a preliminary step, observe that just as in Figure 1.3 the space $D^n \times [0,1]$ (deformation) retracts to $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$ via radial projection from $(0,2)$. It follows that the pair (D^n, S^{n-1}) has the homotopy extension property: the initial data is a map

$$f \cup H : (D^n \times \{0\}) \cup (S^{n-1} \times [0,1]) \longrightarrow Y$$

and precomposing with the retraction $r : D^n \times [0,1] \to (D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$ gives the required extension. Lecture 3

Now, let (X, A) be a relative CW complex, $f : X \to Y$ and $H : A \times [0,1] \to Y$ be given. We will construct compatible maps $H^n : X^n \times [0,1] \to Y$ by induction over n: we start with $H^{-1} := H$.

Suppose H^{n-1} is given. Now

$$X^n \times [0,1] = \frac{(X^{n-1} \times [0,1]) \sqcup (I_n \times D^n \times [0,1])}{(\alpha, x, t) \in I_n \times D^n \times [0,1] \sim (\varphi_\alpha(x), t) \in X^{n-1} \times [0,1]},$$

so the retraction $r : D^n \times [0,1] \to (D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$ for each $\alpha \in I_n$ gives a retraction

$$r^n : X^n \times [0,1] \longrightarrow (X^n \times \{0\}) \cup (X^{n-1} \times [0,1]),$$

[4] So a function $f : X \to Y$ is continuous if and only if all its restrictions $f|_{X^n} : X^n \to Y$ are continuous.

[5] Though this equivalence is subtle if $A \subset X$ is not closed, see Proposition A.18 of Hatcher's book.

and we can set $H^n := (f|_{X^n} \cup H^{n-1}) \circ r^n$. By definition of the topology on X, the map $H' := \cup_{n\geq 0} H^n : X \times [0,1] = \cup_{n\geq 0} X^n \times [0,1] \to Y$ is continuous, and gives the required extension. $\square$

This argument shows the most useful point of working with CW complexes: we can make arguments by induction over cells, or skeleta, and often reduce to proving things for D^n. The following is another example of this strategy.

Lemma 1.8.4 (Compression Lemma). *Let (X, A) be a relative CW complex and (Y, B) be a pair with $B \neq \emptyset$. Suppose that for each n such that X has an n-cell relative to A, $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then any map $f : (X, A) \to (Y, B)$ is homotopic relative to A to a map into B.*

Proof. We will construct compatible homotopies

$$H^n : X^n \times [0,1] \longrightarrow Y$$

relative to A, such that $H^n|_{X^n \times \{0\}} = f|_{X^n}$ and $H^n(X^n \times \{1\}) \subset B$. We start by taking $H^{-1}(x,t) = f(x)$ to be the constant homotopy on X^0.

If H^{n-1} has been defined let I_n index the n-cells of X relative to A. For each $\alpha \in I_n$ we have a map

$$m_\alpha := f|_{\{\alpha\}\times D^n} \cup H^{n-1}|_{\{\alpha\}\times S^{n-1}\times[0,1]} : \{\alpha\} \times \big((D^n \times \{0\}) \cup (S^{n-1} \times [0,1])\big) \longrightarrow Y.$$

The domain is homeomorphic to D^n, and $m_\alpha(S^{n-1} \times \{1\}) \subset H^{n-1}(X^{n-1} \times \{1\}) \subset B$, so m_α represents an element of $\pi_n(Y, B, m_\alpha(\alpha, *, 1))$. As this homotopy group vanishes by assumption, by the compression criterion (Section 1.5) the map m_α is homotopic relative to $\{\alpha\} \times S^{n-1} \times \{1\}$ to a map into B: this homotopy gives an extension to a map

$$M_\alpha : \{\alpha\} \times D^n \times [0,1] \longrightarrow Y$$

satisfying $M_\alpha(\{\alpha\} \times D^n \times \{1\}) \subset B$. As

$$X^n \times [0,1] = \frac{(X^{n-1} \times [0,1]) \sqcup (I_n \times D^n \times [0,1])}{(\alpha, x, t) \in I_n \times D^n \times [0,1] \sim (\varphi_\alpha(x), t) \in X^{n-1} \times [0,1]}$$

we can let $H^n = (H^{n-1} \sqcup \bigsqcup_{\alpha\in I_n} M_\alpha)/\sim$. $\square$

1.9 Weak homotopy equivalences and Whitehead's Theorem

Definition 1.9.1. A map $f : X \to Y$ is a **weak homotopy equivalence** if the functions $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ are bijections for all n and for all basepoints $x_0 \in X$.

More generally, we say that spaces X and Y are **weakly homotopy equivalent** if there exists a zig-zag

$$X \longrightarrow Z_1 \longleftarrow Z_2 \longrightarrow Z_3 \longleftarrow \cdots \longrightarrow Z_n \longleftarrow Y$$

of weak homotopy equivalences.

We have seen (Example Sheet 1 Q1) that homotopy equivalences are weak homotopy equivalences. Our goal is to show that the converse is also true, as long as the spaces involved are CW complexes.

Theorem 1.9.2. *Let (Z, A) be a relative CW complex, $f : X \to Y$ be a weak homotopy equivalence, and a commutative square*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

be given. There there is a map $\hat{g} : Z \to X$ such that

(i) $\hat{g} \circ i = h$,

(ii) *$f \circ \hat{g}$ is homotopic to g, relative to A.*

Proof. Suppose first that $f : X \to Y$ is the inclusion of a subspace, so that (Y, X) is a pair. The long exact sequence

$$\begin{tikzcd} & \cdots \arrow[r] & \pi_{n+1}(Y, X, x_0) \arrow[dll, "\partial"'] \\ \pi_n(X, x_0) \arrow[r, "f_*"] & \pi_n(Y, f(x_0)) \arrow[r, "j_*"] & \pi_n(Y, X, x_0) \arrow[dll, "\partial"'] \\ \pi_{n-1}(X, x_0) \arrow[r, "f_*"] & \pi_{n-1}(Y, f(x_0)) \arrow[r, "j_*"] & \cdots \end{tikzcd}$$

then shows that $\pi_n(Y, X, x_0) = 0$ for all $x_0 \in X$. The claim then follows from the Compression Lemma (Lemma 1.8.4).

For a general map $f : X \to Y$ we reduce to the case above following Section 1.7. Let M_f denote the mapping cylinder of f, with inclusions $inc : X \to M_f$ and $j : Y \to M_f$ and retraction $r : M_f \to Y$. Consider the diagram

I gave a fallacious version of this argument in lectures: here it is now corrent.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"] & X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, equal] \arrow[d, "inc"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \arrow[r, "j"] & M_f \arrow[r, "r"] & Y, \end{tikzcd}$$

in which the left and right squares commute (using that $r \circ j = \mathrm{Id}_Y$) and the middle square commutes up to homotopy. Combining the two leftmost squares gives a new square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"] & X \arrow[d, "inc"] \\ Z \arrow[r, "g' := j \circ g"] & M_f \end{tikzcd}$$

which also commutes up to homotopy. Using the homotopy extension property, we may change g' by a homotopy to a map g'' making the square literally commute. This puts us in a situation to apply the version of the statement proved above, as $inc : X \to M_f$ is the inclusion of a subspace. Thus there is a map $\hat{g}'' : Z \to X$ such that $\hat{g}'' \circ i = h$ and such that $inc \circ \hat{g}'' \simeq g'' \simeq g'$ relative to A. Applying r to this homotopy, it follows that $f \circ \hat{g}''$ is homotopic to g relative to A, as required. □

Corollary 1.9.3 (Whitehead's theorem). *If $f : X \to Y$ is a weak homotopy equivalence between CW complexes, then it is a homotopy equivalence.*

Proof. First apply the previous theorem to

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "Id_Y"] & Y, \end{tikzcd}$$

which provides a map $g : Y \to X$ such that $f \circ g$ is homotopic to Id_Y. We claim that g is a homotopy inverse to f so we must also show that $g \circ f$ is homotopic to Id_X.

As $f \circ g$ is homotopic to Id_Y, there is a homotopy H from $f \circ (g \circ f) = (f \circ g) \circ f$ to $f \circ \mathrm{Id}_X = \mathrm{Id}_Y \circ f$. Thus we may form the commutative square

$$\begin{tikzcd} X \times \{0\} \sqcup X \times \{1\} \arrow[r, "f \circ g \sqcup \mathrm{Id}_X"] \arrow[d, "inc"] & X \arrow[d, "f"] \\ X \times [0,1] \arrow[r, "H"] & Y. \end{tikzcd}$$

The left-hand map is the inclusion of a sub-CW complex, and the right-hand map f is still a weak homotopy equivalence, so by the previous theorem we may find is a map $\hat{H} : X \times [0,1] \to X$ such that $\hat{H}|_{X \times \{0\}} = f \circ g$ and $\hat{H}|_{X \times \{1\}} = \mathrm{Id}_X$, as required. □

There is another consequence of Theorem 1.9.2 with further justifies focussing on the notion of weak homotopy equivalences.

Corollary 1.9.4. *A weak homotopy equivalence induces an isomorphism on homology and cohomology with any coefficients.*

Lecture 4

Proof sketch. It is enough to prove that a weak homotopy equivalence $f : X \to Y$ induces an isomorphism on homology with $\mathbb{Z}$ coefficients: it then follows for (co)homology with any coefficients by Universal Coefficient Theorems. By replacing Y by the mapping cylinder M_f, it is also enough to treat the case where f is the inclusion of a subspace. In this case we need to show that $H_n(Y, X; \mathbb{Z}) = 0$.

Let $[\sum n_\sigma \cdot \sigma] \in C_n(Y, X)$ be a relative cycle, where $\sigma : \Delta^n \to Y$ are singular simplices. We must show that this cycle represents zero in $H_n(Y, X; \mathbb{Z})$. By considering $n_\sigma \cdot \sigma$ as $|n_\sigma|$ copies of σ (or of $-\sigma$), we may glue copies of Δ^n together along faces as follows. If $\tau \subset \sigma$ is a $(n-1)$-dimensional face which is not sent into X, then because $\partial(\sum n_\sigma \cdot \sigma) \in C_{n-1}(X)$ there is another simplex σ' which also has τ as a face. Let

K denote a simplicial complex obtained by gluing together in this manner all $(n-1)$-dimensional faces which are not mapped into X, and let $L \subset K$ denote the union of the unglued $(n-1)$-dimensional faces. By construction there is a map of pairs

$$f : (|K|, |L|) \longrightarrow (Y, X).$$

Now

$$x := \text{sum of all } n\text{-simplices of } K \in C_n(|K|, |L|)$$

is a cycle, and by construction $f_*(x) = [\sum n_\sigma \cdot \sigma] \in H_i(Y, X)$. Now as $(|K|, |L|)$ is a relative CW complex, by Theorem 1.9.2 the map $f : (|K|, |L|) \to (Y, X)$ is homotopic relative to $|L|$ to a map with image in X, and so $f_*(x) = 0$. $\square$

1.10 Connectivity

Definition 1.10.1. For $n \geq 0$ say that a pair (X, A) is n-**connected** if $\pi_0(A) \to \pi_0(X)$ is onto[6] and $\pi_i(X, A, x_0) = 0$ for all $1 \leq i \leq n$ and all $x_0 \in A$. Equivalently, for all $x_0 \in A$ the maps

$$\pi_i(A, x_0) \longrightarrow \pi_i(X, x_0)$$

are epimorphisms for $i \leq n$ and isomorphisms for $i < n$. More generally, say that a map $f : X \to Y$ is n-**connected** if the pair (M_f, X) is n-connected.

Say that a space X is n-**connected** if the pair $(X, \{x_0\})$ is n-connected for all $x_0 \in X$, i.e. $\pi_i(X, x_0) = 0$ for all $0 \leq i \leq n$ and all $x_0 \in X$.

Theorem 1.10.2. *S^n is $(n-1)$-connected.*

Proof. Let $* \in S^n$ be a basepoint, $i < n$, and $f : (S^i, *) \to (S^n, *)$ be a continuous map, representing $[f] \in \pi_i(S^n, *)$. By the Simplicial Approximation Theorem we may change f by a homotopy to a map f' that is simplicial with respect to some triangulations $S^i \cong |K|$ and $S^n \cong |L|$; as $i < n$ the map $f' : S^i \to S^n$ is not surjective, so lands inside some $(\mathrm{int}(D^n), *) \subset (S^n, *)$. As $\mathrm{int}(D^n)$ deformation retracts to $* \in \mathrm{int}(D^n)$, it follows that f' is homotopic, relative to $*$, to a constant map, so $[f] = 0 \in \pi_i(S^n)$. $\square$

Corollary 1.10.3. *The pair (D^n, S^{n-1}) is $(n-1)$-connected.*

Proof. The long exact sequence on homotopy groups for this pair, and the fact that $\pi_i(D^n, *) = 0$, shows that the boundary map

$$\partial : \pi_i(D^n, S^{n-1}, *) \longrightarrow \pi_{i-1}(S^{n-1}, *)$$

is an isomorphism. By the previous theorem the latter group vanishes for $i - 1 < n - 1$, so the former group vanishes for $i < n$. $\square$

Corollary 1.10.4. *If (X, A) is a relative CW complex only having relative cells of dimension $< n$, then any map $f : (X, A) \to (D^n, S^{n-1})$ is homotopic relative to A to a map into S^{n-1}.*

[6] You can interpret this condition as "$\pi_0(X, A, x_0) = 0$", but we did not define such a relative homotopy group.

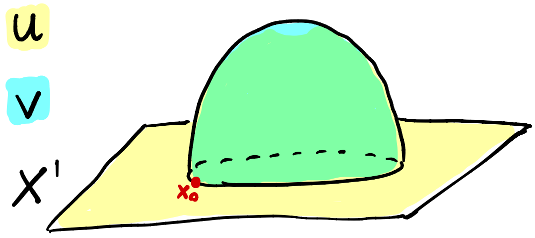
Proof. Apply the Compression Lemma (Lemma 1.8.4). $\square$

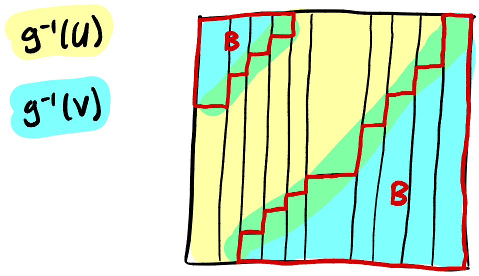
Corollary 1.10.5. *If (X, A) is a relative CW complex only having relative cells of dimension $\geq n$, then it is $(n-1)$-connected.*

Proof. Let $i < n$ and $f : (I^i, \partial I^i, \sqcap^{i-1}) \to (X, A, x_0)$ represent an element $[f] \in \pi_i(X, A, x_0)$, which we must show is zero, i.e. we must find a homotopy compressing f into A.

As I^i is compact so is $f(I^i)$, so it lies in a sub relative CW complex (X', A) having finitely-many cells (see Example Sheet 1 Q5). Thus we may suppose without loss of generality that (X, A) has finitely-many cells. Let $A \subset X' \subset X$ be a sub-CW complex having one fewer relative cell than X. By induction it suffices to show that we can homotope f relative to ∂I^n until it has image in X'.

Thus we may suppose that we are in the following simplified situation: we have a map $g : (I^i, \partial I^i) \to (X, X')$, where $X = X' \cup D^k$ with $k \geq n$ and $i < n$, and we wish to homotope g relative to ∂I^i to that it has image in X'.





Cover $X = X' \cup D^k$ by the open sets $U = X' \cup (D^k \setminus \{0\})$ and $V = \mathrm{int}(D^k)$. Then $g^{-1}(U)$ and $g^{-1}(V)$ form an open cover of I^i, so by the Lesbegue number lemma we may subdivide I^i into small cubes each of which maps into U or into V (or into both). Let $B \subset I^n$ be the "bad cubes": those closed cubes which map into V. Then $g(\partial B) \subset V \cap U$, and so we have a map

$$g|_B : (B, \partial B) \longrightarrow (V, V \cap U) \cong (\mathrm{int}(D^k), \mathrm{int}(D^k) \setminus \{0\}) \simeq (D^k, S^{k-1}).$$

As $(B, \partial B)$ is a relative CW complex only having cells of dimension $\leq i < n$, and $(V, V \cap U)$ is homotopy equivalent to (D^k, S^{k-1}) with $k \geq n$, it follows from Corollary 1.10.4 that $g|_B$ is homotopic relative to ∂B to a map into $U \cap V$, and hence that g is homotopic to a map into U. Finally, U deformation retarcts to X'. $\square$

1.11 Cellular approximation

Corollary 1.11.1 (Cellular approximation)**.** *If $f : (X, A) \to (Y, B)$ is a map between relative CW complexes, then it is homotopic to a map $\bar{f}$ satisfying $\bar{f}(X^n) \subset (Y^n)$ for all $n \geq 0$.*

Proof. See Example Sheet 1 Q7. $\square$

Corollary 1.11.2. *If X is a CW complex and $\varphi : S^{n-1} \to X$ is an attaching map for an n-cell then φ is homotopic to a map into X^{n-1}, so $X \cup_\varphi D^n$ is homotopy equivalent to a CW complex.* □

1.12 CW approximation

From the point of view of weak homotopy equivalence, every space is equivalent to a CW complex:

Theorem 1.12.1. *For any space X there is a CW complex C and a weak homotopy equivalence $f : C \to X$.*

Furthermore, if $g : D \to X$ is another weak homotopy equivalence from a CW complex, there is a weak homotopy equivalence $\phi : C \to D$ such that $g \circ \phi \simeq f$.

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Proof. Suppose that X is path connected (otherwise repeat the below for each path component), and choose a basepoint x_0. We will construct CW complexes $C^0 \subset C^1 \subset C^2 \subset \cdots$ and maps $f^n : C^n \to X$ such that

(i) C^n is n-dimensional, and

(ii) f_n is n-connected, i.e. $f^n_* : \pi_i(C^n, *) \to \pi_i(X, x_0)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$.

Then the map $f := \bigcup_{n \geq 0} f^n : C := \bigcup_{n \geq 0} C^n \to X$ induces an isomorphism on all homotopy groups and so is a weak homotopy equivalence.

We start with $C^0 = \{*\}$ and $f^0(*) = x_0$. Assuming that C^{n-1} has been constructed, let $\{\psi_\alpha : (S^{n-1}, *) \to (C^{n-1}, *)\}_{\alpha \in I_n}$ be a set of maps such that $[\psi_\alpha] \in \pi_{n-1}(C^{n-1}, *)$ generate

$$\mathrm{Ker}(f^{n-1}_* : \pi_{n-1}(C^{n-1}, *) \to \pi_{n-1}(X, x_0)),$$

and let

$$C^n_{prelim} := C^{n-1} \cup \bigcup_{\alpha \in I_n} D^n_\alpha$$

be the CW complex obtained by attaching n-cells to C^{n-1} along the maps ψ_α. As $f^{n-1} \circ \psi_\alpha : (S^{n-1}, *) \to (X, x_0)$ is nullhomotopic, a choice of nullhomotopy gives an extension of this map over the cell D^n_α, and so an extension of f^{n-1} to a map

$$f^n_{prelim} : C^n_{prelim} \longrightarrow X.$$

Consider the factorisation $f^{n-1} : C^{n-1} \xrightarrow{i} C^n_{prelim} \xrightarrow{f^n_{prelim}} X$ as giving

$$f^{n-1}_* : \pi_{n-1}(C^{n-1}, *) \xrightarrow{i_*} \pi_{n-1}(C^n_{prelim}, *) \xrightarrow{(f^n_{prelim})_*} \pi_{n-1}(X, x_0).$$

The map f^{n-1}_* is surjective by assumption. The map i_* is surjective, because we have $\pi_{n-1}(C^n_{prelim}, C^{n-1}, *) = 0$ by an application of cellular approximation (Corollary 1.11.1).

On the other hand the ψ_α map to 0 under i_* by construction, and as these generate the kernel of f_*^{n-1} it follows that $(f_{prelim}^n)_*$ is a bijection. For $i < n-1$ we have

$$f_*^{n-1} : \pi_i(C^{n-1}, *) \xrightarrow{i_*} \pi_i(C_{prelim}^n, *) \xrightarrow{(f_{prelim}^n)_*} \pi_i(X, x_0),$$

the map f_*^{n-1} is an isomorphism by assumption, and the map i_* is an isomorphism as both $\pi_i(C_{prelim}^n, C^{n-1}, *)$ and $\pi_{i+1}(C_{prelim}^n, C^{n-1}, *)$ vanish by cellular approximation (Corollary 1.11.1): it follows that $(f_{prelim}^n)_*$ is an isomorphism too.

Now let $\{\varphi_\alpha : (S^n, *) \to (X, x_0)\}_{\beta \in J_n}$ be a set of maps such that $[\varphi_\beta] \in \pi_n(X, x_0)$ generate this group, let

$$C^n := C_{prelim}^n \vee \bigvee_{\beta \in J_n} S_\beta^n$$

be obtained by attaching an n-cell trivially for each $\beta \in J_n$, and set $f^n := f_{prelim}^n \vee \bigvee \varphi_\beta : C^n \to X$. By construction the map $f_*^n : \pi_n(C^n, *) \to \pi_n(X, x_0)$ is an epimorphism. Consider the maps

$$(f_{prelim}^{n-1})_* : \pi_i(C_{prelim}^n, *) \xrightarrow{i_*} \pi_i(C^n, *) \xrightarrow{f_*^n} \pi_i(X, x_0).$$

The map i_* is a split monomorphism as the inclusion $i : C_{prelim}^n \to C^n$ has a retraction (by collapsing all S_β^n's to the basepoint). As $\pi_i(C^n, C_{prelim}^n, *) = 0$ for $i \le n-1$ by cellular approximation (Corollary 1.11.1), it follows that the map i_* is an isomorphism for $i \le n-1$. Together with the fact that $(f_{prelim}^n)_*$ is also an isomorphism for $i \le n-1$, it follows that f_*^n is an isomorphism in this range too. This finishes the construction of the C^n's and f^n's.

For the last part, apply Theorem 1.9.2 to

$$\begin{array}{ccc} \emptyset & \longrightarrow & D \\ \downarrow & & \downarrow g \\ C & \xrightarrow{f} & X \end{array}$$

to get $\phi : C \to D$ such that $g \circ \phi \simeq f$; as f and g are weak homotopy equivalences, so is ϕ. $\square$

Corollary 1.12.2. *If X is n-connected, there is a weak homotopy equivalence $f : C \to X$ where C has a single 0-cell, and all other cells of dimension $> n$.*

Proof. As in the last proof, but start with $\{*\} = C^0 = C^1 = \cdots = C^n$. $\square$

1.13 Hurewicz's Theorem

By Theorem 1.10.5, if (X, A) is a relative CW complex only having cells of dimension $\ge n$ then $\pi_i(X, A, x_0) = 0$ for $i < n$. What about $\pi_n(X, A, x_0)$? We start by considering the situation where (X, A) consists of a single relative n-cell.

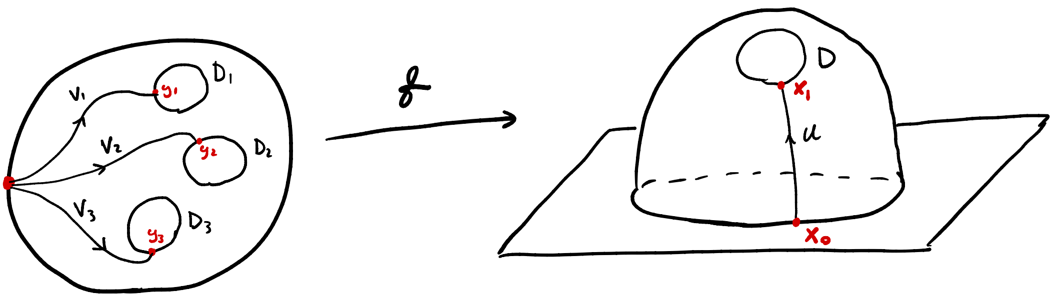
Theorem 1.13.1. *If $X = A \cup D^n$, then the map*

$$\Phi : (D^n, S^{n-1}, *) \longrightarrow (X, A, x_0)$$

given by the n-cell generates $\pi_n(X, A, x_0)$ as a $\mathbb{Z}[\pi_1(A, x_0)]$-module.[7]

Proof. Let $[f] \in \pi_n(X, A, x_0)$ with $n \geq 3$; our goal is to write it as a $\mathbb{Z}$-linear combination of elements $\gamma_\#([\Phi])$ for $\gamma \in \pi_1(A, x_0)$.

We have a map $f : (D^n, S^{n-1}, *) \to (X, A, x_0)$. There is an open set $U = \mathrm{int}(D^n) \subset X$, and so $f^{-1}(U) \subset D^n$ is an open subset contained in the interior. Now $f : f^{-1}(U) \to U$ is a map between open subsets of $\mathbb{R}^n$, so it may be homotoped near $f^{-1}(0)$ so that it is smooth and transverse to $0 \in U$ near this set: continue to call the map f. Then there is a small closed disc $0 \in D \subset U$ such that $f^{-1}(D) = \sqcup_{j=1}^{k} D_j$ is a disjoint union of closed discs, each of which is sent homeomorphically to D by f.



Let x_1 be a point on the boundary of D, and u be a path in $D^n \subset X$ from x_0 to x_1. Let $y_j \in D_j$ be the point that is sent by f to x_1.

Choose paths v_j in D^n from $*$ to y_j, and let $V = A \cup (D^n \setminus \{0\}) \subset X$, another open set. Then the disc D gives an element

$$[\mathrm{inc}_D] \in \pi_n(X, V, x_1)$$

which under the isomorphisms

$$\pi_n(X, A, x_0) \longrightarrow \pi_n(X, V, x_0) \xleftarrow{u_\#} \pi_n(X, V, x_1)$$

corresponds to $[\Phi]$.

Each $f \circ v_j$ is a path from x_0 to x_1 lying in V, but need not be homotopic to the path u. However, it is homotopic to $\gamma_j \cdot u$ for some $[\gamma_j] \in \pi_1(V, x_0) = \pi_1(A, x_0)$. Thus from the geometric description of sum of relative homotopy classes (Remark 1.3.2) we see that

$$[f] = \sum_{j=1}^{k} \pm (\gamma_j)_\#([\Phi]) \in \pi_n(X, V, x_0)$$

as required, where the signs come from whether the homeomorphisms $f|_{D_j} : D_j \xrightarrow{\cong} D$ preserve or reverse orientation. $\square$

Corollary 1.13.2. *If X is obtained from A by attaching a single n-cell along a map $\varphi : (S^{n-1}, *) \to (A, x_0)$, then $\pi_i(A, x_0) \to \pi_i(X, x_0)$ is an isomorphism for all $i < n-1$, and $\pi_{n-1}(A, x_0) \to \pi_{n-1}(X, x_0)$ is an epimorphism with kernel the $\mathbb{Z}[\pi_1(A, x_0)]$-submodule generated by $[\varphi] \in \pi_{n-1}(A, x_0)$.*[8]

Proof. The first two parts follows from the fact that $\pi_i(X, A, x_0) = 0$ for $i < n$, by Corollary 1.10.5. For the last part we consider the portion of the long exact sequence of (X, A) given by

$$\pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \pi_{n-1}(X, x_0) \longrightarrow 0,$$

which has $\partial([\Phi]) = [\varphi]$, and apply the previous theorem. □

Corollary 1.13.3. *If $X = A \cup D^n$ and A is simply connected, then $\pi_n(X, A, x_0)$ is isomorphic to $\mathbb{Z}$, generated by $[\Phi]$.*

Proof. By the theorem this group is generated as a $\mathbb{Z}$-module by $[\Phi]$, so it remains to show that $[\Phi]$ has infinite order. For this consider the Hurewicz homomorphism

$$h : \pi_n(X, A, x_0) \longrightarrow H_n(X, A; \mathbb{Z}) \cong_{excision} H_n(D^n, S^{n-1}; \mathbb{Z}) = \mathbb{Z}$$

and observe that $[\Phi]$ is tautologically sent to the generator $[D^n] \in H_n(D^n, S^{n-1}; \mathbb{Z})$. This does indeed have infinite order. □

Corollary 1.13.4 (Hopf's theorem)**.** *We have $\pi_n(S^n, *) \cong \mathbb{Z}$, generated by the identity map $\mathrm{Id}_{S^n} : S^n \to S^n$. That is, maps $f, g : S^n \to S^n$ are homotopic if and only if they have the same degree.*

Proof. Take $A = \{x_0\}$ in the previous corollary. □

Corollary 1.13.5. *If $X = \bigvee_{\alpha \in J} S^n_\alpha$ is a wedge of n-spheres with $n \geq 2$, then*

$$\pi_n(X, *) \cong \bigoplus_{\alpha \in J} \mathbb{Z},$$

generated by the inclusions $i_\alpha : S^n_\alpha \to X$.

Proof. Suppose first that J is finite. Then the inclusion $\bigvee_{\alpha \in J} S^n_\alpha \to \prod_{\alpha \in J} S^n_\alpha$ is a sub-CW complex, and the relative cells have dimension $\geq 2n$, so this pair is $(2n-1)$-connected. Thus the inclusion induces an isomorphism on $\pi_n(-)$. On the other hand a map into a product is a product of maps, so

$$\pi_n\left(\prod_{\alpha \in J} S^n_\alpha\right) \cong \prod_{\alpha \in J} \pi_n(S^n_\alpha) = \bigoplus_{\alpha \in J} \pi_n(S^n_\alpha) = \bigoplus_{\alpha \in J} \mathbb{Z}.$$

[7] For $n \geq 3$; for $n = 2$ the $\pi_1(A, x_0)$-orbit of $[\Phi]$ generates the non-abelian group $\pi_2(X, A, x_0)$; for $n = 1$ there is no sensible statement.

[8] For $n - 1 \geq 2$; for $n - 1 = 1$ the kernel is the normal subgroup of $\pi_1(A, x_0)$ generated by $[\varphi]$.

Now in general we have $X = \bigcup_{J' \subset J \text{ finite}} (\bigvee_{J'} S^n)$, and as any map from a compact space (such as S^n or $S^n \times [0,1]$) has image in a finite subcomplex, and taking homotopy groups preserves direct limits, the claim follows. $\square$

Theorem 1.13.6 (Hurewicz)**.** *Let X be a path-connected space with basepoint x_0, which is $(n-1)$-connected for some $n \geq 2$. Then $H_i(X;\mathbb{Z}) = 0$ for $0 < i < n$, and the Hurewicz homomorphism*

$$h : \pi_n(X, x_0) \longrightarrow H_n(X;\mathbb{Z})$$

is an isomorphism.

Proof. By Corollary 1.12.2 there is a weak equivalence $f : C \to X$ from a CW complex having a single 0-cell, and no other cells of dimension $< n$. By Corollary 1.9.4 the map $f_* : H_i(C;\mathbb{Z}) \to H_i(X;\mathbb{Z})$ is an isomorphism, but computing with cellular homology gives $H_i(C;\mathbb{Z}) = H_i^{cell}(C;\mathbb{Z}) = 0$ for $0 < i < n$ as claimed.

We now consider the Hurewicz homomorphism in degree n. The inclusion $i : C^{n+1} \to C$ is $(n+1)$-connected, so $i_* : \pi_n(C^{n+1}, *) \to \pi_n(C, *)$ is an isomorphism, and $i_* : H_n^{cell}(C^{n+1};\mathbb{Z}) \to H_n^{cell}(C;\mathbb{Z})$ is clearly an isomorphism too: thus we may suppose that $C = C^{n+1}$ is $(n+1)$-dimensional and apart from a 0-cell has no cells of dimension $< n$.

As any map from a compact set (such as S^n or Δ^n) into C^{n+1} intersects the interiors of finitely-many $(n+1)$-cells, we may suppose without loss of generality that C^{n+1} has finitely-many $(n+1)$-cells: we proceed by induction on the number of $(n+1)$-cells. By the previous corollary the map $h : \pi_n(C^n, *) \to H_n(C^n;\mathbb{Z})$ is an isomorphism, providing the base of the induction. Suppose then that $C^{n+1} = C' \cup_\varphi D^n$ and that the Hurewicz homomorphism for C' is an isomorphism. The Hurewicz homomorphism gives a map of long exact sequences

$$\begin{tikzcd}
\mathbb{Z} \cong \pi_{n+1}(C^{n+1}, C', *) \arrow[r] \arrow[d, "\cong"] & \pi_n(C', *) \arrow[r] \arrow[d, "\cong"] & \pi_n(C^{n+1}, *) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "\cong"] \\
\mathbb{Z} \cong H_{n+1}(C^{n+1}, C';\mathbb{Z}) \arrow[r] & H_n(C';\mathbb{Z}) \arrow[r] & H_n(C^{n+1};\mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where the first vertical map is an isomorphism by Corollary 1.13.3, the next is an isomorphism by assumption, and the other two are isomorphisms by observation: it follows from the 5-lemma that the middle vertical map is an isomorphism too. $\square$

Theorem 1.13.7. *If X is a path-connected space with basepoint x_0, then the Hurewicz homomorphism*

$$h : \pi_1(X, x_0) \longrightarrow H_1(X;\mathbb{Z})$$

is an epimorphism, with kernel the commutator subgroup of $\pi_1(X, x_0)$. That is, this homomorphism is the abelianisation of $\pi_1(X, x_0)$.

Proof sketch. It holds for $X = \bigvee_J S^1$ by direct calculation of both sides: the left-hand side is the free group on J and the right-hand side is the free abelian group on J. Then proceed as in the proof of the last theorem, looking at the effect on both sides of adding a cell. $\square$

1.14 Eilenberg–Mac Lane spaces and cohomology

Definition 1.14.1. A path-connected based space (X, x_0) is an **Eilenberg–Mac Lane space of type** (G, n) if

$$\pi_i(X, x_0) \cong \begin{cases} G & i = n \\ 0 & i \neq n. \end{cases}$$

(Note that if $n \geq 2$ then the group G must necessarily be abelian, but if $n = 1$ it need not be.)

Example 1.14.2. Recall from Q9 on Example Sheet 1 that if $\pi : \overline{X} \to X$ is a covering space then $\pi_* : \pi_i(\overline{X}, \bar{x}_0) \to \pi_i(X, \pi(\bar{x}_0))$ is an isomorphism for $i \geq 2$. Thus if a path-connected based space (X, x_0) has contractible universal cover then $\pi_i(X, x_0) = 0$ for all $i \geq 2$, and so X is an Eilenberg–Mac Lane space of type $(G, 1)$ for $G := \pi_1(X, x_0)$.

In particular,

(i) S^1 is Eilenberg–Mac Lane space of type $(\mathbb{Z}, 1)$, and

(ii) $\mathbb{RP}^\infty$ is an Eilenberg–Mac Lane space of type $(\mathbb{Z}/2, 1)$. △

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Lemma 1.14.3. *If either $n = 1$ and G is any group, or $n \geq 2$ and G is an abelian group, then an Eilenberg–Mac Lane space of type (G, n) exists, and can be taken to be a CW complex.*

Proof. Suppose first that $n \geq 2$ and G is abelian. Let

$$\bigoplus_{\beta \in J} \mathbb{Z} \xrightarrow{f} \bigoplus_{\alpha \in I} \mathbb{Z} \longrightarrow G \longrightarrow 0$$

be a partial free resolution of the abelian group G.[9] Form the CW complex $X'' := \bigvee_{\alpha \in I} S^n$, then form X' by attaching an $(n+1)$-cell to X'' for each $\beta \in J$, along a map φ_β with

$$[\varphi_\beta] = f(1_\beta) \in \bigoplus_{\alpha \in I} \mathbb{Z} = \pi_n(X'', *).$$

Now X' has no cells of dimension $< n$ apart from a 0-cell, so is $(n-1)$-connected, and by construction $H_n(X'; \mathbb{Z}) = (\bigoplus_{\alpha \in I} \mathbb{Z})/\mathrm{im}(f) \cong G$, so by Hurewicz's theorem $\pi_n(X'; *) \cong G$. Now form X from X' by attaching cells of dimension $\geq n+2$ to kill $\pi_i(X', *)$ for all $i > n$: it is then the required Eilenberg–Mac Lane space.

If $n = 1$ we choose a presentation $G = \langle I | J \rangle$ of the group G, let $X'' = \bigvee_{\alpha \in I} S^1$, and form X' by attaching 2-cells along φ_β with $[\varphi_\beta] = \beta \in \mathrm{Free}(I) = \pi_1(X'', *)$. Then proceed as above. □

[9] For example, the free abelian group on G surjects onto G, and its kernel is again free abelian as *any* subgroup of a free abelian group is free abelian.

Let (X, x_0) be an Eilenberg–Mac Lane space of type (G, n). If $n \geq 2$, or more generally if $n \geq 1$ and G is abelian, then by the Hurewicz theorem we have $H_{n-1}(X;\mathbb{Z}) = 0$ (or it is the free abelian group $\mathbb{Z}$ if $n = 1$) and an isomorphism

$$\phi_X : H_n(X;\mathbb{Z}) \xleftarrow{\sim} \pi_n(X, x_0) = G,$$

so by the Universal Coefficient Theorem

$$0 \longrightarrow \cancel{\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}), G)} \longrightarrow H^n(X;G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(H_n(X;\mathbb{Z}), G) \longrightarrow 0$$

the isomorphism ϕ_X corresponds to a canonical cohomology class

$$\iota_X \in H^n(X;G).$$

For any space Y there is therefore a function

$$\begin{aligned} [Y, X] &\longrightarrow H^n(Y;G) \\ [f : Y \to X] &\longmapsto f^*(\iota_X). \end{aligned} \tag{1.14.1}$$

Theorem 1.14.4. *If Y is a CW complex, then the function* (1.14.1) *is a bijection.*

Proof. To show surjectivity, let $[\phi] \in H^n(Y;G) \cong H^n_{cell}(Y;G)$, so that $\phi : C_n^{cell}(Y) \to G$ is a cocycle, i.e. a homomorphism such that $\phi \circ d = 0$. For each n-cell of Y, ϕ gives an element of $G = \pi_n(X, x_0)$. This determines a map

$$\bar{f} : Y^n/Y^{n-1} = \bigvee_{\alpha \in I_n} S^n \longrightarrow X,$$

and therefore a map

$$\bar{f}^n : Y^n \longrightarrow Y^n/Y^{n-1} \xrightarrow{\bar{f}} X.$$

By construction this satisfies $(\bar{f}^n)^*(\iota_X) = [\phi]|_{Y^n} \in H^n(Y^n;G)$. As the restriction map $H^n(Y;G) \to H^n(Y^n;G)$ is injective (by considering cellular cohomology, say), it suffices to show that the map $\bar{f}^n : Y^n \to X$ extends to a map $Y \to X$.

We first extend it to Y^{n+1}. To do so, note that for each $(n+1)$-cell Φ attached along a map $\varphi : S^n \to Y^n$, the composition $[\bar{f}^n \circ \varphi] \in \pi_n(X, x_0) = G$ is precisely $\phi \circ d(\Phi)$, which vanishes as ϕ is a cocycle. Thus $\bar{f}^n$ extends to a map $\bar{f}^{n+1} : Y^{n+1} \to X$. Now Y is obtained from Y^{n+1} by attaching cells of dimension $i > n+1$, but then $\pi_{i-1}(X, x_0) = 0$ so for an i-cell attached along φ the composition $S^{i-1} \xrightarrow{\varphi} Y^{i-1} \xrightarrow{\bar{f}^{i-1}} X$ is nullhomotopic so the map $\bar{f}^{n+1}$ extends to Y.

Injectivity is proved similarly, constructing a nullhomotopy out of a coboundary. □

Corollary 1.14.5. *If (X, x_0) is an Eilenberg–Mac Lane space of type (G, n), and (Z, z_0) is another which is a CW complex, then there is a weak homotopy equivalence $f : Z \to X$.*

Proof. The class $\iota_Z \in H^n(Z;G)$ corresponds by the previous theorem to a map $f : Z \to X$ (which we may homotope so that it sends z_0 to x_0) such that $f^*(\iota_X) = \iota_Z$. That is, the map

$$G \cong \pi_n(Z, z_0) \xrightarrow{f_*} \pi_n(X, x_0) \cong G$$

is the identity map of G: but as X and Z only have this homotopy group non-trivial, f is a weak homotopy equivalence. □

We will therefore generally write $K(G,n)$ for any space which is an Eilenberg–Mac Lane space of type (G,n): by Lemma 1.14.3 such spaces exist for all (G,n), and by Corollary 1.14.5 they are unique up to weak homotopy equivalence.

1.15 Fibrations

Definition 1.15.1. Let $\mathcal{C}$ be a class of spaces. A map $p : E \to B$ has the **homotopy lifting property** with respect to $\mathcal{C}$ if for each $X \in \mathcal{C}$ and for each commutative square

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ X \times [0,1] & \xrightarrow{H} & B \end{array} \tag{1.15.1}$$

there is a homotopy $\tilde{H} : X \times [0,1] \to E$ such that

$$p \circ \tilde{H} = H \quad \text{and} \quad \tilde{H}(-,0) = f(-).$$

If p has the homotopy lifting property with respect to $\mathcal{C} = \{\text{all spaces}\}$ then it is called a **Hurewicz fibration**. If it has the homotopy lifting property with respect to $\mathcal{C} = \{D^0, D^1, D^2, D^3, \ldots\}$ then it is called a **Serre fibration**.

We call B the **base**, E the **total space**, and $p^{-1}(b)$ the **fibre** over $b \in B$.

Example 1.15.2. Let $p = \mathrm{proj}_1 : E := B \times F \to B$ denote projection to the first coordinate. Then given a lifting diagram as above we can take

$$\tilde{H}(x,t) := (H(x,t), \mathrm{proj}_2 \circ f(x)).$$

This satisfies the required properties, so p is a Hurewicz fibration. △

Example 1.15.3. Any covering space $p : \overline{B} \to B$ is a Hurewicz fibration (by the homotopy lifting lemma). △

Example 1.15.4. A Hurewicz fibration is a Serre fibration. △

Example 1.15.5. A composition of Hurewicz (or Serre) fibrations is a Hurewicz (or Serre) fibration. △

Example 1.15.6. Let $p : E \to B$ be a map, and $\phi : B' \to B$ be a map. Let

$$E' = B' \times_B E := \{(b',e) \in B' \times E \mid \phi(b') = p(e)\}$$

and $p' : E' \to B'$ be $p'(b',e) = b'$. Then p' is called the **pullback of p along ϕ**; it fits into a diagram

$$\begin{array}{ccc} E' & \xrightarrow{(b',e)\mapsto e} & E \\ \downarrow p' & & \downarrow p \\ B' & \xrightarrow{\phi} & B \end{array}$$

and has the following universal property: Given maps $X \to E$ and $X \to B'$ which become
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equal in B, there is a unique map $X \to E'$ making the evident diagram commute.

Suppose now that p has the homotopy lifting property with respect to $\mathcal{C}$, and consider a lifting problem

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & E' \arrow[d, "p'"] \\ X \times [0,1] \arrow[r, "H"] & B' \end{tikzcd}$$

with $X \in \mathcal{C}$. We can incorporate this into the larger commutative diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & E' \arrow[r, "(b',e)\mapsto e"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "H"] & B' \arrow[r, "\phi"] & B, \end{tikzcd}$$

and the outer square is a homotopy lifting problem for p, which can be solved as $X \in \mathcal{C}$ and p has the homotopy lifting property with respect to $\mathcal{C}$: let $\widetilde{\phi \circ H} : X \times [0,1] \to E$ be the lift. But by the universal property this produces a lift $\tilde{H} : X \times [0,1] \to E'$ of the original lifting problem. Concretely, it is given by

$$\tilde{H}(x,t) := (H(x,t), \widetilde{\phi \circ H}(x,t)) \in E'.$$

This argument shows that the class of maps having the homotopy lifting property with respect to $\mathcal{C}$ is closed under the formation of pullbacks. In particular, the pullback of a Hurewicz (or Serre) fibration is again a Hurewicz (or Serre) fibration. We often write $\phi^*E := E'$ and $\phi^*p := p' : \phi^*E \to B'$. △

Lemma 1.15.7. *Let $p : E \to B$ be a Serre fibration, (X, A) be a relative CW complex, and a commutative diagram*

$$\begin{tikzcd} X \times \{0\} \cup A \times [0,1] \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "H"] & B \end{tikzcd}$$

be given. Then there is a map $\tilde{H} : X \times [0,1] \to E$ extending f and lifting H.

Proof. Consider first the case $(X, A) \cong (D^n, S^{n-1})$. There is a homeomorphism of pairs

$$(D^n \times [0,1], D^n \times \{0\} \cup S^{n-1} \times [0,1]) \cong (D^n \times [0,1], D^n \times \{0\})$$

which translates this lifting problem to that of (1.15.1) for $X = D^n$, which can be solved as p is a Serre fibration.

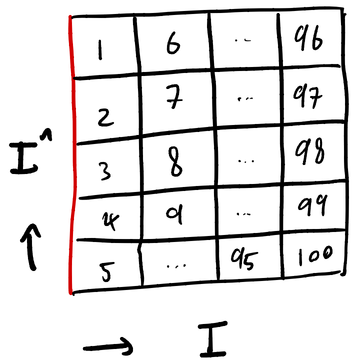
For a general relative CW complex (X, A), we can construct the lift one cell at a time using the above. □

Theorem 1.15.8 (Local-to-global principle for Serre fibrations)**.** *If $p : E \to B$ is a map such that there is an open cover $\{U_\alpha\}_{\alpha \in I}$ with each $p|_{p^{-1}(U_\alpha)} : p^{-1}(U_\alpha) \to U_\alpha$ is a Serre fibration, then p is a Serre fibration.*

Proof. Using a homeomorphism $D^n \cong I^n$, let a lifting problem

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{f} & E \\ \big\downarrow & & \big\downarrow p \\ I^n \times [0,1] & \xrightarrow{H} & B \end{array}$$

be given. Now $\{H^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of $I^n \times [0,1]$, so by the Lesbegue number lemma we may choose a grid fine enough that each cube lies in some $H^{-1}(U_\alpha)$. Order the cubes as shown in the figure below, so that each cube except the rightmost ones has a face to its right labelled *higher*.



Now we can lift the map H on the 1st cube, extending the lift given on its red face, using that this cube has image in some U_α, and that p is a Serre fibration over U_α. Similarly, supposing that the first $(i-1)$ cubes have been compatibly lifted, we may find a lift on the ith cube, extending the lift already given over part of its boundary, as no lift has yet been given on its rightmost face. Continuing in this way gives the desired lift. $\square$

Corollary 1.15.9. *A fibre bundle is a Serre fibration.*

Proof. If $p : E \to B$ is a fibre bundle then by definition there is an open cover $\{U_\alpha\}_{\alpha \in I}$ of B and homeomorphisms $p^{-1}(U_\alpha) \cong U_\alpha \times F$ over U_α, identifying $p|_{p^{-1}(U_\alpha)} : p^{-1}(U_\alpha) \to U_\alpha$ with $\mathrm{proj}_1 : U_\alpha \times F \to U_\alpha$. By Example 1.15.2 such maps are Serre fibrations, so by the previous theorem p is too. $\square$

Theorem 1.15.10. *Let (B, b_0) be a path-connected based space, $p : E \to B$ be a Serre fibration, and $x_0 \in F := p^{-1}(b_0)$. Then the map*

$$p_* : \pi_n(E, F, x_0) \longrightarrow \pi_n(B, b_0)$$

is an isomorphism for all $n \geq 1$, so there is a long exact sequence

$$
\begin{tikzcd}
& \cdots \arrow[r] & \pi_{n+1}(B,b_0) \arrow[dll, rounded corners, "\partial"] \\
\pi_n(F,x_0) \arrow[r, "i_*"] & \pi_n(E,x_0) \arrow[r, "p_*"] & \pi_n(B,b_0) \arrow[dll, rounded corners, "\partial"] \\
\pi_{n-1}(F,x_0) \arrow[r, "i_*"] & \pi_{n-1}(E,x_0) \arrow[r, "j_*"] & \cdots
\end{tikzcd}
$$

Proof. Let $f : (I^n, \partial I^n) \to (B, b_0)$ be a map. Construct the relative lifting problem

$$
\begin{tikzcd}
\sqcap^{n-1} \arrow[r, equal] & I^{n-1} \times \{0\} \cup \partial I^{n-1} \times [0,1] \arrow[r, "\mathrm{const}_{x_0}"] \arrow[d] & E \arrow[d, "p"] \\
& I^{n-1} \times [0,1] \arrow[r, "f"] & B,
\end{tikzcd}
$$

which has a solution $\hat{f} : I^{n-1} \times [0,1] \to E$. By construction this sends ∂I^n to F and $\sqcap^{n-1}$ to x_0, so we have $[\hat{f}] \in \pi_n(E, F, x_0)$, and by construction this satisfies $p_*([\hat{f}]) = [p \circ \hat{f}] = [f]$. Thus p_* is surjective.

Suppose that $[g] \in \pi_n(E, F, x_0)$ satisfies $p_*([g]) = 0$, so that $p \circ g$ is homotopic to const_{x_0}: let $H : I^n \times [0,1] \to B$ be such a homotopy, which sends $\partial I^n \times [0,1]$ to b_0. Now $\mathrm{const}_{x_0} : \sqcap^{n-1} \times [0,1] \to E$ is a lift of $H|_{\sqcap^{n-1} \times [0,1]}$, so we may construct the relative lifting problem

$$
\begin{tikzcd}
I^n \times \{0\} \cup \sqcap^{n-1} \times [0,1] \arrow[r, "g \cup \mathrm{const}_{x_0}"] \arrow[d] & E \arrow[d, "p"] \\
I^n \times [0,1] \arrow[r, "H"] & B.
\end{tikzcd}
$$

This has a solution $\tilde{H} : I^n \times [0,1] \to E$, which is a homotopy relative to $\sqcap^{n-1}$ from g to a map into $F = p^{-1}(b_0)$, which by the Compression Criterion (Section 1.5) shows that $[g] = 0 \in \pi_n(E, F, x_0)$. ☐

Example 1.15.11. Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere. The group $S^1 = U(1)$ of unit complex numbers acts freely on S^{2n+1} with quotient the complex projective space $\mathbb{CP}^n$. The map

$$p : S^{2n+1} \longrightarrow \mathbb{CP}^n$$

is a fibre bundle, so a Serre fibration, with fibre over each point homeomorphic to S^1. Thus there is a long exact sequence

$$\cdots \longrightarrow \pi_i(S^1, x_0) \xrightarrow{i_*} \pi_i(S^{2n+1}, x_0) \xrightarrow{p_*} \pi_i(\mathbb{CP}^n, *) \xrightarrow{\partial} \pi_{i-1}(S^1, x_0) \longrightarrow \cdots$$

Combining this with

$$\pi_i(S^1, x_0) = \begin{cases} \mathbb{Z} & i = 1 \\ 0 & \text{else} \end{cases}$$

and $\pi_i(S^{2n+1}, x_0) = 0$ for $i \leq 2n$, we find that

$$\pi_i(\mathbb{CP}^n, *) = \begin{cases} 0 & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & 2 < i \leq 2n. \end{cases}$$

In particular, for $\mathbb{CP}^\infty = \cup_{n\geq 0}\mathbb{CP}^n$ we find

$$\pi_i(\mathbb{CP}^\infty, *) = \begin{cases} 0 & i \neq 2 \\ \mathbb{Z} & i = 2, \end{cases}$$

so that $\mathbb{CP}^\infty$ is an Eilenberg–Mac Lane space of type $(\mathbb{Z}, 2)$, i.e. $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$.

The map $H := p : S^3 \to S^2 = \mathbb{CP}^1$, with fibre S^1, is known as the **Hopf map**. The sequence

$$0 = \pi_3(S^1, x_0) \xrightarrow{i_*} \pi_3(S^3, x_0) \xrightarrow{H_*} \pi_3(S^2, *) \xrightarrow{\partial} \pi_2(S^1, x_0) = 0$$

shows that $\pi_3(S^2, *) \cong \mathbb{Z}$ generated by $[H]$. △

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1.16 Comparing fibres

Theorem 1.16.1. *Let $p : E \to B$ be a Serre fibration, $b_0, , b_1 \in B$ be points with fibres $F_{b_i} := p^{-1}(b_i)$, and $u : [0,1] \to B$ be a path from b_0 to b_1. Then there is a space F_u and weak homotopy equivalences*

$$F_{b_0} \xrightarrow{i^u_{b_0}} F_u \xleftarrow{i^u_{b_1}} F_{b_1}.$$

Furthermore, if v is a path from b_1 to b_2 then there is a commutative diagram of weak homotopy equivalences

$$\begin{tikzcd}
& & F_{u\cdot v} & & \\
& F_u \arrow[ur] & & F_v \arrow[ul] & \\
F_{b_0} \arrow[ur, "i^u_{b_0}"] \arrow[uurr, bend left, "i^{u\cdot v}_{b_0}"] & & F_{b_1} \arrow[ul, "i^u_{b_1}"'] \arrow[ur, "i^v_{b_1}"] & & F_{b_2}. \arrow[ul, "i^v_{b_2}"'] \arrow[uull, bend right, "i^{u\cdot v}_{b_2}"']
\end{tikzcd}$$

Proof. Define $F_u := u^*E$. There are maps of Serre fibrations

$$\begin{tikzcd}
F_{b_0} \arrow[r] \arrow[d] & F_u \arrow[d, "u^*p"] & F_{b_1} \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & {[0,1]} & \{1\}. \arrow[l]
\end{tikzcd}$$

The long exact sequence on homotopy groups for the fibration u^*p based at a point $x_0 \in F_u$ with $(u^*p)(x_0) = 0$ is

$$\cdots \longrightarrow \pi_{i+1}([0,1],\{0\}) \longrightarrow \pi_i(F_{b_0}, x_0) \longrightarrow \pi_i(F_u, x_0) \longrightarrow \pi_i([0,1],\{0\}) \longrightarrow \cdots$$

and as $[0,1]$ is contractible the middle map is an isomorphism: thus $F_{b_0} \to F_u$ is a weak homotopy equivalence. The map $F_{b_1} \to F_u$ is also a weak homotopy equivalence by the same argument.

For the second part, note that the composition

$$[0,1] \xrightarrow{\cong} [0,1/2] \subset [0,1] \xrightarrow{u \cdot v} B$$

is the path u, so pulling p back along these maps gives a map of Serre fibrations

$$\begin{tikzcd} F_u \arrow[rr] \arrow[d] & & F_{u\cdot v} \arrow[d] \\ {[0,1]} \arrow[r, "\text{rescale}"] & {[0,1/2]} \arrow[r, "inc"] & {[0,1];} \end{tikzcd}$$

the map of total spaces is the required map $F_u \to F_{u \cdot v}$. These fibrations have the same fibres, and their bases are both contractible, so it follows from the map of long exact sequences for these fibrations that the map $F_u \to F_{u \cdot v}$ is a weak equivalence. The map $F_v \to F_{u \cdot v}$ is constructed analogously, and it is then easy to verify that the diagram so obtained commutes. □

By the first part, if B is path-connected then the fibres over different points are all weakly homotopy equivalent to each other. Because of this, we will often say that

$$\text{“}F \xrightarrow{i} E \xrightarrow{p} B \text{ is a fibration sequence,”}$$

meaning that p is a fibration, B is path-connected, and that F is the fibre of p over some point: the fibre over any other point is then weakly homotopy equivalent to F.

Corollary 1.16.2. *Let $p : E \to B$ be a Serre fibration, $b_0, b_1 \in B$ be points with fibres $F_{b_i} := p^{-1}(b_i)$, and $u : [0,1] \to B$ be a path from b_0 to b_1. Then there is an induced isomorphism*

$$u_\# : H_*(F_{b_0}) \xrightarrow{\sim} H_*(F_{b_1}).$$

If v is a path from b_1 to b_2, then $v_\# \circ u_\# = (u \cdot v)_\#$. In particular $\pi_1(B, b_0)$ acts on $H_*(F_{b_0})$. Similarly, it acts on the cohomology of F_{b_0}.

Proof. As weak homotopy equivalences induce isomorphisms on homology, we define $u_\#$ to be the composition of the isomorphisms

$$H_*(F_{b_0}) \xrightarrow{\sim} H_*(F_u) \xleftarrow{\sim} H_*(F_{b_1}).$$

The formula $v_\# \circ u_\# = (u \cdot v)_\#$ then follows from the commutative diagram in the last theorem. □

1.17 Function spaces

Definition 1.17.1. For spaces X and Y, we let $\mathrm{map}(X,Y)$ denote the set of continuous maps $f: X \to Y$. We endow this set with the **compact-open topology**: the topology generated by the subbasis of sets

$$W(K,U) := \{f : X \to Y \mid f(K) \subset U\},$$

ranging over all compact $K \subset X$ and all open $U \subset Y$.

Lemma 1.17.2. *If X is locally compact*[10] *then the evaluation map*

$$\begin{aligned} e_{X,Y} : \mathrm{map}(X,Y) \times X &\longrightarrow Y \\ (f,x) &\longmapsto f(x) \end{aligned}$$

is continuous.

Proof. Let $U \ni f(x)$ be an open neighbourhood. As X is locally compact there is a compact neighbourhood $x \in K \subset f^{-1}(U)$. Then $e_{X,Y}$ sends $W(K,U) \times K$ into U, so the open set $W(K,U) \times \mathrm{int}(K) \ni (f,x)$ lies in U. □

Lemma 1.17.3. *Let $f : Z \times X \to Y$ be continuous. Then its adjoint*

$$\begin{aligned} f^{ad} : Z &\longrightarrow \mathrm{map}(X,Y) \\ z &\longmapsto (x \mapsto f(z,x)) \end{aligned}$$

is continuous.

Proof. We must show that $(f^{ad})^{-1}(W(K,U))$ is open, for $K \subset X$ compact and $U \subset Y$ open. If $f^{ad}(z) \in W(K,U)$ then $\{z\} \times K \subset f^{-1}(U)$. As K is compact there is an open $V \ni z$ such that $V \times K \subset f^{-1}(U)$, but then $z \in V \subset (f^{ad})^{-1}(W(K,U))$. □

Corollary 1.17.4. *If X is locally compact and $f^{ad} : Z \to \mathrm{map}(X,Y)$ is continuous, then $f : Z \times X \to Y$ is continuous.*

Proof. The map f is

$$Z \times X \overset{f^{ad} \times \mathrm{Id}_X}{\longrightarrow} \mathrm{map}(X,Y) \times X \overset{e_{X,Y}}{\longrightarrow} Y.$$

□

Corollary 1.17.5. *If X and Z are locally compact, then the map*

$$\begin{aligned} \alpha : \mathrm{map}(Z \times X, Y) &\longrightarrow \mathrm{map}(Z, \mathrm{map}(X,Y)) \\ f &\longmapsto f^{ad} \end{aligned}$$

is a homeomorphism.

[10] i.e. for every point $x \in X$ and every open neighbourhood $U \ni x$, there is a compact set K with $x \in \mathrm{int}(K) \subset K \subset U$.

Proof. By the last two results this function is well-defined and a bijection. Also, by the last results,

$$\begin{aligned}\alpha \text{ is continuous} &\iff \mathrm{map}(Z \times X, Y) \times Z \longrightarrow \mathrm{map}(X, Y) \text{ is continuous}\\ &\iff \mathrm{map}(Z \times X, Y) \times Z \times X \longrightarrow Y \text{ is continuous}\end{aligned}$$

which it is as $Z \times X$ is locally compact. Similarly α^{-1} is continuous if and only if $\mathrm{map}(Z, \mathrm{map}(X, Y)) \times Z \times X \to Y$ is continuous, but this factors as

$$\mathrm{map}(Z, \mathrm{map}(X, Y)) \times Z \times X \overset{e_{Z,\mathrm{map}(X,Y)} \times \mathrm{Id}_X}{\longrightarrow} \mathrm{map}(X, Y) \times X \overset{e_{X,Y}}{\longrightarrow} Y$$

so is indeed continuous. □

Theorem 1.17.6. *If (X, A) is a relative CW complex and X is locally compact, then the restriction map*

$$\begin{aligned}res : \mathrm{map}(X, Y) &\longrightarrow \mathrm{map}(A, Y)\\ f &\longmapsto f|_A\end{aligned}$$

is a Serre fibration.

Proof. Suppose given a homotopy lifting problem

$$\begin{CD} D^n \times \{0\} @>{f^{ad}}>> \mathrm{map}(X, Y)\\ @VVV @VV{res}V\\ D^n \times [0, 1] @>{H^{ad}}>> \mathrm{map}(A, Y), \end{CD}$$

where f^{ad} and H^{ad} are adjoint to maps $f : X \times D^n \times \{0\} \to Y$ and $H : A \times D^n \times [0, 1] \to Y$. As $(X \times D^n, A \times D^n)$ is again a relative CW complex, it has the homotopy extension property, so H extends to a $\bar{H} : X \times D^n \times [0, 1] \to Y$ starting at f. By adjunction this gives a map

$$\bar{H}^{ad} : D^n \times [0, 1] \longrightarrow \mathrm{map}(X, Y),$$

lifting H^{ad} and extending f^{ad}. □

Example 1.17.7. Consider the relative CW complex $([0, 1], \{0\})$. The theorem shows that the evaluation map

$$\begin{aligned}ev_0 : PX := \mathrm{map}([0, 1], X) &\longrightarrow X\\ \gamma &\longmapsto \gamma(0)\end{aligned}$$

is a Serre fibration, called the **path fibration**. The fibre $ev_0^{-1}(x_0) =: P_{x_0}X$ is the space of paths in X starting at x_0, and this is contractible: the homotopy

$$\begin{aligned}P_{x_0}X \times [0, 1] &\longrightarrow P_{x_0}X\\ (\gamma, t) &\longmapsto \left(s \mapsto \begin{cases}\gamma(s) & 0 \leq s \leq t\\ \gamma(t) & t \leq s \leq 1\end{cases}\right)\end{aligned}$$

is easily checked to be a deformation retraction. △

Lemma 1.17.8. *In fact $ev_0 : PX \to X$ is a Hurewicz fibration.*

Proof. Supposing given a homotopy lifting problem

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "f"] \arrow[d] & PX \arrow[d, "ev_0"] \\ Z \times [0,1] \arrow[r, "H"] & X, \end{tikzcd}$$

with Z an arbitrary space. Define a map $\tilde{H}^{ad} : Z \times [0,1] \times [0,1] \to X$ by the formula

$$\tilde{H}^{ad}(z,t,s) := \begin{cases} H(z, t-2s) & 0 \le s \le t/2 \\ f(z,0)(\frac{s-t/2}{1-t/2}) & t/2 \le s \le 1 \end{cases}.$$

This is continuous by the gluing lemma, so its adjoint $\tilde{H} : Z \times [0,1] \to PX$ is continuous by Lemma 1.17.3. We easily check that $\tilde{H}$ lifts H and extends f. □

The same argument, with more notation, shows that

$$\begin{aligned} ev_0 \times ev_1 : PX &\longrightarrow X \times X \\ \gamma &\longmapsto (\gamma(0), \gamma(1)) \end{aligned}$$

is a Hurewicz fibration.

Let $f : Y \to X$ be a map, and form the pullback

$$\begin{tikzcd} E_f \arrow[r, equal] & (f \times \mathrm{Id}_X)^* PX \arrow[r] \arrow[d] & PX \arrow[d, "ev_0 \times ev_1"] \\ & Y \times X \arrow[r, "f \times \mathrm{Id}_X"] & X \times X. \end{tikzcd}$$

Unravelling the definition, we have

$$E_f \cong \{(y,\gamma) \in Y \times PX \mid \gamma(0) = f(y)\},$$

and the map $(y,\gamma) \mapsto (y, \gamma(1)) : E_f \to Y \times X$, being the pullback of $ev_0 \times ev_1$, is a Hurewicz fibration. As the projection $Y \times X \to X$ is also a Hurewicz fibration, the map

$$\begin{aligned} p_f : E_f &\longrightarrow X \\ (y,\gamma) &\longmapsto \gamma(1) \end{aligned}$$

Lecture 10

is a Hurewicz fibration.

On the other hand, the maps $y \mapsto (y, \mathrm{const}_{f(y)}) : Y \to E_f$ and $(y,\gamma) \mapsto y : E_f \to Y$ are easily seen to be homotopy inverses. In total, the diagram

$$\begin{tikzcd} & E_f \arrow[d, "p_f"] \\ Y \arrow[ur, "y \mapsto (y, \mathrm{const}_{f(y)})"] \arrow[r, "f"] & X \end{tikzcd}$$

shows that the map $f : Y \to X$ may be replaced up to homotopy equivalence with a Hurewicz fibration $p_f : E_f \to X$. We call the fibre

$$p_f^{-1}(x_0) = \{(y,\gamma) \in Y \times PX \mid \gamma(0) = f(y), \gamma(1) = x_0\}$$

the **homotopy fibre** of f at the point $x_0 \in X$. The long exact sequence on homotopy groups for the fibration p_f, along with the homotopy equivalence $Y \simeq E_f$, gives a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{n+1}(X,x_0) \arrow[dll, "\partial"'] \\
\pi_n(p_f^{-1}(x_0),(y_0,\mathrm{const}_{x_0})) \arrow[r] & \pi_n(Y,y_0) \arrow[r, "f_*"] & \pi_n(X,x_0) \arrow[dll, "\partial"'] \\
\pi_{n-1}(p_f^{-1}(x_0),(y_0,\mathrm{const}_{x_0})) \arrow[r] & \pi_{n-1}(Y,y_0) \arrow[r, "f_*"] & \cdots
\end{tikzcd}$$

Applied to the inclusion map $f : \{x_0\} \to X$, we get

$$E_f = \{\gamma \in PX \mid \gamma(0) = x_0\} = P_{x_0}X$$

and

$$p_f^{-1}(x_0) = \{\gamma \in PX \mid \gamma(0) = x_0, \gamma(1) = x_0\} =: \Omega_{x_0}X$$

the **loop space** of X based at x_0. The long exact sequence on homotopy groups for the fibration p_f shows that

$$\partial : \pi_n(X,x_0) \xrightarrow{\sim} \pi_{n-1}(\Omega_{x_0}X, \mathrm{const}_{x_0})$$

is an isomorphism for $n \geq 1$. In this sense ΩX has the analogous effect on homotopy groups as the suspension ΣX does on homology groups.

As the space $\Omega_{x_0}X$ is again based, at const_{x_0}, we can iterate this construction to form $\Omega^k_{x_0}X$. Unravelling definitions shows that

$$\Omega^k_{x_0}X \cong \{f \in \mathrm{map}(I^k, X) \mid f(\partial I^k) = x_0\}.$$

If x_1 lies in the same path-component as x_0 then Theorem 1.16.1 applied to the fibres of the fibration

$$ev_0 \times ev_1 : PX \longrightarrow X \times X$$

over (b_0, b_0) and (b_1, b_1) shows that $\Omega_{x_1}X$ and $\Omega_{x_0}X$ are weakly equivalent[11], so if X is path-connected then we often just write ΩX to mean the loop space taken at some basepoint.

[11] In fact we easily see that they are homotopy equivalent: conjugating by a path u from x_0 to x_1 gives a map $\Omega_{x_0}X \to \Omega_{x_1}X$, conjugating by the reverse path gives a map $\Omega_{x_1}X \to \Omega_{x_0}X$, and the two compositions are easily seen to be homotopic to the identity.

1.18 The Moore–Postnikov tower

Lemma 1.18.1. *Let $f : (A, a_0) \to (X, x_0)$ be a map of path-connected and based spaces, and $n \geq 0$. There is a relative CW complex (Z_n, A) and an extension $g_n : Z_n \to X$ of f such that*

(i) (Z_n, A) only has relative cells of dimension $> n$, so in particular is n-connected,

(ii) the map $(g_n)_* : \pi_n(Z_n, a_0) \to \pi_n(X, x_0)$ is injective, and for all $i > n$ the maps $(g_n)_* : \pi_i(Z_n, a_0) \to \pi_i(X, x_0)$ are isomorphisms.[12]

To emphasise: the factorisation

$$f : A \longrightarrow Z_n \xrightarrow{g_n} X$$

has the first map an isomorphism on homotopy groups in degrees $< n$, the second map an isomorphism in degrees $> n$, and in degree n

$$f_* : \pi_n(A, a_0) \longrightarrow \pi_n(Z_n, a_0) \longrightarrow \pi_n(X, x_0)$$

is the (unique up to isomorphism) factorisation of f_* as an epimorphism followed by a monomorphism.

Proof. First construct (T^n, A) by attaching $(n+1)$-cells to A to kill

$$\ker(f_* : \pi_n(A, a_0) \to \pi_n(X, x_0)),$$

and using the nullhomotopies in X to extend the map f to a $g_n^n : T^n \to X$. Now construct relative CW complexes $T^n \subset T^{n+1} \subset T^{n+2} \subset \cdots$ with $g_n^i : T^i \to X$ such that

(i) (T^i, A) only has relative cells of dimensions $n + 1 \leq * \leq i + 1$

(ii) the map $\pi_j(g_n^i)$ is injective for $n \leq j < i + 1$ and surjective for $n < j \leq i + 1$.

This can be done exactly as in the proof of the CW approximation theorem: if T^{i-1} has been constructed, first attach $(i+1)$-cells to kill $\ker(f_* : \pi_i(T^{i-1}) \to \pi_i(X))$, and then wedge on $(i+1)$-spheres to generate $\pi_{i+1}(X)$.

Finally, let $Z_n := \bigcup_{i \geq n} T^i$. ☐

If $g_n : Z_n \to X$ and $g_{n+1} : Z_{n+1} \to X$ are maps given by the lemma, consider the commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Z_n \arrow[d, "g_n"] \\
Z_{n+1} \arrow[r, "g_{n+1}"] & X.
\end{tikzcd}$$

The relative CW complex (Z_{n+1}, A) only has cells of dimension $\geq n + 2$. On the other hand we have a long exact sequence

$$\cdots \pi_i(Z_n) \longrightarrow \pi_i(X) \longrightarrow \pi_i(X, Z_n) \longrightarrow \pi_{i-1}(Z_n) \longrightarrow \pi_{i-1}(X) \cdots,$$

[12] This property is often phrased as “the map $g_n : Z_n \to X$ is n-co-connected”.

the left-hand map is surjective for $i > n$, so for $i \geq n+1$, and the right-hand map is injective for $i-1 \geq n$, so for $i \geq n+1$. Thus $\pi_i(X, Z_n) = 0$ for $i \geq n+1$. It then follows from the compression lemma that there is a map

$$p_{n+1} : Z_{n+1} \longrightarrow Z_n$$

which is the identity on A, such that $g_n \circ p_{n+1}$ is homotopic to g_{n+1} relative to A. In total we obtain a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & Z_2 \arrow[d, "p_2"] \arrow[ddr, "g_2"] & \\
 & Z_1 \arrow[d, "p_1"] \arrow[dr, "g_1"] & \\
f : A \arrow[r] \arrow[ur] \arrow[uur] & Z_0 \arrow[r, "g_0"] & X
\end{tikzcd}$$

where the left-hand triangles commute, and the right-hand triangles commute up to homotopy. By redefining $g_n' = g_0 \circ p_1 \circ p_2 \circ \cdots \circ p_n$ we may assume that the right-hand triangles commute too. This is the **Moore–Postnikov tower** of the map $f : A \to X$.

Example 1.18.2. For a space X consider the map $f : X \to \{*\}$. Its Moore–Postnikov tower gives a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_2 \arrow[d, "p_2"] \\
 & X_1 \arrow[d, "p_1"] \\
X \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uur, "f_2"] & X_0
\end{tikzcd}$$

where $\pi_i(X_n) = 0$ for $i \geq n$ and

$$(f_n)_* : \pi_i(X, x_0) \longrightarrow \pi_i(X_n, f_n(x_0)) \text{ is an isomorphism for } i < n.$$

This is the **Postnikov tower** of X.

It is perhaps suggestive to use "interval" notation and write $X_n = X[0, n-1]$: this space has the same homotopy groups as X in the range of degrees $0 \leq * \leq n-1$ and trivial homotopy groups outside this range. $\triangle$

Example 1.18.3. Choose a basepoint $x_0 \in X$, and consider $f : \{x_0\} \to X$. Its Moore–Postnikov tower gives a diagram

$$
\begin{tikzcd}
& \vdots \arrow[d] & \\
& Z_2 \arrow[d, "p_2"] \arrow[ddr, "g_2"] & \\
& Z_1 \arrow[d, "p_1"] \arrow[dr, "g_1"] & \\
\{x_0\} \arrow[r] \arrow[uur] \arrow[ur] & Z_0 \arrow[r, "g_0"] & X
\end{tikzcd}
$$

where Z_n is n-connected, and $(g_n)_* : \pi_i(Z_n, x_0) \to \pi_i(X, x_0)$ is an isomorphism for $i > n$. This is the **Whitehead tower** of X (at the basepoint x_0).

It is again suggestive to write $Z_n = X[n+1, \infty]$: this space has the same homotopy groups as X in the range of degrees $n + 1 \le * < \infty$ and trivial homotopy groups outside this range. △

In the Whitehead tower consider the long exact sequence

$$
\begin{tikzcd}
\pi_i(Z_{n+1}) \arrow[r, "(p_{n+1})_*"] \arrow[d, "(g_{n+1})_*"] & \pi_i(Z_n) \arrow[r] \arrow[d, "(g_n)_*"] & \pi_i(Z_n, Z_{n+1}) \arrow[r, "\partial"] & \pi_{i-1}(Z_{n+1}) \arrow[r, "(p_{n+1})_*"] \arrow[d, "(g_{n+1})_*"] & \pi_{i-1}(Z_n) \arrow[d, "(g_n)_*"] \\
\pi_i(X) \arrow[r, equal] & \pi_i(X) & & \pi_{i-1}(X) \arrow[r, equal] & \pi_{i-1}(X).
\end{tikzcd}
$$

If $i \le n$ then $\pi_i(Z_n) = 0 = \pi_{i-1}(Z_{n+1})$, so $\pi_i(Z_n, Z_{n+1}) = 0$. If $i \ge n+2$ then the vertical maps are all isomorphisms, so $\pi_i(Z_n, Z_{n+1}) = 0$ too. In degree $n+1$ we have $\pi_{n+1}(Z_{n+1}) = 0$, $\pi_{n+1}(Z_n) = \pi_{n+1}(X)$ and $\pi_n(Z_{n+1}) = 0$, and so

$$\pi_i(Z_n, Z_{n+1}, x_0) \cong \begin{cases} \pi_{n+1}(X, x_0) & i = n+1 \\ 0 & \text{else.} \end{cases}$$

In other words, the homotopy fibre of $p_n : Z_{n+1} \to Z_n$ is a $K(\pi_{n+1}(X, x_0), n)$.

Similarly, for the Postnikov tower of X we find

$$\pi_i(X_n, X_{n+1}, x_0) \cong \begin{cases} \pi_n(X, x_0) & i = n+1 \\ 0 & \text{else,} \end{cases}$$

so the homotopy fibre of $p_n : X_{n+1} \to X_n$ is a $K(\pi_n(X, x_0), n)$.

Both of these become easier to remember using the "interval" notation: the first says that the homotopy fibre of $X[n, \infty] \to X[n-1, \infty]$ is a $K(\pi_{n-1}(X, x_0), n-2)$, and the second says that the homotopy fibre of $X[0, n] \to X[0, n-1]$ is a $K(\pi_n(X, x_0), n)$.

1.19 A motivating strategy

At this point we have developed various techniques for manipulating homotopy groups of spaces, but we do not yet know how to compute many of them. We know Hopf's theorem, that $\pi_n(S^n) \cong \mathbb{Z}$ generated by the identity map, and in Example 1.15.11 we calculated that $\pi_3(S^2) \cong \mathbb{Z}$ generated by the Hopf map $H : S^3 \to S^2$.

If X is a simply-connected space and $H_i(X;\mathbb{Z}) = 0$ for $0 < i < n$ then the Hurewicz theorem shows that

$$h : \pi_n(X, x_0) \xrightarrow{\sim} H_n(X;\mathbb{Z}).$$

As we are quite good at calculating homology, this means that we can often calculate the *lowest* non-trivial homotopy group.

If X is a path-connected space and

$$\cdots \longrightarrow X[n,\infty] \longrightarrow X[n-1,\infty] \longrightarrow \cdots \longrightarrow X[3,\infty] \longrightarrow X[2,\infty] \longrightarrow X[1,\infty] \simeq X$$

is the Whitehead tower of X, then as $X[n,\infty]$ is $(n-1)$-connected we have isomorphisms

$$\pi_n(X) \xleftarrow{\sim} \pi_n(X[n,\infty]) \xrightarrow{\sim} H_n(X[n,\infty];\mathbb{Z}),$$

so we can calculate $\pi_n(X)$ as the lowest non-trivial homology group of the space $X[n,\infty]$. The problem is that $X[n,\infty]$ is a space that we have shown exists by highly inexplicit means, and we have no idea how to express its homology in terms of e.g. the homology of X.

However, supposing that we know the homology of X we could *try* to inductively calculate the homology of the spaces $X[n,\infty]$ (and hence by the above the homotopy groups of X) as follows. As the homotopy fibre of $X[n,\infty] \to X[n-1,\infty]$ is a $K(\pi_{n-1}(X), n-2)$, and we may suppose that we know $\pi_{n-1}(X)$ by induction, the strategy would have some hope if

(i) given a fibration $p : E \to B$ with fibre F, we had a mechanism to calculate the homology of E given the homology of B and of F, and

(ii) given an abelian group G we knew how to calculate the homology of $K(G,n)$.

Developing tools to do this will be the rest of the course: the first is the **Serre spectral sequence**, and the second is the subject of **cohomology operations**. These tools will have broad applications, well beyond calculating homotopy groups.

Chapter 2

The Serre spectral sequence and applications

Lecture 11

2.1 Spectral sequences

Definition 2.1.1. A **bigraded abelian group** $A_{\bullet,\bullet}$ is an abelian group A with a decomposition $A = \bigoplus_{p,q\in\mathbb{Z}} A_{p,q}$. A **degree** (a,b) **map** $f : A_{\bullet,\bullet} \to B_{\bullet,\bullet}$ of bigraded abelian groups is a homomorphism $f : A \to B$ such that $f(A_{p,q}) \subset B_{p+a,q+b}$.

Definition 2.1.2. A (homological) **spectral sequence** is a sequence $E^1_{\bullet,\bullet}, E^2_{\bullet,\bullet}, E^3_{\bullet,\bullet}, \ldots$ of bigraded abelian groups, called **pages**, equipped with maps

$$d^r : E^r_{\bullet,\bullet} \longrightarrow E^r_{\bullet,\bullet} \text{ of degree } (-r, r-1)$$

such that $d^r \circ d^r = 0$, so that d^r is a differential, and $E^{r+1}_{\bullet,\bullet} = H(E^r_{\bullet,\bullet}, d^r)$. That is,

$$E^{r+1}_{p,q} = \frac{\ker(d^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{im}(d^r : E^r_{p+r,q-r+1} \to E^r_{p,q})}.$$

Remark 2.1.3. All the gradings I have given are optional and variable.

Definition 2.1.4. An **exact couple of type** r consists of bigraded abelian groups $E_{\bullet,\bullet}$ and $A_{\bullet,\bullet}$ and maps

$$\begin{aligned} i : A_{\bullet,\bullet} &\longrightarrow A_{\bullet,\bullet} \text{ of degree } (1,-1) \\ j : A_{\bullet,\bullet} &\longrightarrow E_{\bullet,\bullet} \text{ of degree } (-r,r) \\ k : E_{\bullet,\bullet} &\longrightarrow A_{\bullet,\bullet} \text{ of degree } (-1,0) \end{aligned}$$

so that the triangle

$$\begin{tikzcd} A_{\bullet,\bullet} \arrow[rr, "i"] & & A_{\bullet,\bullet} \arrow[dl, "j"] \\ & E_{\bullet,\bullet} \arrow[ul, "k"] & \end{tikzcd}$$

is exact at each vertex (i.e. $\mathrm{im}(i) = \ker(j)$, and so on).

In this case $d := j \circ k$, which has degree $(-r-1, r)$, is a differential on $E_{\bullet,\bullet}$, as $d \circ d = j \circ (k \circ j) \circ k$ and $k \circ j = 0$.

The **derived couple** of such an exact couple is given by

$$A'_{\bullet,\bullet} := \mathrm{im}(i) \qquad E'_{\bullet,\bullet} := \frac{\ker(d)}{\mathrm{im}(d)}$$

$$i' := i|_{A'_{\bullet,\bullet}} \qquad j'(i(a)) := [j(a)] \qquad k'([e]) := k(e).$$

Theorem 2.1.5. *The derived couple of an exact couple of type r is well-defined and is an exact couple of type $(r+1)$.*

Proof. First show that $j' : A'_{\bullet,\bullet} \to E'_{\bullet,\bullet}$ is well-defined. If $i(a) = i(b)$, then $i(a-b) = 0$ so $a - b = k(c)$. Then

$$j(a) = j(b) + j(k(c)) = j(b) + d(c)$$

so $[j(a)] = [j(b)] \in E'_{\bullet,\bullet} = \frac{\ker(d)}{\mathrm{im}(d)}$.

Now show that $k' : E'_{\bullet,\bullet} \to A'_{\bullet,\bullet}$ is well-defined. If $[e] = [f] \in E'_{\bullet,\bullet}$, then $d(e) = d(f) = 0$, and $e = f + d(g) = f + j(k(g))$. Then

$$k(e) = k(f) + k(j(k(g))) = k(f)$$

as $k \circ j = 0$.

That the resulting triangle is exact at each vertex I leave as an exercise.

Finally, if $i(a) \in A_{p,q}$ then $a \in A_{p-1,q+1}$, so $j(a) \in E_{p-1-r,q+1+r}$, so j' has degree $(-r-1, r+1)$. The map i' has degree $(1,-1)$, and k' has degree $(-1,0)$. Thus the derived couple is indeed an exact couple of type $(r+1)$. $\square$

In particular, if $(A_{\bullet,\bullet}, E_{\bullet,\bullet}, i, j, k)$ is an exact couple of type 0, then letting $E^r_{\bullet,\bullet}$ be the $(r-1)$st derived $E_{\bullet,\bullet}$, and d^r be the $(r-1)$st derived d, gives a spectral sequence.

Example 2.1.6 (The spectral sequence of a filtered space)**.** Let

$$\emptyset \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

be a sequence of subspaces. Using the long exact sequences for the pairs (X_n, X_{n-1}) for homology with arbitrary coefficients (which we omit from the notation), let

$$\begin{aligned}
A_{p,q} &:= H_{p+q}(X_p)\\
E_{p,q} &:= H_{p+q}(X_p, X_{p-1})\\
i &: H_{p+q}(X_p) \longrightarrow H_{(p+1)+(q-1)}(X_{p+1}) \text{ of degree } (1,-1)\\
j &: H_{p+q}(X_p) \longrightarrow H_{p+q}(X_p, X_{p-1}) \text{ of degree } (0,0)\\
k &: H_{p+q}(X_p, X_{p-1}) \longrightarrow H_{(p-1)+q}(X_{p-1}) \text{ of degree } (-1,0).
\end{aligned}$$

This is an exact couple of type 0 (it is exact by the exactness of the long exact sequence of a pair), so gives a spectral sequence with

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}).$$

The form of this spectral sequence is shown in Figure 2.1.

The differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is given by the composition

$$H_{p+q}(X_p, X_{p-1}) \xrightarrow{\partial} H_{(p-1)+q}(X_{p-1}) \longrightarrow H_{(p-1)+q}(X_{p-1}, X_{p-2}). \qquad \triangle$$

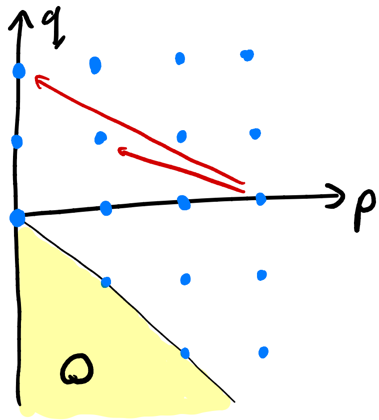


Figure 2.1 The homological spectral sequence for a filtered space.

Example 2.1.7. Let X be a CW complex, $\emptyset \subset X^0 \subset X^1 \subset \cdots$ be its skeletal filtration. Then we have

$$E^1_{p,q} = H_{p+q}(X^p, X^{p-1}) = \begin{cases} C^{cell}_p(X) & \text{if } q = 0 \\ 0 & \text{else,} \end{cases}$$

and

$$C^{cell}_p(X) = H_p(X^p, X^{p-1}) \xrightarrow{\partial} H_{(p-1)}(X^{p-1}) \longrightarrow H_{p-1}(X^{p-1}, X^{p-2}) = C^{cell}_{p-1}(X)$$

is by definition the cellular boundary map. Thus

$$E^2_{p,q} = \begin{cases} H^{cell}_p(X) & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

The differential d^r for $r \geq 2$ changes the q-degree, so must necessarily be zero: thus the above describes $E^r_{p,q}$ for all $r \geq 2$. $\triangle$

Theorem 2.1.8 (Convergence). *Let $\emptyset \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ be a sequence of subspaces such that any simplex in X lies in some X_n, and let $\{(E^r_{\bullet,\bullet}, d^r)\}_{r\geq 1}$ be the associated spectral sequence.*

If $r \geq p+1$ then $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is zero, so $E^{r+1}_{p,q}$ is a quotient of $E^r_{p,q}$. Write

$$E^\infty_{p,q} := \varinjlim(E^{p+1}_{p,q} \twoheadrightarrow E^{p+2}_{p,q} \twoheadrightarrow E^{p+3}_{p,q} \twoheadrightarrow \cdots).$$

Then there is a filtration

$$0 \leq F^0 H_d(X) \leq F^1 H_d(X) \leq F^2\ H_d(X) \leq \cdots$$

of $H_d(X)$ such that

(i) $\bigcup_{n\geq 0} F^n H_d(X) = H_d(X)$*, and*

(ii) $\frac{F^p H_{p+q}(X)}{F^{p-1} H_{p+q}(X)} \cong E^\infty_{p,q}$.

We say the spectral sequence converges to $H_*(X)$, and write "$E^1_{p,q} \Rightarrow H_{p+q}(X)$".

Lecture 12

Proof. Firstly, $E^r_{p-r,q+r-1}$ is a subquotient of $E^1_{p-r,q+r-1} = H_{p+q-1}(X_{p-r}, X_{p-r-1})$, and this vanishes if $p - r < 0$. Thus $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is indeed zero if $r \geq p+1$.

Now, define

$$F^p H_{p+q}(X) := \mathrm{im}(H_{p+q}(X_p) \to H_{p+q}(X)).$$

By our assumption that any simplex in X lies in some X_n, we have $\bigcup_{n\geq 0} F^n H_{p+q}(X) = H_{p+q}(X)$ as required.

To prove property (ii) we first establish some intermediate results.

Claim: If $x \in A_{p,q}$ then $j(x) \in E_{p,q} = E^1_{p,q}$ is a cycle with respect to every d^r.

Proof of claim. We have $d(j(x)) = j(k(j(x))) = 0$ as $k \circ j = 0$. More generally, if $j(x)$ survives until $E^r_{p,q}$ then

$$d^r([j(x)]) = j^r \circ k^r([j(x)]) = j^r([k(j(x))])$$

which also vanishes as $k \circ j = 0$. □

This defines a homomorphism $j : A_{p,q} \to E^r_{p,q}$ for all $r \geq 1$, so a homomorphism

$$j_\infty : A_{p,q} \longrightarrow E^\infty_{p,q}.$$

Claim: The homomorphism j_∞ is onto.

Proof of claim. An element of $E^\infty_{p,q}$ is represented by an $x \in E_{p,q} = E^1_{p,q}$ such that $d^r([x]) = 0$ for all $r \geq 1$. Thus $j^r(k^r([x])) = 0$, so $k^r([x]) = i^{\circ r}(a_r)$ for some $a_r \in A_{\bullet,\bullet}$ (as it is the image under i of an element which lies in the image of $i^{\circ r-1} : A_{\bullet,\bullet} \to A_{\bullet,\bullet}$). But $k^r([x]) = k(x)$, so we find that

$$k(x) \in \mathrm{im}(i^{\circ r} : A_{p-r-1,q+r} \to A_{p-1,q})$$

for all r. But $A_{p-r-1,q+r} = 0$ for $p - r - 1 < 0$, so for all $r \gg 0$, and hence $k(x) = 0$, so $x = j(y)$, so $[x] = j_\infty(y)$. □

Claim: $\ker(j_\infty) = i(A_{p-1,q+1}) + \bigcup_s \ker(i^{\circ s})$.

Proof of claim. If $j_\infty(x) = 0$ then $[j(x)] = 0 \in E^r_{p,q}$ for some r. Thus

$$[j(x)] = j^{r-1} \circ k^{r-1}(y_r) \in E^{r-1}_{p,q} \text{ for some } y_r.$$

Now $k^{r-1}(y_r) \in A^{r-1}_{\bullet,\bullet}$, so $k^{r-1}(y_r) = i^{\circ r-2}(a_{r-1})$ for some $a_{r-1} \in A_{\bullet,\bullet}$, and thus $i^{\circ i-1}(a_{r-1}) = 0$, so $a_{r-1} \in \ker(i^{\circ r-1})$. Now

$$[j(x)] = j^{r-1}(i^{\circ r-2}(a_{r-1})) = [j(a_{r-1})] \in E^{r-1}_{p,q},$$

so

$$[j(x - a_{r-1})] = 0 \in E^{r-1}_{p,q}.$$

Continuing in this way, get

$$j(x - a_{r-1} - a_{r-2} - \cdots - a_1) = 0 \in E^1_{p,q}$$

with $a_s \in \ker(i^{\circ s})$, so

$$x = i(y) + a_{r-1} + a_{r-2} + \cdots + a_1$$

as required. □

We now prove that $\frac{F^p H_{p+q}(X)}{F^{p-1} H_{p+q}(X)} \cong E^\infty_{p,q}$. By the above we have

$$E^\infty_{p,q} \cong \frac{A_{p,q}}{i(A_{p-1,q+1}) + \bigcup_s \ker(i^{\circ s})} \cong \frac{A_{p,q}/\bigcup_s \ker(i^{\circ s})}{i(A_{p-1,q+1}/\bigcup_s \ker(i^{\circ s}))},$$

but

$$\bigcup_s \ker(i^{\circ s}) = \ker(H_{p+q}(X_p) \to H_{p+q}(X)),$$

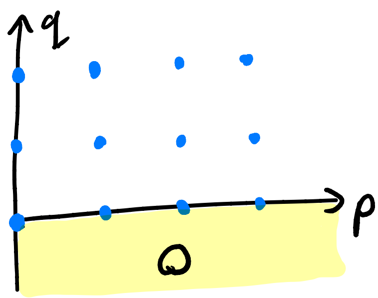
because a homology class on X_p vanishes in X if and only it vanishes in some X_{p+s}, again using the assumption that every simplex in X lies in some X_n. The fact that

$$F^p H_{p+q}(X) = \operatorname{im}(H_{p+q}(X_p) \to H_{p+q}(X)) \cong \frac{H_{p+q}(X_p)}{\ker(H_{p+q}(X_p) \to H_{p+q}(X))}$$

finishes the proof. □

In practice the spectral sequences we will look at have even better vanishing properties: they satisfy

$$E^1_{p,q} = 0 \text{ for } p < 0 \text{ or } q < 0.$$



In this case there are only finitely-many non-zero groups $E^\infty_{p,q}$ along each diagonal $p+q = d$, so the filtration $F^\bullet H_d(X)$ has finite length, and $\{E^\infty_{p,q}\}_{p+q=d}$ gives a composition series for $H_d(X)$.

Example 2.1.9. Returning to Example 2.1.7, on the skeletal filtration of a CW complex X, we have

$$E^\infty_{p,q} \cong \begin{cases} H^{cell}_p(X) & \text{if } q = 0 \\ 0 & \text{else,} \end{cases}$$

so by the convergence theorem $H_p(X)$ has a filtration with a single nontrivial filtration quotient, $H_p^{cell}(X)$, and hence

$$H_p(X) \cong H_p^{cell}(X).$$

This gives a new proof that cellular homology calculates singular homology. △

2.2 The Serre spectral sequence

Theorem 2.2.1. *Let $p : E \to B$ be a Hurewicz fibration over a CW complex B with a single 0-cell $b_0 \in B$ and fibre $F := p^{-1}(b_0)$, such that $\pi_1(B, b_0)$ acts trivially (for the action of Corollary 1.16.2) on $H_*(F; G)$.*

Then there is a spectral sequence $\{(E^r_{\bullet,\bullet}, d^r)\}$ with

$$E^2_{p,q} \cong H_p(B; H_q(F; G))$$

and a filtration of $H_*(E; G)$ such that $\frac{F^p H_{p+q}(E;G)}{F^{p-1} H_{p+q}(E;G)} \cong E^\infty_{p,q}$.

Corollary 2.2.2. *Let $p : E \to B$ be a Serre fibration over a path-connected space B, such that $\pi_1(B, b_0)$ acts trivially on the homology of $F := p^{-1}(b_0)$ with G-coefficients. Then there is a spectral sequence with precisely the same properties.*

Proof. Let $\alpha : C \to B$ be a CW approximation, where C has a single 0-cell c_0 and $a(c_0) = b_0$. Construct the following commutative diagram

$$\begin{tikzcd}
p^{-1}_{\alpha^* p}(c_0) \arrow[d] & F \arrow[l, "\simeq_{weak}"'] \arrow[r, equal] \arrow[d] & F \arrow[d] \\
E_{\alpha^* p} \arrow[d, "p_{\alpha^* p}"] & \alpha^* E \arrow[l, "\simeq"'] \arrow[r, "\simeq_{weak}"] \arrow[d, "\alpha^* p"] & E \arrow[d, "p"] \\
C \arrow[r, equal] & C \arrow[r, "\alpha"] & B
\end{tikzcd}$$

by first forming the pullback $\alpha^* p : \alpha^* E \to C$, which is again a Serre fibration, and whose fibre over c_0 is again F, and then replacing the map $\alpha^* p$ by a Hurewicz fibration as in Section 1.17. The replacement map $\alpha^* E \to E_{\alpha^* p}$ is a homotopy equivalence, so the induced map $F \to p^{-1}_{\alpha^* p}(c_0)$ on fibres is a weak homotopy equivalence (by the 5-lemma applied to the map of long exact sequences of homotopy groups for the Serre fibrations $\alpha^* p$ and $p_{\alpha^* p}$). Similarly, as α is a weak equivalence the map $\alpha^* E \to E$ is too (by the 5-lemma applied to the map of long exact sequence for the fibrations p and $\alpha^* p$). In particular all the horizontal maps induce isomorphisms on homology, and the left-hand column is a Hurewicz fibre sequence to the theorem above applies to it: this gives the analogous spectral sequence for the right-hand column. □

Example 2.2.3. Consider the Hopf fibration $H : S^3 \to S^2$, whose fibres are all homeomorphic to S^1. This is a fibre bundle, and so a Serre fibration. The group $\pi_1(S^2, *)$ is trivial so necessarily acts trivially on $H_*(S^1;\mathbb{Z})$. We have

$$E^2_{p,q} = H_p(S^2; H_q(S^1;\mathbb{Z})) = \begin{cases} \mathbb{Z} & (p,q) = (0,0),(2,0),(0,1),(2,1) \\ 0 & \text{otherwise} \end{cases}$$

as shown in Figure 2.2.

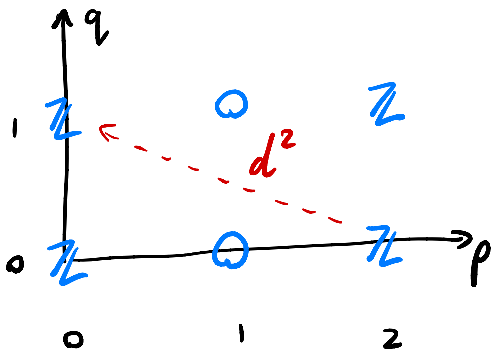


Figure 2.2 The homological Serre spectral sequence for the Hopf fibration.

This spectral sequence converges to

$$H_*(S^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0,3 \\ 0 & \text{otherwise,} \end{cases}$$

so the copies of $\mathbb{Z}$ at $E^2_{2,0}$ and $E^2_{0,1}$ cannot survive to $E^\infty_{\bullet,\bullet}$. Thus the differential

$$d^2 : E^2_{2,0} \cong \mathbb{Z} \longrightarrow E^2_{0,1} \cong \mathbb{Z}$$

must be an isomorphism, as this is the only way these groups can die. △

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Example 2.2.4. Let $n > 1$ and consider the path fibration $\gamma \mapsto \gamma(1) : P_*S^n \to S^n$, with fibre over $* \in S^n$ given by $\Omega_*S^n = \Omega S^n$. As $n > 1$ the space S^n is simply-connected, so its fundamental group acts trivially on $H_*(\Omega_*S^n;\mathbb{Z})$. Thus we have a spectral sequence

$$E^2_{p,q} = H_p(S^n; H_q(\Omega S^n;\mathbb{Z})) \Rightarrow H_{p+q}(P_*S^n;\mathbb{Z}).$$

Now P_*S^n is contractible, so we must have $E^\infty_{p,q} = 0$ for $p+q > 0$ (and $E^\infty_{0,0} \cong \mathbb{Z}$).

We will reverse-engineer this spectral sequence, using as input just $H_*(S^n;\mathbb{Z})$, that the spectral sequence converges to zero in degree $p+q > 0$, and that ΩS^n is path connected, which follows from the long exact sequence on homotopy groups for the path fibration shows that ΩS^n is path-connected. It will be useful to refer to the chart at the end of the example throughout.

As ΩS^n is path-connected we have $H_0(\Omega S^n;\mathbb{Z}) \cong \mathbb{Z}$, and we can determine the bottom row of $E^2_{\bullet,\bullet}$ to be

$$E^2_{p,0} = H_p(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0,n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $H_i(\Omega S^n;\mathbb{Z}) \neq 0$ for $0 < i < n-1$, and let i be minimal with this property. Then $E^2_{0,i} = H_0(S^n; H_i(\Omega S^n;\mathbb{Z})) \neq 0$. The differential entering $E^r_{0,i}$ is $d^r : E^r_{r,i-r+1} \to E^r_{0,i}$, and $E^r_{r,i-r+1}$ is a subquotient of

$$E^2_{r,i-r+1} = H_r(S^n; H_{i-r+1}(\Omega S^n;\mathbb{Z}))$$

but this vanishes as $i - r + 1 < i$ for $r \geq 2$, and we supposed that i was minimal. Thus $E^\infty_{0,i} = E^2_{0,i} \neq 0$, which is impossible as the spectral sequence converges to zero for $p + q > 0$. Thus

$$H_i(\Omega S^n;\mathbb{Z}) = 0 \text{ for } 0 < i < n-1.$$

Now in degree n there is a unique possible non-zero differential leaving position $(n, 0)$, which is

$$d^n : E^n_{n,0} = E^2_{n,0} = \mathbb{Z} \longrightarrow E^n_{0,n-1} = H_{n-1}(\Omega S^n;\mathbb{Z}).$$

This must be injective, but is must also be surjective as no other differential can kill the group at position $(0, n-1)$. Thus

$$H_{n-1}(\Omega S^n;\mathbb{Z}) \cong \mathbb{Z}.$$

But now we can fill in the entire row $q = n-1$, and we find that $E^2_{n,n-1} \cong \mathbb{Z}$ too. The pattern of rows now repeats exactly, giving

$$H_q(\Omega S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & q = 0, n-1, 2(n-1), 3(n-1), \ldots \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence is therefore as shown in Figure 2.3. $\triangle$

Proof of Theorem 2.2.1. Let $\{b_0\} = B^0 \subset B^1 \subset B^2 \subset \cdots$ be the skeleta of B, and filter E by $E_n := p^{-1}(B^n)$. Note that any simplex in E lies in some E_n, as any simplex in B lies in some B^n. Following Example 2.1.6 there is an associated spectral sequence, with

$$E^1_{p,q} = H_{p+q}(E_p, E_{p+1}; G),$$

and by Theorem 2.1.8 this spectral sequence converges to $H_{p+q}(E; G)$. To prove the theorem we must therefore show that

$$E^2_{p,q} \cong H_p(B; H_q(F; G))$$

under the assumption that $\pi_1(B, b_0)$ acts trivially on $H_q(F; G)$. To do that, we will compute the homology of the chain complex $(E^1_{\bullet,\bullet}, d^1)$. Let us omit the coefficients G from the notation: it plays no role.

If $\{i_\alpha : D^p \to B^p\}_{\alpha \in I_p}$ are the characteristic maps of the p-cells of B, then we have Hurewicz fibrations

$$p_\alpha := i_\alpha^* p : E_\alpha := i_\alpha^* E \longrightarrow D^p.$$

Let $\partial E_\alpha = p_\alpha^{-1}(S^{p-1})$, so that $\partial E_\alpha \to S^{p-1}$ is also a fibration.

Claim: The natural map

$$\bigoplus_{\alpha \in I_p} H_*(E_\alpha, \partial E_\alpha) \longrightarrow H_*(E_p, E_{p-1})$$

is an isomorphism.

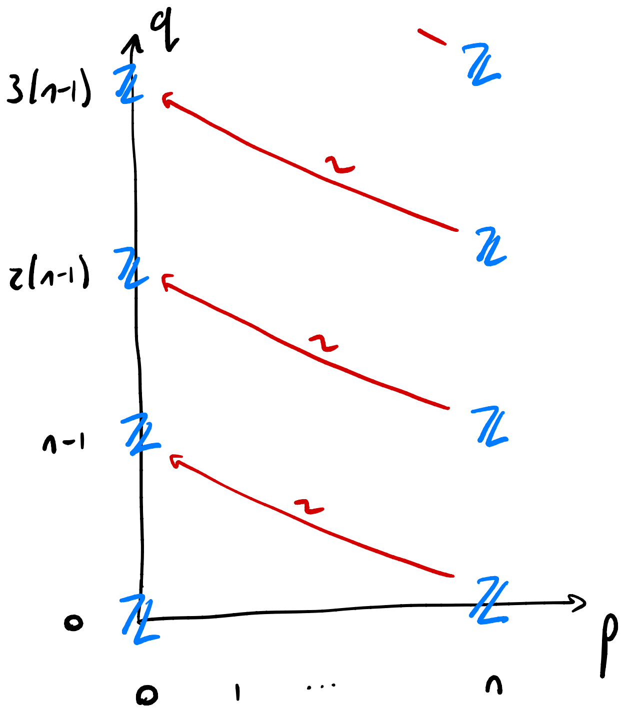


Figure 2.3 The homological Serre spectral sequence for the path fibration of S^n.

Proof of claim. There is an open neighbourhood of E^{p-1} in E^p which weakly[1] deformation retracts to E^{p-1} (this is true for $B^{p-1} \subset B^p$, by taking the complements of the centres of the discs: then use the homotopy lifting property). Thus by excision

$$H_*(E_p, E_{p-1}) \xrightarrow{\sim} H_*(E_p/E_{p-1}, *),$$

but we also have

$$E_p/E_{p-1} \cong \bigvee_{\alpha \in I_p} E_\alpha/\partial E_\alpha. \qquad \square$$

Now consider a single fibration $p_\alpha : E_\alpha \to D^p$, let $0 \in D^p$ be a basepoint, and set $F_\alpha := p_\alpha^{-1}(0)$. Consider the homotopy lifting problem

$$\begin{tikzcd}
S^{p-1} \times F_\alpha \times \{0\} \arrow[r, "\mathrm{proj}"] \arrow[d] & F_\alpha \arrow[r, "inc"] & E_\alpha \arrow[d] \\
S^{p-1} \times F_\alpha \times [0,1] \arrow[r, "\mathrm{proj}"] & S^{p-1} \times [0,1] \arrow[r, "(v,t)\mapsto t\cdot v"] & D^p.
\end{tikzcd}$$

[1] A weak deformation retraction of a space X to a subspace A is a homotopy from the identity map of X to a map into A, which restricts to a homotopy of maps from A into A. It is “weak" in that it need not fix A pointwise.

The lifted homotopy $\tilde{H} : S^{p-1} \times F_\alpha \times [0,1] \to D^p$ satisfies $\tilde{H}(v, x, 0) = x$ for all $v \in S^{p-1}$, so it descends to a map ϕ of fibrations

$$\begin{array}{ccc} D^p \times F_\alpha & \xrightarrow{\phi} & E_\alpha \\ \big\downarrow & & \big\downarrow \\ D^p & = & D^p. \end{array}$$

As the fibres over $0 \in D^p$ of both fibrations are F_α, it follows from the 5-lemma that ϕ is a weak homotopy equivalence. Restricting to $S^{p-1} \subset D^p$ gives a map $\partial\phi : S^{p-1} \times F_\alpha \to \partial E_\alpha$ which is a weak homotopy equivalence for the same reason. Thus we have

$$\begin{aligned} H_*(E_\alpha, \partial E_\alpha) &\cong H_*(D^p \times F_\alpha, S^{p-1} \times F_\alpha) \\ &\cong H_{*-p}(F_\alpha) \end{aligned}$$

by the Künneth theorem for pairs. We therefore have

$$E^1_{p,q} \cong \bigoplus_{\alpha \in I_p} H_q(F_\alpha).$$

Choosing a path from $0 \in \mathrm{int}(D^p) \subset B$ to b_0, Corollary 1.16.2 gives an isomorphism $H_*(F_\alpha) \xrightarrow{\sim} H_*(F)$. Different choices of paths in principle give different isomorphisms but by our assumption that $\pi_1(B, b_0)$ acts trivially on the homology of F in the current situation they do not: thus we may canonically identify $H_*(F_\alpha)$ with $H_*(F)$.

Thus we may write

$$E^1_{p,q} \cong \bigoplus_{\alpha \in I_p} H_q(F) = C^{cell}_p(B) \otimes H_q(F). \tag{2.2.1}$$

Claim: Under this isomorphism $d^1 = d^{cell} \otimes 1$.

Proof of claim. Before starting the proof proper, we make two observations about naturality:

(i) If $f : C \to B$ is a cellular map, C also having a single 0-cell c_0, then the filtration of f^*E given by $(f^*E)_p = (f^*p)^{-1}(C^p)$ is compatible with the filtration of E by E_p, giving a map of exact couples and hence a map of spectral sequences. Under the isomorphism (2.2.1) the induced map on $E^1_{\bullet,\bullet}$ is $f_\# \otimes 1$.

(ii) If $g : E' \to E$ is a map of fibrations over B, with $h : F' \to F$ the induced map on fibres over b_0, then $g(E'_p) \subset E_p$, so again get a map of exact couples and hence of spectral sequences. Under the isomorphism (2.2.1) the induced map on $E^1_{\bullet,\bullet}$ is $1 \otimes h_*$.

Now, as it is enough to check the claim on basis elements of $C^{cell}_p(B)$, by (i) we may suppose that $B = D^p$, having a single 0-cell, a $(p-1)$-cell, and a p-cell. Then, using the equivalence $\phi : D^p \times F_\alpha \to E_\alpha$ over D^p, by (ii) it is enough to check the claim for the trivial fibration $D^p \times F_\alpha \to D^p$, where we can write $F = F_\alpha$. In this case we have

$$\begin{tikzcd}
E^1_{p,q} \arrow[d, equal] & & E^1_{p-1,q} \arrow[d, equal] \\
H_{p+q}(D^p \times F, S^{p-1} \times F) \arrow[rr, "d^r"] \arrow[dr, "\partial"'] & & H_{p+q-1}(S^{p-1} \times F, \{b_0\} \times F) \\
& H_{p+q-1}(S^{p-1} \times F) \arrow[ur] &
\end{tikzcd}$$

which by naturality of the Künneth theorem for pairs is

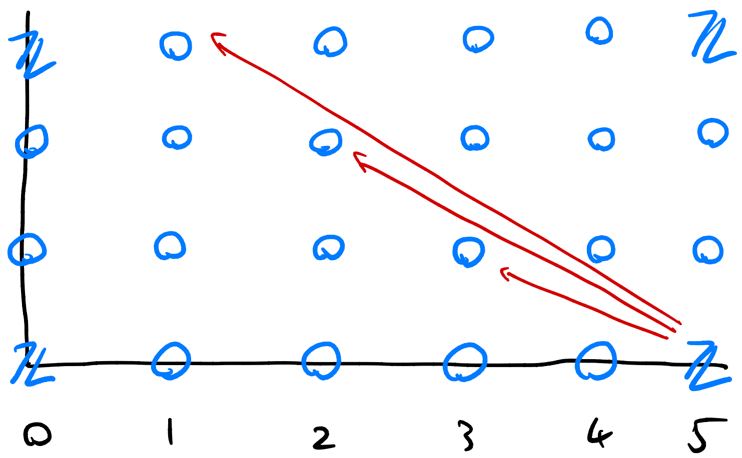
$$H_p(D^p, S^{p-1}) \otimes H_q(F) \xrightarrow{\partial \otimes 1} H_{p-1}(S^{p-1}) \otimes H_q(F) \xrightarrow{\sim} H_{p-1}(S^{p-1}, *) \otimes H_q(F)$$

as required. □

Having identified $(E^1_{\bullet,q}, d^1)$ with the cellular chain complex $C^{cell}_\bullet(B; H_q(F;G))$, we find that $E^2_{p,q}$ is isomorphic to $H_p(B; H_q(F;G))$ as required. □

Example 2.2.5. Acting on the unit vector $(1,0,0) \in \mathbb{C}^3$ gives a map $q : SU(3) \to S^5$, which is easily checked to be a fibre bundle and hence a Serre fibration. The fibre over $(1,0,0) \in S^5$ is the stabiliser of this vector, so $SU(2) \cong S^3$. Thus we have a spectral sequence

$$E^2_{p,q} = H_p(S^5; H_q(S^3)) \Rightarrow H_{p+q}(SU(3)).$$



By considering the chart, there is no space for non-trivial differentials, and in each total degree $p + q = d$ there is at most one non-trivial group. Thus

$$H_n(SU(3); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, 3, 5, 8 \\ 0 & \text{otherwise.} \end{cases}$$ △

Lecture 14

2.3 The Serre spectral sequence in cohomology

There is similarly a spectral sequence for the *cohomology* of a filtered space, with two changes: the indexing is slightly different, and the matter of convergence is, in general, more complicated.

If $\emptyset \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ is a sequence of subspaces, let

$$A^{p,q} = H^{p+q}(X, X_p; G) \qquad E^{p,q} = H^{p+q}(X_p, X_{p-1}; G)$$

so the long exact sequence on cohomology for the triple[2] (X, X_p, X_{p-1}) gives maps

$$\begin{aligned} i : A^{p,q} &\longrightarrow A^{p-1,q+1} \text{ of degree } (-1,1) \\ j : A^{p,q} &\longrightarrow E^{p+1,q-1} \text{ of degree } (1,-1) \\ k : E^{p,q} &\longrightarrow A^{p,q+1} \text{ of degree } (0,1) \end{aligned}$$

Based on this one can invent the notion of a **cohomological exact couple**, which is precisely the same as an exact couple but with a different convention on degrees: the above gives a cohomological exact couple of type 0 (by definition), and its rth derived couple will have type r. The corresponding spectral sequence has differentials

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-(r-1)},$$

so have *minus* the degree of the homological differentials. Its form is shown in Figure 2.4.

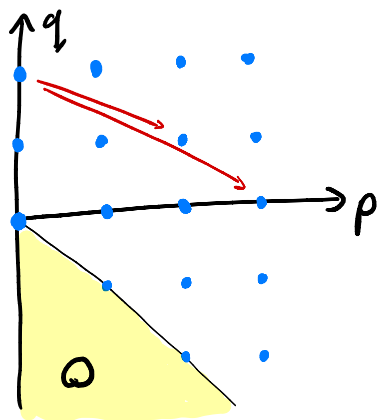


Figure 2.4 The cohomological spectral sequence for a filtered space.

In this situation each position (p, q) can only be the target of finitely-many differentials, but can be the source of infinitely-many. Thus for $r \gg 0$ we have inclusions $E_r^{p,q} \supset E_{r+1}^{p,q} \supset E_{r+2}^{p,q} \supset \cdots$, and we define

$$E_\infty^{p,q} := \bigcap_{r \gg 0} E_r^{p,q} = \varprojlim(\cdots \hookrightarrow E_{r+2}^{p,q} \hookrightarrow E_{r+1}^{p,q} \hookrightarrow E_r^{p,q}).$$

[2]This is the long exact sequence on cohomology associated to the short exact sequence of cochain complexes $0 \to C^*(X, X_p) \to C^*(X, X_{p-1}) \to C^*(X_p, X_{p-1}) \to 0$ given by the evident maps.

Inverse limits are more subtle than direct limits, and in general one has to analyse this quite carefully to understand what it means to converge to $H^*(X)$. In the case of the cohomological Serre spectral sequence the following will suffice.

Theorem 2.3.1. *If $\emptyset \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ is a sequence of subspaces such that any simplex in X lies in some X_n and*

$$E_1^{p,q} = H^{p+q}(X_p, X_{p-1}; G) = 0 \textit{ unless } p \geq 0 \textit{ and } q \geq 0,$$

then defining a filtration by

$$F^p H^n(X;G) := \mathrm{im}(H^n(X, X_{p-1}; G) \to H^n(X;G))$$

we have

(i) $\bigcup_{p\geq 0} F^p H^n(X;G) = H^n(X;G)$, $\bigcap_{p\geq 0} F^p H^n(X;G) = 0$, *and*

(ii) $E_\infty^{p,q} \cong \frac{F^p H^{p+q}(X;G)}{F^{p+1}H^{p+q}(X;G)}$.

Proof. Let us omit the coefficients G from the notation. We have $F^0H^n(X) = H^n(X)$, which gives the first part of (i). For the second part consider

$$\begin{array}{ccccccc}
\cdots \xrightarrow{3} H^n(X, X_{p+1}) & \xrightarrow{2} & H^n(X, X_p) & \xrightarrow{1} & H^n(X, X_{p-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & H^n(X_{p+1}, X_p) & & H^n(X_p, X_{p-1}). & &
\end{array}$$

From the long exact sequence of the triple (X, X_p, X_{p-1}) and our assumption that $H^n(X_p, X_{p-1}) = 0$ if $n-p<0$, we see that the map "1" is an epimorphism for $p \geq n+1$, and by the analogous argument the maps "2" and "3" and so on are then all epimorphisms too.

Suppose then that $p \geq n+1$, let $\varphi_{p-1} \in C^n(X, X_{p-1})$ be a relative cocycle, which we consider as a homomorphism $\varphi_{p-1} : C_n(X;\mathbb{Z}) \to G$ which vanishes on $C_n(X_{p-1};\mathbb{Z})$. Choose a sequence

$$\varphi_p \in C^n(X, X_p), \quad \varphi_{p+1} \in C^n(X, X_{p+1}), \quad \ldots$$

of relative cocycles whose cohomology classes correspond under the maps "1", "2", "3", and so on. Now we must have

$$\varphi_{p-1} = \varphi_p + \rho_{p-1} \circ d \text{ for some } \rho_{p-1} \in C^{n-1}(X, X_{p-1}),$$

as $[\varphi_{p-1}] = [\varphi_p] \in H^n(X, X_{p-1})$, and similarly there must be cochains

$$\rho_p \in C^{n-1}(X, X_p), \quad \rho_{p+1} \in C^{n-1}(X, X_{p+1}), \quad \rho_{p+2} \in C^{n-1}(X, X_{p+2}), \quad \ldots$$

such that $\varphi_i = \varphi_{i+1} + \rho_i \circ d$. Consider

$$\phi : \rho_{p-1} + \rho_p + \rho_{p+1} + \cdots : C_{n-1}(X;\mathbb{Z}) \longrightarrow G,$$

which is well-defined as any $x \in C_{n-1}(X;\mathbb{Z})$ lies in $C_{n-1}(X_{p-1+r};\mathbb{Z})$ for some $r \gg 0$, so all but finitely-many ρ_i's vanish on it.

Now we observe that if $x \in C_n(X_q;\mathbb{Z})$ then

$$\begin{aligned}\varphi_{p-1}(x) &= \rho_{p-1}(dx) + \rho_p(dx) + \cdots + \cancel{\rho_q(dx)} + \cancel{\rho_{q+1}(dx)} + \cdots \\ &= \phi(dx),\end{aligned}$$

and every element of $C_n(X;\mathbb{Z})$ lies in some $C_n(X_q;\mathbb{Z})$ so $\varphi_{p-1} = \phi \circ d$, so $[\varphi_{p-1}] = 0 \in H^n(X, X_{p-1})$. As φ_{p-1} was arbitrary, $H^n(X, X_{p-1}) = 0$, and hence $F^pH^n(X;G) = 0$, which finishes the proof of (i).

Part (ii) is just as in Theorem 2.1.8. □

Theorem 2.3.2. *Let $p : E \to B$ be a Serre fibration over a path-connected space with fibre $F := p^{-1}(b_0)$ so that $\pi_1(B, b_0)$ acts trivially on $H^*(F;G)$. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p(B; H^q(F;G))$$

and with $E_\infty^{p,q} \cong \frac{F^pH^{p+q}(E;G)}{F^{p+1}H^{p+q}(E;G)}$, for a certain descending filtration $F^\bullet H^n(E;G)$ having $F^0H^n(E;G) = H^n(E;G)$ and $F^mH^n(E;G) = 0$ for $m > n$.

Proof. Using the above convergence result this is just as for the homology Serre spectral sequence, with a little care regarding direct sums vs. direct products. □

2.4 Multiplicative structure

The great advantage of the cohomological Serre spectral sequence of a fibration sequence $F \to E \to B$ is that it relates the cup-product structure on $H^*(E)$ to that on $H^*(B)$ and $H^*(F)$. The following encapsulates all the necessary properties. We do not include the proof: it is no more difficult than what we have do so far, but is not very enlightening.

Theorem 2.4.1. *Let R be a commutative ring, $p : E \to B$ be a Serre fibration with path-connected base, and fibre $F = p^{-1}(b_0)$ such that $\pi_1(B, b_0)$ acts trivially on $H^*(F;R)$. Then the Serre spectral sequence $\{(E_r^{\bullet,\bullet}, d_r)\}_{r\geq 2}$ admits product maps*

$$- \cdot - : E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'} \text{ for } r \geq 2$$

such that

(i) $E_r^{\bullet,\bullet}$ *is a bigraded ring for each $r \geq 2$,*

(ii) $d_r : E_r^{\bullet,\bullet} \to E_r^{\bullet,\bullet}$ *is a* ***derivation****, i.e.*

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{\deg(x)} x \cdot d_r(y),$$

where $\deg(x) = p + q$ *when* $x \in E_r^{p,q}$,

(iii) $E_{r+1}^{\bullet,\bullet} = H(E_r^{\bullet,\bullet}, d_r)$ *as bigraded rings,*

(iv) *the isomorphisms* $E_2^{p,q} \cong H^p(B; H^q(F;R))$ *assemble to an isomorphism of bigraded rings, where the latter is given the product*

$$H^p(B; H^q(F;R)) \otimes H^{p'}(B; H^{q'}(F;R)) \overset{\smile}{\longrightarrow} H^{p+p'}(B; H^q(F;R) \otimes H^{q'}(F;R))$$
$$\overset{\smile}{\longrightarrow} H^{p+p'}(B; H^{q+q'}(F;R)),$$

(v) *the filtration* $F^\bullet H^*(E;R)$ *satisfies*

$$F^m H^n(E;R) \smile F^{m'} H^{n'}(E;R) \subset F^{m+m'} H^{n+n'}(E;R),$$

so the cup product induces well-defined maps

$$\frac{F^m H^n(E;R)}{F^{m+1}H^n(E;R)} \otimes \frac{F^{m'} H^{n'}(E;R)}{F^{m'+1}H^{n'}(E;R)} \longrightarrow \frac{F^{m+m'} H^{n+n'}(E;R)}{F^{m+m'+1}H^{n+n'}(E;R)},$$

(vi) *the isomorphisms* $E_\infty^{p,q} \cong \frac{F^p H^{p+q}(E;R)}{F^{p+1}H^{p+q}(E;R)}$ *assemble to an isomorphism of bigraded rings.*

Remark 2.4.2. When the Universal Coefficient Theorem applies to let us write

$$H^p(B; H^q(F;R)) \cong H^p(B;R) \otimes_R H^q(F;R)$$

for all p and q, the multiplication in (iv) is given by

$$(x \otimes y) \cdot (x' \otimes y') = (-1)^{\deg(y)\deg(x')}(x \smile x') \otimes (y \smile y').$$

This sign may seem strange at first, but it is necessary in order to make the Künneth isomorphism $H^*(B;R) \otimes_R H^*(F;R) \xrightarrow{\sim} H^*(B \times F;R)$ into a ring isomorphism.

Example 2.4.3. Let us revisit the fibration sequence

$$\Omega S^n \longrightarrow P_* S^n \longrightarrow S^n$$

from Example 2.2.4, with $n > 1$. We showed there that $H_*(\Omega S^n;\mathbb{Z})$ is $\mathbb{Z}$ in degrees divisible by $(n-1)$ and 0 otherwise; by the Universal Coefficient Theorem $H^*(\Omega S^n;\mathbb{Z})$ has the same description. The cohomological Serre spectral sequence must therefore be as shown below.

Let us be concrete about generators. Use the Universal Coefficient Theorem to write

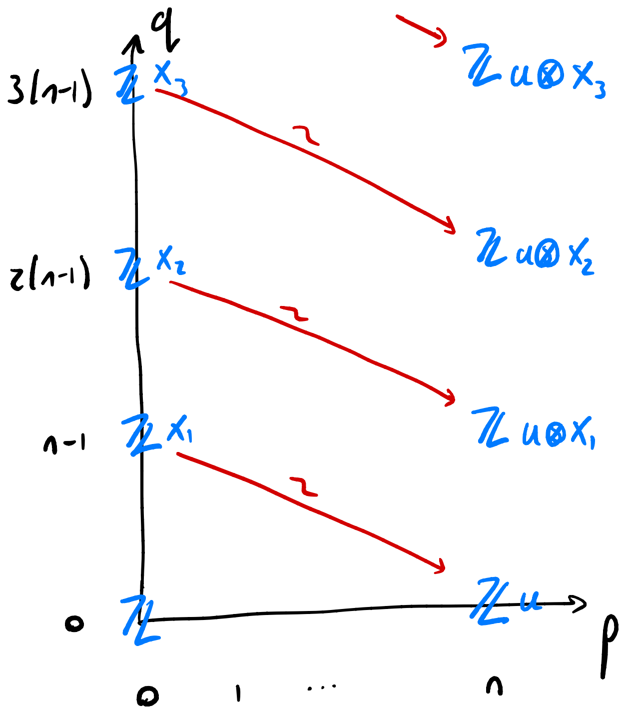
$$E_2^{p,q} = H^p(S^n;\mathbb{Z}) \otimes H^q(\Omega S^n;\mathbb{Z}).$$

Let $u \in H^n(S^n;\mathbb{Z})$ be the standard generator, giving an element $u \otimes 1 \in E_2^{n,0}$ and let $x_1 \in H^{n-1}(\Omega S^n;\mathbb{Z})$ be such that

$$d_n(1 \otimes x_1) = u \otimes 1.$$

More generally, assuming that $x_{i-1} \in H^{(i-1)(n-1)}(\Omega S^n;\mathbb{Z})$ has been chosen, define $x_i \in H^{i(n-1)}(\Omega S^n;\mathbb{Z})$ to be such that $d_n(1 \otimes x_i) = u \otimes x_{i-1}$. We have therefore chosen preferred generators for all the cohomology groups of ΩS^n, and so have

$$x_i \smile x_j = A(i,j) x_{i+j} \tag{2.4.1}$$



for some integers $A(i,j)$. What are they?

Using the multiplicative properties of the Serre spectral sequence we calculate

$$\begin{aligned} d_n(1 \otimes x_i \smile x_j) &= d_n((1 \otimes x_i) \cdot (1 \otimes x_j)) \\ &= (u \otimes x_{i-1}) \cdot (1 \otimes x_j) + (-1)^{i(n-1)}(1 \otimes x_i) \cdot (u \otimes x_{j-1}) \\ &= u \otimes (x_{i-1} \smile x_j) + (-1)^{i(n-1)}(-1)^{i(n-1)(n)} u \otimes (x_i \smile x_{j-1}) \\ &= u \otimes (x_{i-1} \smile x_j) + (-1)^{i(n-1)} u \otimes (x_i \smile x_{j-1}), \end{aligned}$$

using that $(n-1)(n)$ is even. By the ansatz (2.4.1) this gives the recurrence relation

$$A(i,j) = A(i-1,j) + (-1)^{i(n-1)} A(i,j-1).$$

Case 1: $(n-1)$ **is even.** Then we have $A(i,j) = A(i-1,j) + A(i,j-1)$ which is solved by $A(i,j) = \binom{i+j}{i}$. Thus the cup-product structure on $H^*(\Omega S^n; \mathbb{Z})$ is given by

$$x_i \smile x_j = \binom{i+j}{i} x_{i+j}.$$

This is known as a free **divided power algebra** $\Gamma_{\mathbb{Z}}[x_1]$ on the class x_1.

Case 2: $(n-1)$ **is odd.** Then we have $A(i,j) = A(i-1,j) + (-1)^i A(i,j-1)$. One may verify that

$$A(2i+1, 2j+1) = 0, \quad A(2i, 2j) = \binom{i+j}{i}$$

$$A(2i, 2j+1) = \binom{i+j}{i} \quad A(2i+1, 2j) = \binom{i+j}{i}$$

satisfies this recurrence.

This may be understood a little as follows. As $x_1 \in H^{n-1}(\Omega S^n; \mathbb{Z})$ has odd degree, by the graded-commutativity of the cup product we have $x_1^2 = -x_1^2$ and so $2x_1^2 = 0$, but $H^{2(n-1)}(\Omega S^n; \mathbb{Z}) = \mathbb{Z}\{x_2\}$ is torsion-free, so $x_1^2 = 0$. This fits with $A(1,1) = 0$ above. Now the above says that $x_1 \smile x_{2i} = x_{2i+1}$. The above is then saying that we have a ring isomorphism

$$H^*(\Omega S^n; \mathbb{Z}) = \mathbb{Z}[x_1]/(x_1^2) \otimes \Gamma_{\mathbb{Z}}[x_2]. \qquad \triangle$$

Lecture 15

Example 2.4.4. The long exact sequence on homotopy groups for the fibration sequence

$$\Omega K(\mathbb{Z}, 3) \longrightarrow P_* K(\mathbb{Z}, 3) \longrightarrow K(\mathbb{Z}, 3),$$

and the fact that $P_* K(\mathbb{Z}, 3)$ is contractible, shows that $\Omega K(\mathbb{Z}, 3)$ is a $K(\mathbb{Z}, 2)$.[3] Thus it is weakly equivalent to $\mathbb{CP}^\infty$ and so we know its cohomology ring: we have $H^*(K(\mathbb{Z}, 2); \mathbb{Z}) = \mathbb{Z}[\iota_2]$ for a class ι_2 of degree 2. We can take this to be the class we described in Section 1.14, which also gives a $\iota_3 \in H^3(K(\mathbb{Z}, 3); \mathbb{Z})$. We will explain how to produce the chart shown in Figure 2.5 for the cohomological Serre spectral sequence of this fibration

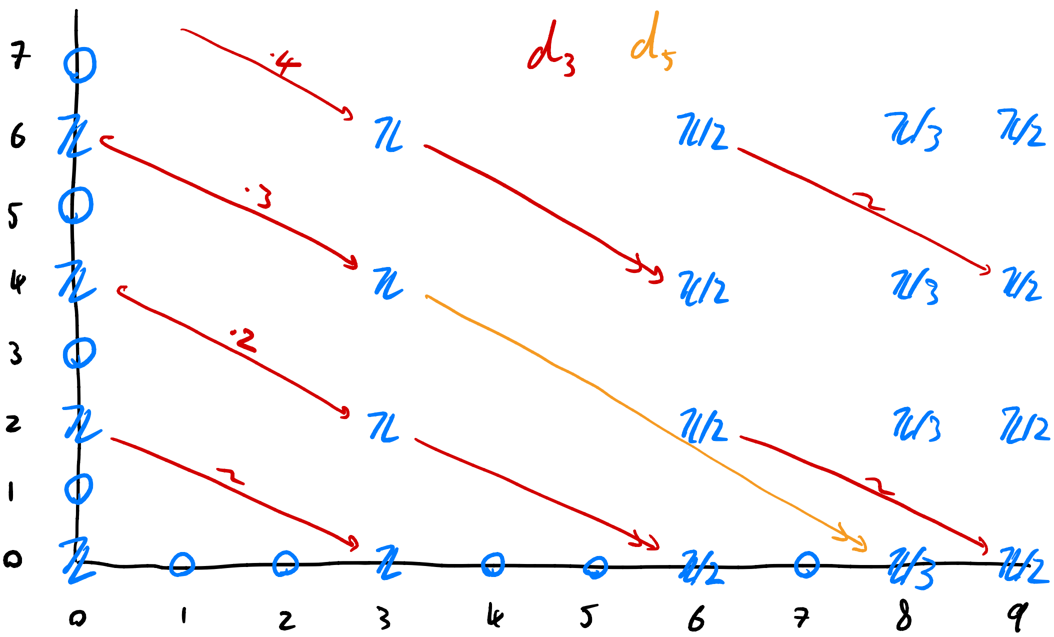


Figure 2.5 The cohomological Serre spectral sequence for the path fibration of $K(\mathbb{Z}, 3)$.

We have $H^i(K(\mathbb{Z}, 3); \mathbb{Z}) = 0$ for $0 < i < 3$ and that $H^3(K(\mathbb{Z}, 3); \mathbb{Z}) = \mathbb{Z}\{\iota_3\}$, by the Universal Coefficient Theorem and the Hurewicz Theorem. This lets us complete the

[3] This is completely general: $\Omega K(G, n) \simeq K(G, n-1)$.

E_2-page for $p \leq 3$. As the cohomological Serre spectral sequence converges to zero in positive degrees, we must have $d_3(1 \otimes \iota_2) = \pm \iota_3 \otimes 1$; by rechoosing generators we can suppose it is $+\iota_3 \otimes 1$. By the derivation property we then have $d_3(1 \otimes \iota_2^2) = 2\iota_3 \otimes \iota_2$, and more generally

$$d_3(1 \otimes \iota_2^n) = n \cdot \iota_3 \otimes \iota_2^{n-1},$$

which completes the first column of differentials. We also deduce that $H^i(K(\mathbb{Z}, 3); \mathbb{Z}) = 0$ for $3 < i < 6$, as there is no way these groups could die in the spectral sequence.

As ι_3 has odd degree, we have $2\iota_3^2 = 0$. We also have

$$d_3(\iota_3 \otimes \iota_2) = \iota_3^2$$

by the derivation property. In order to leave nothing in $E_\infty^{3,2}$ we must therefore have $H^6(K(\mathbb{Z}, 3); \mathbb{Z}) = \mathbb{Z}/2\{\iota_3^2\}$. This lets us complete the E_2-page for $p \leq 6$. By the derivation property again

$$d_3(\iota_3 \otimes \iota_2^n) = n \cdot \iota_3^2 \otimes \iota_2^{n-1} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \iota_3^2 \otimes \iota_2^{n-1} & \text{if } n \text{ is odd,} \end{cases}$$

which completes the second column of differentials. We also see $H^7(K(\mathbb{Z}, 3); \mathbb{Z}) = 0$ as there is no way this group could die in the spectral sequence.

We have found that $E_4^{3,4} = \mathbb{Z}/3\{\iota_3 \otimes \iota_2^2\}$, and the only way this can die is if the differential

$$d_5 : E_5^{3,4} \longrightarrow E_5^{8,0}$$

is injective: similarly, the only way the group at $(8, 0)$ can die is if this group is surjective. Thus we have

$$H^8(K(\mathbb{Z}, 3); \mathbb{Z}) \cong \mathbb{Z}/3.$$

In the same way, $d_3 : E_3^{6,2} = \mathbb{Z}/2\{\iota_3^2 \otimes \iota_2\} \to E_3^{9,0}$ must be an isomorphism, so $H^9(K(\mathbb{Z}, 3); \mathbb{Z}) \cong \mathbb{Z}/2$. In total, we have calculated

n	0	1	2	3	4	5	6	7	8	9	10
$H^n(K(\mathbb{Z}, 3); \mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	$\mathbb{Z}/2$	?

and so, using the Universal Coefficient Theorem backwards,

n	0	1	2	3	4	5	6	7	8	9	10
$H_n(K(\mathbb{Z}, 3); \mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	$\mathbb{Z}/2$	?	?

describes the homology of $K(\mathbb{Z}, 3)$ in degrees ≤ 8. △

Example 2.4.5. Let $f : S^3 \to K(\mathbb{Z}, 3)$ be a map that generates $\pi_3(K(\mathbb{Z}, 3)) \cong \mathbb{Z}$. Then there is a fibration $p : E_f \to K(\mathbb{Z}, 3)$ with fibre F, and a weak equivalence $w : S^3 \xrightarrow{\sim} E_f$ such that $p \circ w = f$.

The long exact sequence on homotopy groups

$$\cdots \longrightarrow \pi_5(K(\mathbb{Z},3)) = 0 \xrightarrow{\partial}$$
$$\pi_4(F) \xrightarrow{\sim} \pi_4(S^3) \longrightarrow \pi_4(K(\mathbb{Z},3)) = 0 \xrightarrow{\partial}$$
$$\pi_3(F) \xrightarrow{0} \pi_3(S^3) = \mathbb{Z} \xrightarrow{\sim} \pi_3(K(\mathbb{Z},3)) = \mathbb{Z} \longrightarrow \cdots$$

shows that F is 3-connected and that $\pi_4(S^3) \cong \pi_4(F)$: thus, by the Hurewicz theorem we have $\pi_4(S^3) \cong \pi_4(F) \cong H_4(F;\mathbb{Z})$.

Considering the homology Serre spectral sequence

$$E^1_{p,q} \cong H_p(K(\mathbb{Z},3); H_q(F;\mathbb{Z})) \Rightarrow H_{p+q}(E_f;\mathbb{Z}) \cong H_{p+q}(S^3;\mathbb{Z})$$

we see that the differential

$$d^5 : E^4_{0,4} \longrightarrow E^4_{5,0} = \mathbb{Z}/2$$

must be an isomorphism, as its kernel or cokernel would contribute to $H_*(S^3;\mathbb{Z})$ in degree $* \in \{4,5\}$, which is impossible. Thus $H_4(F;\mathbb{Z}) \cong \mathbb{Z}/2$, so by the discussion above $\pi_4(S^3) \cong \mathbb{Z}/2$.

Together with Hopf's theorem $\pi_n(S^n) \cong \mathbb{Z}$, and the Hopf fibration sequence $S^1 \to S^3 \to S^2$, we now know the following homotopy groups of spheres

n	1	2	3	4
$\pi_n(S^3)$	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
$\pi_n(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$

△

2.5 mod $\mathcal{C}$ theory

Definition 2.5.1. A **Serre class** $\mathcal{C}$ is a class of abelian groups closed under the operations of taking subgroups, quotients, and forming extensions.

Example 2.5.2.

(i) The class of finitely-generated abelian groups (using that $\mathbb{Z}$ is Noetherian).

(ii) The class of torsion abelian groups where each element is annihilated by products of prime numbers in some set $\mathcal{P}$.

(iii) The class of finite abelian groups in the above class. △

Definition 2.5.3. For a Serre class $\mathcal{C}$, a map $f : A \to B$ of abelian groups is

(i) **injective mod $\mathcal{C}$** if $\operatorname{Ker}(f) \in \mathcal{C}$,

(ii) **surjective mod** $\mathcal{C}$ if $B/\mathrm{Im}(f) \in \mathcal{C}$,

(iii) an **isomorphism mod** $\mathcal{C}$ if both the above hold.

Lemma 2.5.4. *Suppose in addition that $A \otimes B, \mathrm{Tor}(A, B) \in \mathcal{C}$ when $A, B \in \mathcal{C}$. (The three examples above have this property.) Let $p : E \to B$ be a Serre fibration with path-connected base, and fibre $F = p^{-1}(b_0)$ such that $\pi_1(B, b_0)$ acts trivially on $H_*(F; \mathbb{Z})$, then if any two of*

$$\widetilde{H}_*(F; \mathbb{Z}), \qquad \widetilde{H}_*(E; \mathbb{Z}), \qquad \widetilde{H}_*(B; \mathbb{Z})$$

lie in $\mathcal{C}$ then so does the third. Lecture 16

Proof. Suppose first that $\widetilde{H}_*(F; \mathbb{Z}), \widetilde{H}_*(B; \mathbb{Z}) \in \mathcal{C}$, and consider the Serre spectral sequence

$$E^2_{p,q} = H_p(B; H_q(F; \mathbb{Z})) \Rightarrow H_{p+q}(E; \mathbb{Z}).$$

As the class $\mathcal{C}$ is closed under extensions, to see that $H_{p+q}(E; \mathbb{Z}) \in \mathcal{C}$ for $p + q > 0$ it suffices to show that $E^\infty_{p,q} \in \mathcal{C}$ for $p + q > 0$. As the class $\mathcal{C}$ is closed under forming subquotients, it suffices to show that $E^2_{p,q} \in \mathcal{C}$ for $p + q > 0$. The Universal Coefficient Theorem gives

$$0 \longrightarrow H_p(B; \mathbb{Z}) \otimes H_q(F; \mathbb{Z}) \longrightarrow H_p(B; H_q(F; \mathbb{Z})) \longrightarrow \mathrm{Tor}(H_{p-1}(B; \mathbb{Z}), H_q(F; \mathbb{Z})) \longrightarrow 0$$

so as $\mathcal{C}$ is closed under $\otimes$ and Tor (and $\mathbb{Z} \otimes A = A$ and $\mathrm{Tor}(\mathbb{Z}, A) = 0$) it follows that we do indeed have $E^2_{p,q} \in \mathcal{C}$ for $p + q > 0$.

Suppose now that $\widetilde{H}_*(F; \mathbb{Z}), \widetilde{H}_*(E; \mathbb{Z}) \in \mathcal{C}$. The lower corner of the Serre spectral sequences gives an exact sequence

$$H_2(B; \mathbb{Z}) \xrightarrow{d^2} H_1(F; \mathbb{Z}) \longrightarrow H_1(E; \mathbb{Z}) \longrightarrow H_1(B; \mathbb{Z}) \longrightarrow 0,$$

so $H_1(B; \mathbb{Z})$ is a quotient of $H_1(E; \mathbb{Z})$ and so lies in $\mathcal{C}$. To generalise this, suppose then that $H_p(B; \mathbb{Z}) \in \mathcal{C}$ for $0 < p < k$. We have an exact sequence

$$0 \longrightarrow E^{r+1}_{k,0} \longrightarrow E^r_{k,0} \xrightarrow{d^r} E^r_{k-r,r-1}$$

and $E^r_{k-r,r-1}$ is a subquotient of $E^2_{k-r,r-1} = H_{k-r}(B; H_{r-1}(F; \mathbb{Z}))$. As we have supposed that $H_{k-r}(B; \mathbb{Z}), H_{k-r-1}(B; \mathbb{Z}) \in \mathcal{C}$, by an application of the Universal Coefficient Theorem as above we find that $E^2_{k-r,r-1} \in \mathcal{C}$ and hence $E^r_{k-r,r-1} \in \mathcal{C}$. Thus if $E^{r+1}_{k,0} \in \mathcal{C}$ then $E^r_{k,0} \in \mathcal{C}$. As $\widetilde{H}_k(E; \mathbb{Z}) \in \mathcal{C}$ so are its filtration quotients, so $E^{k+1}_{k,0} = E^\infty_{k,0} \in \mathcal{C}$. Thus by downwards induction we find that $E^2_{k,0} = H_k(B; \mathbb{Z}) \in \mathcal{C}$ as required.

Under the assumption that $\widetilde{H}_*(B; \mathbb{Z}), \widetilde{H}_*(E; \mathbb{Z}) \in \mathcal{C}$, similar reasoning shows that $\widetilde{H}_*(F; \mathbb{Z}) \in \mathcal{C}$. □

Lemma 2.5.5. *Let $\mathcal{C}$ be either*

(i) the class of finitely-generated abelian groups, or

(ii) the class of finite $\mathcal{P}$-torsion abelian groups.

If $A \in \mathcal{C}$ then $H_i(K(A,n);\mathbb{Z}) \in \mathcal{C}$ for all $i, n > 0$.

Proof. By considering the fibration sequences

$$K(A,n-1) \simeq \Omega K(A,n) \longrightarrow P_*K(A,n) \longrightarrow K(A,n),$$

with $P_*K(A,n) \simeq *$, the previous lemma reduces the claim to the case $n = 1$.

Finitely-generated: Then $A = \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$ so we can take $K(A,1) = (S^1)^r \times K(\mathbb{Z}/n_1, 1) \times \cdots \times K(\mathbb{Z}/n_k, 1)$, so by the Künneth theorem we are reduced to showing that each $\widetilde{H}_i(K(\mathbb{Z}/n,1);\mathbb{Z})$ is finitely-generated.

The class $n \cdot \iota_2 \in H^2(K(\mathbb{Z},2);\mathbb{Z})$ corresponds to a map $f_n : K(\mathbb{Z},2) \to K(\mathbb{Z},2)$, which on $\pi_2(-)$ induces multiplication by n. It follows from the long exact sequence on homotopy groups that the homotopy fibre of f_n is $K(\mathbb{Z}/n,1)$, so there is a homotopy fibre sequence

$$K(\mathbb{Z}/n,1) \longrightarrow K(\mathbb{Z},2) \xrightarrow{f_n} K(\mathbb{Z},2).$$

As $K(\mathbb{Z},2) \simeq \mathbb{CP}^\infty$ has finitely-generated homology groups, it follows from the previous lemma that $K(\mathbb{Z}/n,1)$ does too.

Finite $\mathcal{P}$-torsion: Then $A = \mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}$ for $p_j \in \mathcal{P}$, so as above it suffices to show that each $\widetilde{H}_i(K(\mathbb{Z}/p^n,1);\mathbb{Z})$ is finite p-torsion. We have the homotopy fibre sequence

$$K(\mathbb{Z}/p^n,1) \xrightarrow{i} K(\mathbb{Z},2) \xrightarrow{f_{p^n}} K(\mathbb{Z},2)$$

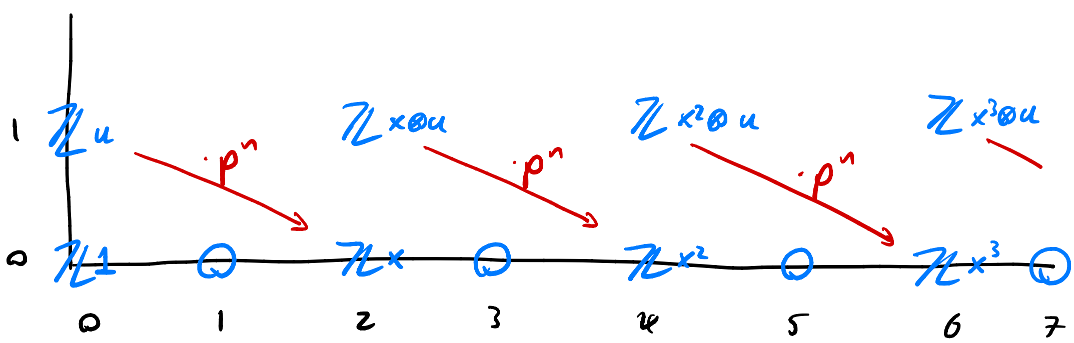
and by the long exact sequence on homotopy groups the homotopy fibre of i is a $K(\mathbb{Z},1) \simeq S^1$, so we have a homotopy fibre sequence

$$S^1 \longrightarrow K(\mathbb{Z}/p^n,1) \xrightarrow{i} K(\mathbb{Z},2). \tag{2.5.1}$$

By the Hurewicz theorem we have $H_1(K(\mathbb{Z}/p^n,1);\mathbb{Z}) \cong \mathbb{Z}/p^n$, and so by the Universal Coefficient Theorem we have

$$H^1(K(\mathbb{Z}/p^n,1);\mathbb{Z}) = 0 \qquad H^2(K(\mathbb{Z}/p^n,1);\mathbb{Z}) \supset \mathbb{Z}/p^n.$$

Thus the cohomology Serre spectral sequence for (2.5.1) must take the following form



so that

$$H^i(K(\mathbb{Z}/p^n,1);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/p^n & i>0 \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

By the Universal Coefficient Theorem (backwards!) each $\widetilde{H}_i(K(\mathbb{Z}/p^n,1);\mathbb{Z})$ is therefore finite p-torsion, as required. □

Theorem 2.5.6 (mod $\mathcal{C}$ Hurewicz)**.** *Let $\mathcal{C}$ be either*

(i) the class of finitely-generated abelian groups, or

(ii) the class of finite $\mathcal{P}$-torsion abelian groups.

If X is 1-connected and $\pi_i(X,x_0) \in \mathcal{C}$ for $0<i<n$, then $H_i(X;\mathbb{Z}) \in \mathcal{C}$ for $0<i<n$ and

$$h : \pi_n(X,x_0) \longrightarrow H_n(X;\mathbb{Z})$$

is an isomorphism mod $\mathcal{C}$.

Proof. Consider the Postnikov tower of X,

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_2 \arrow[d, "p_2"] \\
 & X_1 \arrow[d, "p_1"] \\
X \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uur, "f_2"] & X_0
\end{tikzcd}$$

and recall that the homotopy fibre of $p_k : X_k \to X_{k-1}$ is a $K(\pi_k(X,x_0),k)$.

So if $\pi_i(X,x_0) \in \mathcal{C}$ for $0<i<n$, then by induction using the homotopy fibre sequences

$$K(\pi_i(X,x_0),i) \longrightarrow X_i \xrightarrow{p_i} X_{i-1}$$

we find that $\widetilde{H}_*(X_{n-1};\mathbb{Z}) \in \mathcal{C}$. As $f_{n-1} : X \to X_{n-1}$ is n-connected, we have

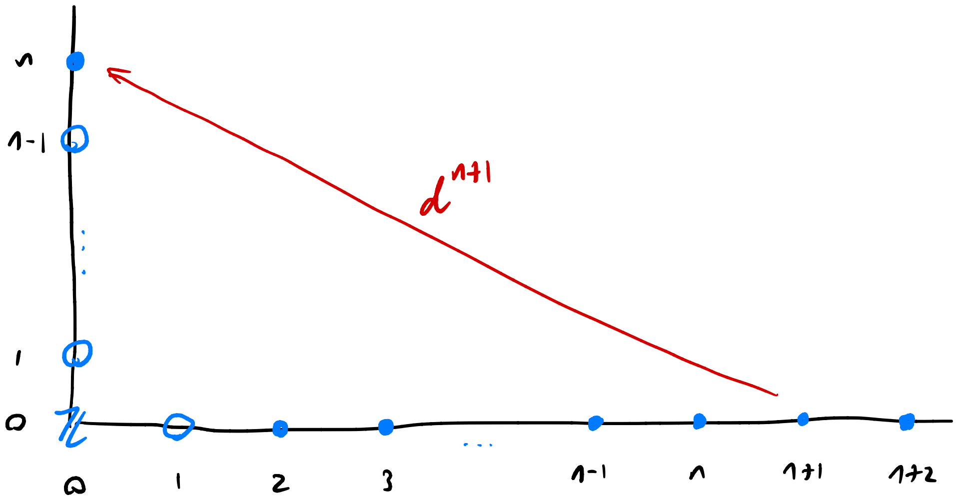
$$H_i(X;\mathbb{Z}) \xrightarrow{\sim} H_i(X_{n-1};\mathbb{Z}) \text{ for } i<n$$

so $H_i(X;\mathbb{Z}) \in \mathcal{C}$ for $0<i<n$. Lecture 17

The map $f_n : X \to X_n$ is $(n+1)$-connected, so an isomorphism on nth homology. The homology Serre spectral sequence for the homotopy fibre sequence

$$K(\pi_n(X,x_0),n) \longrightarrow X_n \xrightarrow{p_n} X_{n-1}$$

has the form



so gives an exact sequence

$$\begin{tikzcd} H_{n+1}(X_{n-1};\mathbb{Z}) \arrow[r, "d^{n+1}"] & H_n(K(\pi_n(X,x_0),n);\mathbb{Z}) \arrow[r] \arrow[d, equal] & H_n(X_n;\mathbb{Z}) \arrow[r] & H_n(X_{n-1};\mathbb{Z}) \\ & \pi_n(X,x_0) \arrow[ur, "h"] & & \end{tikzcd}$$

where the two outer terms lie in $\mathcal{C}$: thus the kernel and cokernel of map h lie in $\mathcal{C}$, so h is an isomorphism mod $\mathcal{C}$. □

Corollary 2.5.7. *$\pi_i(S^n)$ is a finitely-generated abelian group for all i and n.*

Proof. For $n = 1$ we know these homotopy groups, so suppose $n > 1$. Suppose not, and let i be minimal such that $\pi_i(S^n)$ is not finitely-generated. Then $h : \pi_i(S^n) \to H_i(S^n;\mathbb{Z})$ is an isomorphism mod finitely-generated abelian groups, so $H_i(S^n;\mathbb{Z})$ is also not finitely-generated, a contradiction. □

More generally, this argument shows that if X is a 1-connected space with each $H_i(X;\mathbb{Z})$ finitely-generated, then each $\pi_i(X, x_0)$ is also finitely-generated.

Corollary 2.5.8. *The groups $\pi_i(S^3)$ are finite for $i > 3$.*

Proof. Let $f : S^3 \to K(\mathbb{Z}, 3)$ represent a generator of $H^3(S^3;\mathbb{Z})$, and X denote its homotopy fibre. By Lemmas 2.5.4 and 2.5.5 the homology groups of X are finitely-generated.

Let $\mathcal{F}$ denote the Serre class of finite abelian groups. We have

$$\begin{aligned} \pi_i(S^3) \text{ finite for } i > 3 &\iff \pi_i(X) \text{ finite for all } i \text{ (by LES of homotopy groups)} \\ &\iff \widetilde{H}_i(X;\mathbb{Z}) \text{ finite for all } i \text{ (by mod } \mathcal{F} \text{ Hurewicz Theorem)} \\ &\iff \widetilde{H}_i(X;\mathbb{Q}) = \widetilde{H}_i(X;\mathbb{Z}) \otimes \mathbb{Q} = 0 \text{ for all } i \end{aligned}$$

where the last implication is because the homology of X is finitely-generated.

To show that X has trivial $\mathbb{Q}$-homology, we consider the $\mathbb{Q}$-cohomology Serre spectral sequence for homotopy fibre sequence

$$K(\mathbb{Z},2) \simeq \Omega K(\mathbb{Z},3) \longrightarrow P_* K(\mathbb{Z},3) \longrightarrow K(\mathbb{Z},3)$$

which takes the form shown in Figure 2.6, showing that

$$H^i(K(\mathbb{Z},3);\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0,3 \\ 0 & \text{else.} \end{cases}$$

It follows that the map $f : S^3 \longrightarrow K(\mathbb{Z},3)$ is an isomorphism on $H^*(-;\mathbb{Q})$, and therefore from the Serre spectral sequence for the homotopy fibre sequence $X \to S^3 \xrightarrow{f} K(\mathbb{Z},3)$ that $H_*(X;\mathbb{Q}) = \mathbb{Q}$, so $\widetilde{H}_*(X;\mathbb{Q}) = 0$ as required. □

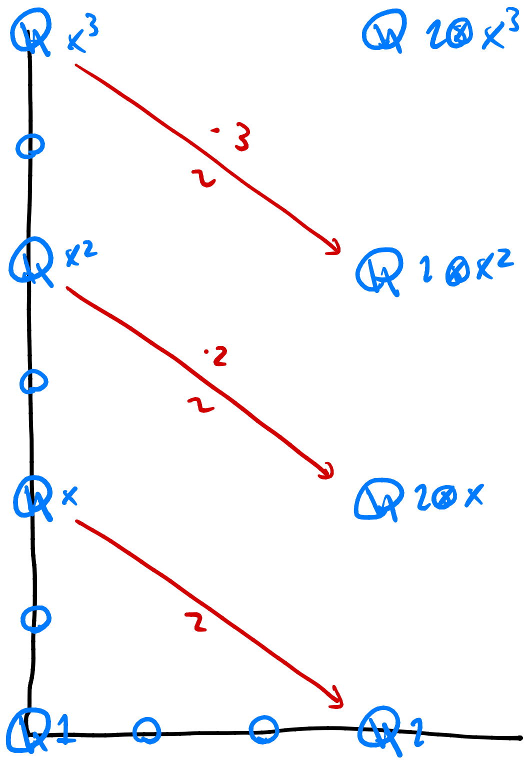


Figure 2.6 The cohomological Serre spectral sequence for the path fibration of $K(\mathbb{Z},3)$ with $\mathbb{Q}$-coefficients.

2.6 The transgression

For $F \to E \xrightarrow{p} B$ a Serre fibration of path-connected spaces, consider

$$\begin{tikzcd} & H_i(E,F) \arrow[r, "\partial"] \arrow[d, "p_*"] & H_{i-1}(F) \\ H_i(B) \arrow[r, "j_*"] & H_i(B,b_0). & \end{tikzcd}$$

Say that $(x,y) \in H_i(B) \times H_{i-1}(F)$ is a **transgressive pair** if there is a $z \in H_i(E,F)$ such that

$$p_*(z) = j_*(x) \qquad \text{and} \qquad \partial(z) = y.$$

This gives a partially-defined and many-valued "function"

$$H_i(B) \dashrightarrow H_{i-1}(F),$$

called the **transgression**. The following lemma relates it to those differentials in the Serre spectral sequence going from the horizontal to the vertical edge.

Lemma 2.6.1. *A pair $(x,y) \in H_i(B) \times H_{i-1}(F)$ is transgressive if and only if in the Serre spectral sequence the class $x \in H_i(B) = H_i(B;H_0(F)) = E^2_{i,0}$ survives until $E^i_{i,0}$ and*

$$d^i(x) = [y] \in E^i_{0,i-1}, \text{ a quotient of } E^2_{0,i-1} = H_0(B;H_{i-1}(F)) = H_{i-1}(F).$$

Proof. Let $z \in H_i(E,F)$ exhibit (x,y) as a transgressive pair, recall that $E_n := p^{-1}(B^n)$, and consider

$$\begin{tikzcd} H_i(E_i,F) \arrow[r, two heads] & H_i(E,F) \arrow[r] & H_i(B,b_0) \\ \bar{z} \arrow[r, mapsto] & z \arrow[r, mapsto] & j_*(x), \end{tikzcd}$$

where the first map is surjective as $H_i(E,E_i) = 0$, and so a $\bar{z}$ can indeed be chosen. The spectral sequence comes from the exact couple

$$\begin{tikzcd} \bigoplus_{p,q} H_{p+q-1}(E_{p-1}) \arrow[rr, "i"] & & \bigoplus_{p,q} H_{p+q}(E_p) \arrow[dl, "j"] \\ & \bigoplus_{p,q} H_{p+q}(E_p,E_{p-1}) \arrow[ul, "k"] & \end{tikzcd}$$

and the class $x \in E^2_{i,0}$ is represented by the image $\bar{\bar{z}}$ of $\bar{z}$ under

$$H_i(E_i,F) = H_i(E_i,E_0) \longrightarrow H_i(E_i,E_{i-1}) = E^1_{i,0}.$$

Thus $d^r(x)$ is given by $j(i^{-1})^r k(\bar{\bar{z}})$. But $k(\bar{\bar{z}}) = \partial(\bar{\bar{z}}) \in H_{i-1}(E_{i-1})$ and we know this lifts to $\partial(\bar{z}) \in H_{i-1}(F) = H_{i-1}(E_0)$. This shows that $d^r(x) = 0$ for $r < i$, and that $d^i(x) = j\partial(\bar{z}) = j\partial(z)$, but this is $j : H_{i-1}(E_0) \xrightarrow{\sim} H_{i-1}(E_0,\emptyset)$, so $d^i(x)$ is represented by $\partial(z) = y \in H_{i-1}(F)$.

The reverse direction is similar. $\square$

There is a similar discussion in cohomology. A pair $(x, y) \in H^{i+1}(B) \times H^i(F)$ is called transgressive if there is a $z \in H^{i+1}(B, b_0)$ such that in the diagram

$$\begin{tikzcd}
 & H^{i+1}(E,F) & H^i(F) \arrow[l, "\delta"'] \\
H^{i+1}(B) & H^{i+1}(B,b_0). \arrow[l, "j^*"'] \arrow[u, "p^*"] &
\end{tikzcd}$$

we have $j^*(z) = x$ and $p^*(z) = \delta(y)$. As in the lemma, this is equivalent to $y \in H^i(F) = E_2^{0,i}$ surviving until $E_{i+1}^{0,i}$ and satisfying $d_{i+1}(y) = [x]$. Lecture 18

2.7 Freudenthal's suspension theorem

Theorem 2.7.1. *Let X be an $(n-1)$-connected based space. Then the suspension map*

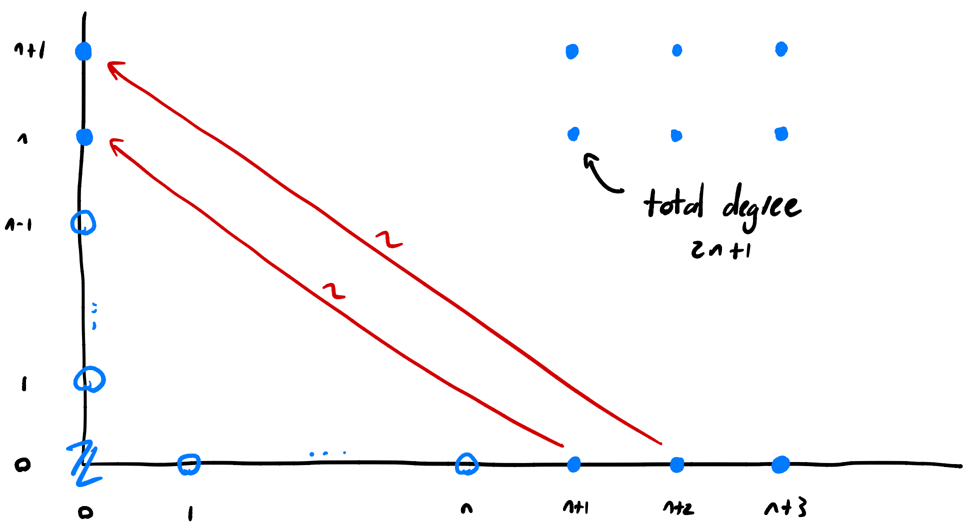
$$\begin{aligned}
\Sigma : \pi_i(X) &\longrightarrow \pi_{i+1}(\Sigma X) \\
[f : S^i \to X] &\longmapsto [\Sigma f : \Sigma S^i = S^{i+1} \to \Sigma X]
\end{aligned}$$

is an epimorphism for $i \leq 2n-1$ and an isomorphism for $i \leq 2n-2$.

Proof. For $n = 1$ this is just Theorem 1.13.7, so suppose that $n - 1 \geq 1$. Consider the path fibration

$$\Omega\Sigma X \longrightarrow P_*\Sigma X \longrightarrow \Sigma X,$$

and note that ΣX is n-connected (by the Hurewicz theorem and the suspension isomorphism $\widetilde{H}_*(\Sigma X) \cong \widetilde{H}_{*-1}(X)$), so $\Omega\Sigma X$ is $(n-1)$-connected. The Serre spectral sequence has the form



Thus the differentials $d^i : H_i(\Sigma X) \longrightarrow H_{i-1}(\Omega\Sigma X)$ are isomorphisms for $i-1 < 2n$. These differentials are transgressions so are given by

$$\begin{tikzcd}
 & H_i(P_*\Sigma X, \Omega\Sigma X) \arrow[r, "\sim"] \arrow[d, "p_*"] & H_{i-1}(\Omega\Sigma X) \\
H_i(\Sigma X) \arrow[r, "\sim"] & H_i(\Sigma X, *) &
\end{tikzcd}$$

and hence

$$p_* : H_i(P_*\Sigma X, \Omega\Sigma X) \longrightarrow H_i(\Sigma X, *)$$

is an isomorphism for $i \le 2n$.

Consider the map

$$\begin{aligned} f : CX &\longrightarrow P_*\Sigma X \\ [t,x] &\longmapsto (s \mapsto [st, x]) \end{aligned}$$

where we consider CX and ΣX as quotients of $[0,1] \times X$. Then there is a commutative square

$$\begin{tikzcd}
X \arrow[r, "\hat{f}"] \arrow[d, hook, "x\mapsto[x,1]"'] & \Omega\Sigma X \arrow[d, hook] \\
CX \arrow[r, "f"] & P_*\Sigma X
\end{tikzcd}$$

where $\hat{f}(x)(s) = [s, x]$. The map on long exact sequences on homology gives

$$\begin{tikzcd}
H_i(CX, X) \arrow[r, "\sim"] \arrow[d, "(f,\hat{f})_*"] \arrow[dd, bend right=60, "\sim"'] & H_{i-1}(X) \arrow[d, "\hat{f}_*"] \\
H_i(P_*\Sigma X, \Omega\Sigma X) \arrow[r, "\sim"] \arrow[d, "p_*"] & H_{i-1}(\Omega\Sigma X) \\
H_i(\Sigma X, *) &
\end{tikzcd}$$

where the curved arrow is an isomorphism by excision. As the map p_* is an isomorphism for $i \le 2n$, it follows that

$$\hat{f}_* : H_*(X) \longrightarrow H_*(\Omega\Sigma X)$$

is an isomorphism for $* \le 2n-1$.

As $n-1 \ge 1$, $\Omega\Sigma X$ is 1-connected. Letting F denote the homotopy fibre of the map $\hat{f}$, it then follows from the Serre spectral sequence that $\widetilde{H}_*(F) = 0$ for $* \le 2n-2$. By the Hurewicz theorem (as $\pi_1(F)$ must be a quotient of $\pi_2(\Omega\Sigma X)$ and so abelian) it follows that F is $(2n-2)$-connected, so by the long exact sequence

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(X) \xrightarrow{\hat{f}_*} \pi_i(\Omega\Sigma X) \xrightarrow{\partial} \pi_{i-1}(F) \longrightarrow \cdots$$

the map $\hat{f}_* : \pi_i(X) \longrightarrow \pi_i(\Omega\Sigma X) \cong \pi_{i+1}(\Sigma X)$ is an epimorphism for $i \le 2n-1$ and an isomorphism for $i \le 2n-2$. $\square$

Corollary 2.7.2. *The map $\Sigma : \pi_i(S^3) \to \pi_{i+1}(S^4)$ is an epimorphism for $i \leq 5$ and an isomorphism for $i \leq 4$, so in particular*

$$\mathbb{Z}/2 = \pi_4(S^3) \xrightarrow{\sim} \pi_5(S^4).$$

As we continue the range in Freudenthal's theorem only gets better, so $\pi_{n+1}(S^n) = \mathbb{Z}/2$ for all $n \geq 3$.

By Freudenthal's theorem $\pi_{i+n}(\Sigma^n X)$ is independent of n as long as $n \gg 0$: we call its stable value $\pi_i^s(X)$, the ith **stable homotopy group** of X. Tautologically it satisfies $\pi_i^s(\Sigma X) \cong \pi_{i-1}^s(X)$, and in particular $\pi_i^s(S^n) = \pi_{i-n}^s(S^0)$, so there is only a 1-parameter family of "stable homotopy groups of spheres", abbreviated $\pi_i^s := \pi_i^s(S^0)$. We have calculated

$$\pi_0^s = \mathbb{Z} \qquad \pi_1^s = \mathbb{Z}/2$$

and the next few are

i	0	1	2	3	4	5	6	7	8
π_i^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

There is not an obvious pattern: these groups are a topic of active research.

Chapter 3

Cohomology operations

3.1 Steenrod squares

Theorem 3.1.1. *There are natural (for maps of pairs) homomophisms*

$$\mathrm{Sq}^i : H^n(X, A; \mathbb{Z}/2) \longrightarrow H^{n+i}(X, A; \mathbb{Z}/2), \quad i \geq 0$$

satisfying

(i) $\mathrm{Sq}^0 = \mathrm{Id}$,

(ii) $\mathrm{Sq}^i(x) = x^2$ *if* $i = |x|$,

(iii) $\mathrm{Sq}^i(x) = 0$ *if* $i > |x|$,

(iv) $\mathrm{Sq}^k(x \smile y) = \sum_{i+j=k} \mathrm{Sq}^i(x) \smile \mathrm{Sq}^j(y)$,

(v) $\delta \circ \mathrm{Sq}^i = \mathrm{Sq}^i \circ \delta$ *for the connecting map* $\delta : H^n(A; \mathbb{Z}/2) \to H^{n+1}(X, A; \mathbb{Z}/2)$,

(vi) $\sigma \circ \mathrm{Sq}^i = \mathrm{Sq}^i \circ \sigma$ *for the suspension isomorphism* $\sigma : \widetilde{H}^n(X; \mathbb{Z}/2) \to \widetilde{H}^{n+1}(\Sigma X; \mathbb{Z}/2)$,

(vii) Sq^1 *is the Bockstein operation associated to the exact sequence*

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

of coefficients.

Sometimes we write $\mathrm{Sq} = \sum_{i=0}^{\infty} \mathrm{Sq}^i$. Then $\mathrm{Sq}(x)$ still makes sense, as the sum is finite by (iii), and (iv) may then be expressed as $\mathrm{Sq}(x \smile y) = \mathrm{Sq}(x) \smile \mathrm{Sq}(y)$. We will come back to construct the Sq^i and prove this theorem in Section 3.4, but will first give
Lecture 19 several applications which only use these properties.

Example 3.1.2. Recall that $\mathbb{CP}^2 = S^2 \cup_h D^4$ for $h : S^3 \to S^2$ the Hopf map. As a ring we have $H^*(\mathbb{CP}^2; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^3)$ and so

$$\mathrm{Sq}^2(x) = x^2 \neq 0.$$

Now $\Sigma^n \mathbb{CP}^2$ has cohomology

i	0	1	$\cdots$	$n+1$	$n+2$	$n+3$	$n+4$
$H^i(\Sigma^n \mathbb{CP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	0	$\cdots$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
generator	1		$\cdots$		$\sigma^n(x)$		$\sigma^n(x^2)$

and we have

$$\mathrm{Sq}^2(\sigma^n(x)) = \sigma^n(\mathrm{Sq}^2(x)) = \sigma^n(x^2) \neq 0.$$

It follows that $\Sigma^n\mathbb{CP}^2 = S^{n+2} \cup_{\Sigma^n h} D^{n+4} \not\simeq S^{n+2} \vee S^{n+4}$, and so $\Sigma^n h : S^{n+3} \to S^{n+2}$ is *not* homotopic to a constant map for any $n \geq 0$.

In particular, in Example 2.4.5 we had calculated $\pi_4(S^3) \cong \mathbb{Z}/2$ as abstract groups, but it now follows that this must be generated by the suspension Σh of the Hopf map. Using Corollary 2.7.2 it then follows that $\pi_{n+1}(S^n)$ is generated by the appropriate suspension of the Hopf map for any $n \geq 3$. △

Lemma 3.1.3. *In $H^*(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{n+1})$ we have*

$$\mathrm{Sq}^i(x^k) = \binom{k}{i} x^{k+i},$$

where the binomial coefficient is taken modulo 2.

Proof. We have

$$\begin{aligned}\mathrm{Sq}(x) &= \mathrm{Sq}^0(x) + \mathrm{Sq}^1(x) + \cancel{\mathrm{Sq}^2(x)} + \cdots \\ &= x + x^2\end{aligned}$$

using properties (i), (ii), and (iii) of the Steenrod squares, so by property (iv) we have

$$\mathrm{Sq}(x^k) = \mathrm{Sq}(x)^k = (x + x^2)^k = \sum_{i=0}^{k} \binom{k}{i} x^{k+i}$$

as required. □

Lemma 3.1.4. *In $H^*(\mathbb{CP}^n; \mathbb{Z}/2) = \mathbb{Z}/2[y]/(y^{n+1})$ we have*

$$\mathrm{Sq}^{2i}(y^k) = \binom{k}{i} y^{k+i},$$

where the binomial coefficient is taken modulo 2, and $\mathrm{Sq}^{2i+1}(y^k) = 0$. □

The following elementary lemma is convenient for making calculations with binomial coefficients modulo 2.

Lemma 3.1.5. *If $n = \sum_{i=0}^{\ell} n_i 2^i$ and $k = \sum_{i=0}^{\ell} k_i 2^i$ are the binary expansions, then*

$$\binom{n}{k} = \binom{n_0}{k_0}\binom{n_1}{k_1} \cdots \binom{n_\ell}{k_\ell} \mod 2.$$

□

The first few Steenrod squares in $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ can be visualised as in Figure 3.1. In particular we see that every x^n may be linked to x by a zig-zag of Sq^1's and Sq^2's, so there can be no decomposition $\mathbb{RP}^\infty \simeq X \vee Y$ with both X and Y having nontrivial $\mathbb{Z}/2$-cohomology. Similarly for $\Sigma^n\mathbb{RP}^\infty$ with any $n \geq 0$.

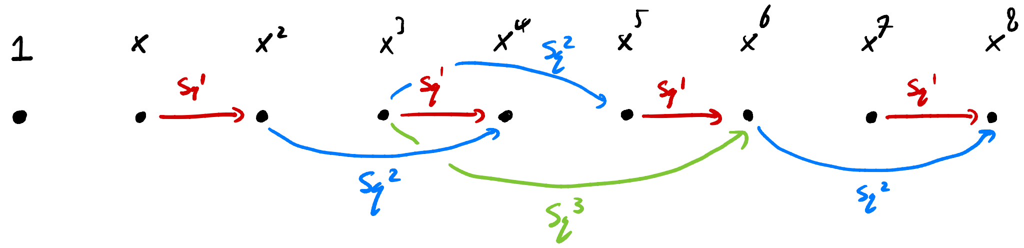


Figure 3.1 Steenrod operations in $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2)$.

Example 3.1.6. Let $n+1 = 2^r(2s+1)$, and consider $\mathbb{RP}^n/\mathbb{RP}^k$. If $n-2^r > k$ there is a class $z \in H^{n-2^r}(\mathbb{RP}^n/\mathbb{RP}^k;\mathbb{Z}/2)$ which under the quotient map $q : \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^k$ satisfies $q^*(z) = x^{n-2^r}$. Thus

$$q^*(\mathrm{Sq}^{2^r}(z)) = \mathrm{Sq}^{2^r}(x^{n-2^r}) = \binom{n-2^r}{2^r}x^n = \binom{2^{r+1}s-1}{2^r}x^n = x^n \neq 0$$

using Lemma 3.1.5, and so $\mathrm{Sq}^{2^r}(z) \neq 0 \in H^n(\mathbb{RP}^n/\mathbb{RP}^k;\mathbb{Z}/2) = \mathbb{Z}/2$.

If the map

$$f : \mathbb{RP}^n/\mathbb{RP}^k \longrightarrow \mathbb{RP}^n/\mathbb{RP}^{n-1} = S^n$$

had a right inverse up to homotopy, $g : S^n \to \mathbb{RP}^n/\mathbb{RP}^k$, then

$$g^* : H^n(\mathbb{RP}^n/\mathbb{RP}^k;\mathbb{Z}/2) \longrightarrow H^n(S^n;\mathbb{Z}/2)$$

would be an isomorphism, so we would have $\mathrm{Sq}^{2^r}(g^*(z)) = g^*(\mathrm{Sq}^{2^r}(z)) \neq 0$. But $g^*(z)$ lies in the group $H^{n-2^r}(S^n;\mathbb{Z}/2)$ which vanishes: this is a contradiction.

Thus the map $f : \mathbb{RP}^n/\mathbb{RP}^k \to S^n$ does not have a right inverse if $n-2^r > k$. △

Theorem 3.1.7 (Kudo's trangesssion theorem)**.** *Let $p : E \to B$ be a Serre fibration with path-connected base, and fibre $F = p^{-1}(b_0)$ such that $\pi_1(B,b_0)$ acts trivially on $H^*(F;\mathbb{Z}/2)$. If $y \in H^i(F;\mathbb{Z}/2) = E^2_{0,i}$ survives until $E^{i+1}_{0,i}$ and then $d^{i+1}(y) = [x]$, then $\mathrm{Sq}^n(y)$ survives until $E^{i+n+1}_{0,i+n}$ and $d^{n+i+1}(\mathrm{Sq}^n(y)) = [\mathrm{Sq}^n(x)]$.*

Proof. Under the assumptions (x,y) is a transgressive pair, so there is a $z \in H^{i+1}(B,b_0)$ so that in

$$\begin{tikzcd} & H^{i+1}(E,F) & H^i(F) \arrow[l, "\delta"'] \\ H^{i+1}(B) & H^{i+1}(B,b_0). \arrow[l, "j^*"'] \arrow[u, "p^*"] & \end{tikzcd}$$

we have $j^*(z) = x$ and $p^*(z) = \delta(y)$. But then $\mathrm{Sq}^n(z)$ exhibits $(\mathrm{Sq}^n(x), \mathrm{Sq}^n(y))$ as being a transgressive pair. □

Lecture 20

3.2 Vector fields on spheres

For a 1-dimensional subspace $\ell \subset \mathbb{R}^n$, let r_ℓ denote reflection in the line ℓ: so $r_\ell|_\ell = -\mathrm{Id}$ and $r_\ell|_{\ell^\perp} = \mathrm{Id}$. Let $r_0 = r_{(1,0,\ldots,0)}$. Then $r_0 r_\ell$ is an orthogonal transformation of determinant 1, so defines a map

$$\begin{aligned} J_n : \mathbb{RP}^{n-1} &\longrightarrow SO(n) \\ \ell &\longmapsto r_0 r_\ell. \end{aligned}$$

If $\ell \in \mathbb{RP}^{k-1} \subset \mathbb{RP}^{n-1}$, then $J_n(\ell)$ fixes the last $(n-k)$ basis vectors, so lands in $SO(k) \subset SO(n)$. Ths we get map

$$q_{n,k} : \mathbb{RP}^{n-1}/\mathbb{RP}^{k-1} \longrightarrow SO(n)/SO(k).$$

The two instances of "/" have different meanings: on the left $\mathbb{RP}^{n-1}/\mathbb{RP}^{k-1}$ denotes collapsing $\mathbb{RP}^{k-1}$ to a point, whereas on the right $SO(n)/SO(k)$ denotes taking the orbits of the $SO(k)$-action.

Lemma 3.2.1. *If $k = n-1$ then this is a map $q_{n,n-1} : S^{n-1} \to S^{n-1}$ and induces an isomorphism on $H_*(-;\mathbb{Z})$ (i.e. it has degree ± 1).*

Proof. The homeomorphism $SO(n)/SO(n-1) \overset{\cong}{\to} S^{n-1}$ is given by sending a matrix in $SO(n)$ to its last column, considered as a unit vector in $\mathbb{R}^n$. In other words, it is given by acting on the last basis vector $SO(n)$.

Thus the composition

$$\mathbb{RP}^{n-1} \longrightarrow \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \overset{q_{n,n-1}}{\longrightarrow} SO(n)/SO(n-1) \cong S^{n-1}$$

sends ℓ to $r_0 r_\ell(e_n)$. The preimage of $-e_n$ under this map consists of those ℓ such that $r_0 r_\ell(e_n) = -e_n$, so $r_\ell(e_n) = -e_n$. There is only one such ℓ, namely $\ell = \langle e_n \rangle$, so the map $q_{n,n-1}$ has degree ± 1. $\square$

Proposition 3.2.2. *The map $q_{n,k}$ induces an isomorphism on $H^*(-;\mathbb{Z})$ for $* \leq n-1$ as long as $n \leq 2k+1$.*

Proof. When $k = n-1$ we have proved this in the last lemma, so we proceed by downwards induction on k. Consider the commutative diagram

$$\begin{tikzcd} \mathbb{RP}^k/\mathbb{RP}^{k-1} \arrow[r, "\cong"] \arrow[d, hook] & SO(k+1)/SO(k) \arrow[d] \\ \mathbb{RP}^{n-1}/\mathbb{RP}^{k-1} \arrow[r, "q_{n,k}"] \arrow[d, "\text{quotient}"] & SO(n)/SO(k) \arrow[d, "\pi"] \\ \mathbb{RP}^{n-1}/\mathbb{RP}^k \arrow[r, "q_{n,k+1}"] & SO(n)/SO(k+1) \end{tikzcd}$$

where the right-hand column is a fibration sequence *but the left-hand column is not.*

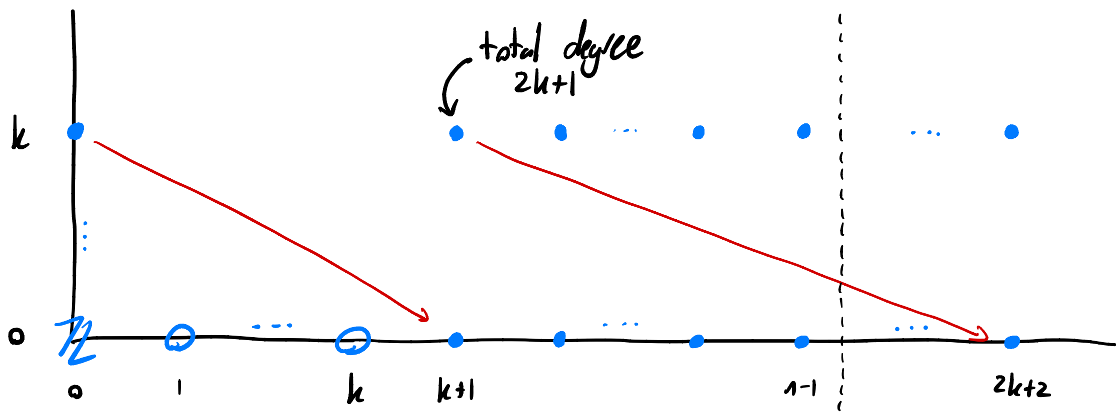


Figure 3.2

By inductive hypothesis the map $q_{n,k+1}$ is an isomorphism in cohomology in degrees $* \leq n-1$, so the Serre spectral sequence for the right-hand column takes the form shown in Figure 3.2.

In degrees $*+1 \leq 2k+1$ we obtain a map of exact sequences

$$\begin{tikzcd}
{} \arrow[r, "d^{k+1}"] & H^*(\frac{SO(n)}{SO(k+1)}) \arrow[r] \arrow[d, "q^*_{n,k+1}"] & H^*(\frac{SO(n)}{SO(k)}) \arrow[r] \arrow[d, "q^*_{n,k}"] & H^*(\frac{SO(k+1)}{SO(k)}) \arrow[r, "d^{k+1}"] \arrow[d, "\cong"] & H^{*+1}(\frac{SO(n)}{SO(k+1)}) \arrow[d, "q^*_{n,k+1}"] \\
{} \arrow[r, "\delta"] & H^*(\frac{\mathbb{RP}^{n-1}}{\mathbb{RP}^k}) \arrow[r] & H^*(\frac{\mathbb{RP}^{n-1}}{\mathbb{RP}^{k-1}}) \arrow[r] & H^*(\frac{\mathbb{RP}^k}{\mathbb{RP}^{k-1}}) \arrow[r, "\delta"] & H^{*+1}(\frac{\mathbb{RP}^{n-1}}{\mathbb{RP}^k}),
\end{tikzcd}$$

where the rightmost square commutes by the zig-zag description of the transgression (Section 2.6) and the other squares clearly commute. By the 5-lemma we find that $q^*_{n,k}$ is an isomorphism for $* \leq n-1$, as long as $n-1 \leq 2k$. $\square$

As $\mathbb{RP}^{n-1}/\mathbb{RP}^{k-1}$ and $SO(n)/SO(k)$ are easily seen to be 1-connected, we deduce

Corollary 3.2.3. *If $n \leq 2k+1$ then the map $q_{n,k}$ is $(n-1)$-connected.*

Theorem 3.2.4. *Let $n+1 = 2^r(2s+1)$. Then S^n does not admit 2^r linearly-independent vector fields.*

Proof. Note if $s=0$ then S^n clearly does not admit $2^r = n+1$ linearly independent vector fields, and if $r=0$ then n is even and so admits no nonvanishing vector fields. Thus we may suppose that $2^r \leq n/2$.

Consider the map
$$\pi : SO(n+1)/SO(n-k) \longrightarrow S^n$$
given by evaluating at the last basis vector. There is a bijection

$$\begin{aligned}
\{\text{orthonormal } (k+1)\text{-tuples in } \mathbb{R}^{n+1}\} &\xleftarrow{\sim} SO(n+1)/SO(n-k) \\
Ae_{n-k+1}, Ae_{n-k+2}, \ldots Ae_{n+1} &\longleftarrow A
\end{aligned}$$

under which the map π corresponds to recording the $(k+1)$-st orthogonal vector.

If S^n admits k linearly independent vector fields then by applying the Gram–Schmidt process it admits k orthonormal vector fields $s_1, \ldots s_k$, and so

$$\begin{aligned} s : S^n &\longrightarrow \{\text{orthonormal } (k+1)\text{-tuples in } \mathbb{R}^{n+1}\} = SO(n+1)/SO(n-k) \\ x &\longmapsto (s_1(x), s_2(x), \ldots, s_k(x), x) \end{aligned}$$

satisfies $\pi \circ s = \mathrm{Id}_{S^n}$.

If $n+1 \leq 2(n-k)+1$ (i.e. if $2k \leq n$) then by Corollary 3.2.3 the map $q_{n+1,n-k} : \mathbb{RP}^n/\mathbb{RP}^{n-k-1} \to SO(n+1)/SO(n-k)$ is n-connected, and so there exists a map

$$s' : S^n \longrightarrow \mathbb{RP}^n/\mathbb{RP}^{n-k-1}$$

such that $q_{n+1,n-k} \circ s' \simeq s$. This map s' would be right inverse up to homotopy to the map $\mathbb{RP}^n/\mathbb{RP}^{n-k-1} \to \mathbb{RP}^n/\mathbb{RP}^{n-1} = S^n$, but by Example 3.1.6 this means that $n - 2^r \leq n-k-1$, so $k < 2^r$. □

In fact if $n+1 = 2^r(2s+1)$ and $r = c+4d$ with $0 \leq c < 4$, then S^n admits $2^c + 8d - 1$ linearly-independent vector fields and no more. Lecture 21

3.3 Wu and Stiefel–Whitney classes

Let M be a closed compact n-dimensional manifold, so the map

$$\begin{aligned} H^i(M;\mathbb{Z}/2) &\longrightarrow \mathrm{Hom}(H^{n-i}(M;\mathbb{Z}/2), \mathbb{Z}/2) \\ x &\longmapsto \langle x \smile -, [M]\rangle \end{aligned}$$

is an isomorphism by Poincaré duality. There is a linear map given by

$$\begin{aligned} H^{n-i}(M;\mathbb{Z}/2) &\longrightarrow \mathbb{Z}/2 \\ y &\longmapsto \langle \mathrm{Sq}^i(y), [M]\rangle \end{aligned}$$

which therefore corresponds under the Poincaré duality isomorphism to a class $v_i \in H^i(M;\mathbb{Z}/2)$, uniquely determined by

$$\langle v_i \smile y, [M]\rangle = \langle \mathrm{Sq}^i(y), [M]\rangle \qquad \forall y \in H^{n-i}(M;\mathbb{Z}/2).$$

This is the ith **Wu class** of M. We write

$$v := 1 + v_1 + v_2 + \cdots \in \bigoplus_{i=0}^{\infty} H^i(M;\mathbb{Z}/2)$$

for the formal sum, the **total Wu class**. We then write

$$w := \mathrm{Sq}(v) = \sum_{i,j} \mathrm{Sq}^i(v_j)$$

for the **total Stiefel–Whitney class** of M, with components $w = 1 + w_1 + w_2 + \cdots$.

Example 3.3.1. On $\mathbb{RP}^n$ we have $\mathrm{Sq}^i(x^{n-i}) = \binom{n-i}{i} x^n$, and so $v_i = \binom{n-i}{i} x^i$. Thus $v = \sum_{i=0}^{\infty} \binom{n-i}{i} x^i$, and so

$$w = \mathrm{Sq}(v) = \sum_{i=0}^{\infty} \binom{n-i}{i} x^i (1+x)^i = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{n-i}{i} \binom{i}{j} x^{i+j}.$$

It is an exercise with binomal coefficients modulo 2 to see that this is $(1+x)^{n+1}$, so $w_i = \binom{n+1}{i} x^i$. △

Remark 3.3.2. Note that $\mathrm{Sq}^i : H^{n-i}(M;\mathbb{Z}/2) \to H^n(M;\mathbb{Z}/2)$ vanishes if $i > n-i$, by property (iii) of Steenrod squares. Thus $v_i = 0$ for $2i > n$. Thus there are *fewer* Wu classes than Stiefel–Whitney classes, which means that Stiefel–Whitney classes must satisfy certain relations.

For example, if $n = 4$ then $v = 1 + v_1 + v_2$ and so

$$w = 1 + (v_1) + (v_2 + v_1^2) + (\mathrm{Sq}^1(v_2)) + (v_2^2).$$

Thus we have

$$\begin{aligned} w_3 &= \mathrm{Sq}^1(v_2) = \mathrm{Sq}^1(v_2 + v_1^2) = \mathrm{Sq}^1(w_2) \\ w_4 &= w_2 + w_1^2. \end{aligned}$$

So if $w_1 = w_2 = 0$ then $w_3 = w_4 = 0$ too.

Suppose that M is a smooth manifold and is embedded in $\mathbb{R}^{n+k}$ with normal bundle ν, and let $M \subset U \subset \mathbb{R}^{n+k}$ be a tubular neighbourhood. There is a collapse map

$$c : S^{n+k} \longrightarrow \frac{S^{n+k}}{S^{n+k} \setminus U} = U^+$$

and the space U^+ may be identified with the **Thom space** $\mathrm{Th}(\nu) = \frac{D(\nu)}{S(\nu)}$. Recall that the Thom class is a class $u \in \widetilde{H}^k(\mathrm{Th}(\nu);\mathbb{Z}/2)$, and the (co)homology Thom isomorphisms are the maps

$$\begin{aligned} &H^i(M;\mathbb{Z}/2) \cong H^i(D(\nu);\mathbb{Z}/2) \xrightarrow{-\smile u} H^{i+k}(D(\nu), S(\nu);\mathbb{Z}/2) \cong \widetilde{H}^{i+k}(\mathrm{Th}(\nu);\mathbb{Z}/2) \\ &\widetilde{H}_{i+k}(\mathrm{Th}(\nu);\mathbb{Z}/2) \cong H_{i+k}(D(\nu), S(\nu);\mathbb{Z}/2) \xrightarrow{u \frown -} H_i(D(\nu);\mathbb{Z}/2) \cong H_i(M;\mathbb{Z}/2). \end{aligned}$$

Abusing notation slightly, we write

$$- \smile u : H^i(M;\mathbb{Z}/2) \xrightarrow{\sim} \widetilde{H}^{i+k}(\mathrm{Th}(\nu);\mathbb{Z}/2).$$

Proposition 3.3.3. *We have*

$$\mathrm{Sq}(u) = \frac{1}{w} \smile u \in \widetilde{H}^*(\mathrm{Th}(\nu);\mathbb{Z}/2).$$

Proof. As $\mathrm{Sq}^0 = \mathrm{Id}$ the operator Sq is formally invertible: call its inverse Sq^{-1}. Let us write $c_*[S^{n+k}] =: [Th]$, so that $u \frown [Th] = [M]$ under the homology Thom isomorphism.

The defining property of the Wu class gives

$$\begin{aligned}\langle \mathrm{Sq}(x) \smile u, [Th]\rangle &= \langle \mathrm{Sq}(x), u \frown [Th]\rangle \\ &= \langle \mathrm{Sq}(x), [M]\rangle \\ &= \langle v \smile x, [M]\rangle \\ &= \langle v \smile x \smile u, [Th]\rangle\end{aligned}$$

for any x, and the left-hand side is $\langle \mathrm{Sq}(x \smile \mathrm{Sq}^{-1}(u)), [Th]\rangle$. Now $[Th] = c_*[S^{n+k}]$ so this is

$$\langle c^*(\mathrm{Sq}(x \smile \mathrm{Sq}^{-1}(u))), [S^{n+k}]\rangle = \langle \mathrm{Sq}(c^*(x \smile \mathrm{Sq}^{-1}(u))), [S^{n+k}]\rangle.$$

But in $H^*(S^{n+k}; \mathbb{Z}/2)$ we have $\mathrm{Sq} = \mathrm{Id}$ (there is no space for any other operations) so this is

$$\langle c^*(x \smile \mathrm{Sq}^{-1}(u)), [S^{n+k}]\rangle = \langle x \smile \mathrm{Sq}^{-1}(u), [Th]\rangle.$$

In total we obtain the identity

$$\langle x \smile \mathrm{Sq}^{-1}(u), [Th]\rangle = \langle v \smile x \smile u, [Th]\rangle$$

for all $x \in H^*(M; \mathbb{Z}/2)$, so by the Thom isomorphism and Poincaré duality we have $\mathrm{Sq}^{-1}(u) = v \smile u$, and hence $u = \mathrm{Sq}(v) \smile \mathrm{Sq}(u)$ giving $\mathrm{Sq}(u) = \frac{1}{w} \smile u$ as required. $\square$

Corollary 3.3.4. *If M^n embeds into $\mathbb{R}^{n+k}$ then $[\frac{1}{w}]_i = 0$ for all $i \geq k$.*

Proof. If $i > k$ then $\mathrm{Sq}^i(u) = 0$ as $|u| = k$, so $[\frac{1}{w}]_i = 0$.

If $i = k$ then $\mathrm{Sq}^i(u) = u^2 = [\frac{1}{w}]_k \smile u$. If $[\frac{1}{w}]_k \neq 0$ then by Poincaré duality there is an $x \in H^{n-k}(M; \mathbb{Z}/2)$ such that

$$\begin{aligned}1 = \langle x \smile [\tfrac{1}{w}]_k, [M]\rangle &= \langle x \smile [\tfrac{1}{w}]_k \smile u, [Th]\rangle \\ &= \langle x \smile u \smile u, [Th]\rangle \\ &= \langle c^*(x \smile u) \smile c^*(u), [S^{n+k}]\rangle\end{aligned}$$

which is a contradiction as all nontrivial cup products in $H^*(S^{n+k}; \mathbb{Z}/2)$ vanish. $\square$

Example 3.3.5. On the manifold $\mathbb{RP}^{2^k}$ by Example 3.3.1 we have

$$\begin{aligned}w &= (1+x)^{2^k+1} \\ &= (1+x)(1+x)^{2^k} \\ &= (1+x)(1+x^{2^k})\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{w} &= \frac{1}{(1+x)(1+x^{2^k})} \\ &= (1 + x + x^2 + x^3 + \cdots)(1 + x^{2^k} + x^{2\cdot 2^k} + \cdots) \\ &= 1 + x + x^2 + \cdots + x^{2^k-1} \in H^*(\mathbb{RP}^{2^k}; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{2^k+1}).\end{aligned}$$

Thus $\mathbb{RP}^{2^k}$ *does not* embed into $\mathbb{R}^{2^{k+1}-1}$.[1] △

3.4 Constructing the Steenrod squares

Let us write $K_n = K(\mathbb{Z}/2, n)$ in this section, and take (co)homology with $\mathbb{Z}/2$ coefficients. Recall that there is a $\iota_n \in H^n(K_n)$ such that

$$\begin{aligned} [X, K_n] &\xrightarrow{\sim} H^n(X) \\ f &\longmapsto f^*(\iota_n) \end{aligned}$$

is a bijection for any CW-complex X. We are only interested in $n > 0$, so ι_n may be represented by a reduced cohomology class $\iota_n \in H^n(K_n, *)$. If (X, x_0) is a based CW-complex, the same argument shows that

$$\begin{aligned} [X, K_n]_* &\xrightarrow{\sim} \widetilde{H}^n(X) \\ f &\longmapsto f^*(\iota_n), \end{aligned}$$

Lecture 22 where $[-,-]_*$ denotes the homotopy classes of *based* maps.

Suppose that we are given a class $\theta\iota_n \in \widetilde{H}^{n+i}(K_n)$. Then for any based CW-complex we have an operation

$$\begin{aligned} \theta : \widetilde{H}^n(X) &\longrightarrow \widetilde{H}^{n+i}(X) \\ f^*(\iota_n) &\longmapsto f^*(\theta\iota_n) \end{aligned}$$

which by construction is natural for maps of based CW-complexes. We can promote it to an operation on the cohomology of all based spaces via CW approximation.

Question: Under what conditions on the class $\theta\iota_n$ is the function θ a homomorphism?

To answer this question, note that $\iota_n \otimes 1 + 1 \otimes \iota_n \in \widetilde{H}^n(K_n \times K_n)$ corresponds to a based map

$$\mu_n : K_n \times K_n \longrightarrow K_n,$$

well-defined up to homotopy.

Lemma 3.4.1. *This map satisfies*

(i) $\mu_n(*, -) \simeq \mathrm{Id} \simeq \mu_n(-, *) : K_n \to K_n$,

(ii) $\mu_n(\mu_n(-,-),-) \simeq \mu_n(-, \mu_n(-,-)) : K_n \times K_n \times K_n \to K_n$.

Proof. We are claiming that certain maps to K_n are homotopic, which is the case if they pull back ι_n to the same class.

For (i) the composition

$$K_n = K_n \times \{*\} \xrightarrow{Id \times inc} K_n \times K_n \xrightarrow{\mu_n} K_n$$

[1] It is a theorem of H. Whitney than any smooth n-manifold may be smoothly embedded in $\mathbb{R}^{2n}$.

pulls back ι_n to

$$(Id \times inc)^*(\mu_n)^*(\iota_n) = (Id \times inc)^*(\iota_n \otimes 1 + 1 \otimes \iota_n) = \iota_n$$

so this composition is homotopic to the identity; similarly for $\mu_n(*, -)$.

For (ii), $\mu_n(\mu_n(-,-),-)$ is the composition

$$K_n \times K_n \times K_n \xrightarrow{\mu_n \times Id} K_n \times K_n \xrightarrow{\mu_n} K_n$$

so it pulls back ι_n to

$$(\mu_n \times Id)^*(\mu_n)^*(\iota_n) = (\mu_n \times Id)^*(\iota_n \otimes 1 + 1 \otimes \iota_n) = (\iota_n \otimes 1 + 1 \otimes \iota_n) \otimes 1 + 1 \otimes \iota_n.$$

The map $\mu_n(-, \mu_n(-,-))$ does too. □

Corollary 3.4.2. *The operation θ is a homomorphism if and only if*

$$\mu_n^*(\theta\iota_n) = \theta\iota_n \otimes 1 + 1 \otimes \theta\iota_n \in \widetilde{H}^{n+i}(K_n \times K_n). \tag{3.4.1}$$

Proof. The function θ is defined by naturality and $\theta(\iota_n) = \theta\iota_n$, so if it is a homomorphism then

$$\begin{aligned}\mu_n^*(\theta\iota_n) &= \mu_n^*(\theta(\iota_n)) = \theta(\mu_n^*(\iota_n)) = \theta(\iota_n \otimes 1 + 1 \otimes \iota_n)\\ &= \theta(\pi_1^*(\iota_n) + \pi_2^*(\iota_n))\\ &= \theta(\pi_1^*(\iota_n)) + \theta(\pi_2^*(\iota_n))\\ &= \pi_1^*(\theta\iota_n) + \pi_2^*(\theta\iota_n)\\ &= \theta\iota_n \otimes 1 + 1 \otimes \theta\iota_n.\end{aligned}$$

Conversely, if $f^*(\iota_n), g^*(\iota_n) \in \widetilde{H}^n(X)$ then the composition

$$X \xrightarrow{f \times g} K_n \times K_n \xrightarrow{\mu_n} K_n$$

pulls back ι_n to

$$(f \times g)^*(\mu_n)^*(\iota_n) = (f \times g)^*(\iota_n \otimes 1 + 1 \otimes \iota_n) = f^*(\iota_n) + g^*(\iota_n).$$

Thus assuming (3.4.1) holds we have

$$\begin{aligned}\theta(f^*(\iota_n) + g^*(\iota_n)) &:= (f \times g)^*(\mu_n)^*(\theta\iota_n)\\ &= (f \times g)^*(\theta\iota_n \otimes 1 + 1 \otimes \theta\iota_n)\\ &= f^*(\theta\iota_n) + g^*(\theta\iota_n)\\ &= \theta(f^*(\iota_n)) + \theta(g^*(\iota_n))\end{aligned}$$

as required. □

Definition 3.4.3. Say that $x \in \widetilde{H}^*(K_n)$ is **primitive** if $\mu_n^*(x) = x \otimes 1 + 1 \otimes x$.

Lemma 3.4.4. *Under the identification $\Omega K_{n+1} \simeq K_n$, $\Omega\mu_{n+1} \simeq \mu_n$.*

Proof. Let

$$K_n \simeq \Omega K_{n+1} \longrightarrow P_* K_{n+1} \xrightarrow{\pi} K_{n+1}$$

be the path fibration. Then $P_*(K_{n+1} \times K_{n+1}) = P_*(K_{n+1}) \times P_*(K_{n+1})$ and there is a map of fibrations

$$\begin{tikzcd}
\Omega K_{n+1} \times \Omega K_{n+1} \arrow[r, "\Omega\mu_{n+1}"] \arrow[d] & \Omega K_{n+1} \arrow[d] \\
P_*(K_{n+1} \times K_{n+1}) \arrow[r, "P_*\mu_{n+1}"] \arrow[d, "\pi\times\pi"] & P_* K_{n+1} \arrow[d, "\pi"] \\
K_{n+1} \times K_{n+1} \arrow[r, "\mu_{n+1}"] & K_{n+1}
\end{tikzcd}$$

In the Serre spectral sequence for the fibration π, the class

$$\iota_n \in H^n(K_n) \cong H^n(\Omega K_{n+1}) = E^2_{0,n}$$

transgresses to $\iota_{n+1} \in H^{n+1}(K_{n+1}) = E^2_{n+1,0}$. Thus there is a commutative square

$$\begin{tikzcd}
H^n(K_n) \arrow[r, "d^{n+1}"] \arrow[d, "(\Omega\mu_{n+1})^*"] & H^{n+1}(K_{n+1}) \arrow[d, "\mu^*_{n+1}"] \\
H^n(K_n \times K_n) \arrow[r, "\bar{d}^{n+1}"] & H^{n+1}(K_{n+1} \times K_{n+1})
\end{tikzcd}$$

where the horizontal maps are isomorphisms, and $\bar{d}^{n+1}$ is the differential in the Serre spectral sequence $\{\bar{E}^r_{*,*}\}$ for the fibration $\pi \times \pi$. We have

$$\mu^*_{n+1} d^{n+1}(\iota_n) = \mu^*_{n+1}(\iota_{n+1}) = \iota_{n+1} \otimes 1 + 1 \otimes \iota_{n+1}$$

and

$$\iota_{n+1} \otimes 1 + 1 \otimes \iota_{n+1} = \bar{d}^{n+1}(\iota_n \otimes 1 + 1 \otimes \iota_n),$$

so $(\Omega\mu_{n+1})^*(\iota_n) = \iota_n \otimes 1 + 1 \otimes \iota_n$, so $\Omega\mu_{n+1} \simeq \mu_n$ as this is its defining property. □

Theorem 3.4.5. *If $\theta\iota_n \in H^{i+n}(K_n)$ is primitive, transgresses in the Serre spectral sequence for*

$$K_n \simeq \Omega K_{n+1} \longrightarrow P_* K_{n+1} \xrightarrow{\pi} K_{n+1},$$

and $i \le n$, then it transgresses to a unique *class, called $\theta\iota_{n+1} \in H^{n+1+i}(K_{n+1})$, which is also primitive.*

Proof. From the Serre spectral sequence for π as shown in Figure 3.3 we see that the lowest degree in which transgressions are not unique is $2n+2$. As $n+i+1 \le 2n+1$ by assumption, the transgression of $\theta\iota_n$ is indeed unique.

To see that $\theta\iota_{n+1}$ is primitive, we consider the map of Serre spectral sequences for the map of fibrations in the previous lemma. We find a commutative diagram

$$\begin{tikzcd}
H^{n+i}(K_n) \supset E^{n+i+1}_{0,n+i} \arrow[r, "d^{n+i+1}"] \arrow[d, "\mu^*_n"] & H^{n+i+1}(K_{n+1}) \arrow[d, "\mu^*_{n+1}"] \\
H^{n+i}(K_n \times K_n) \supset \bar{E}^{n+i+1}_{n+i+1,0} \arrow[r, "\bar{d}^{n+i+1}"] & H^{n+i+1}(K_{n+1} \times K_{n+1}).
\end{tikzcd}$$

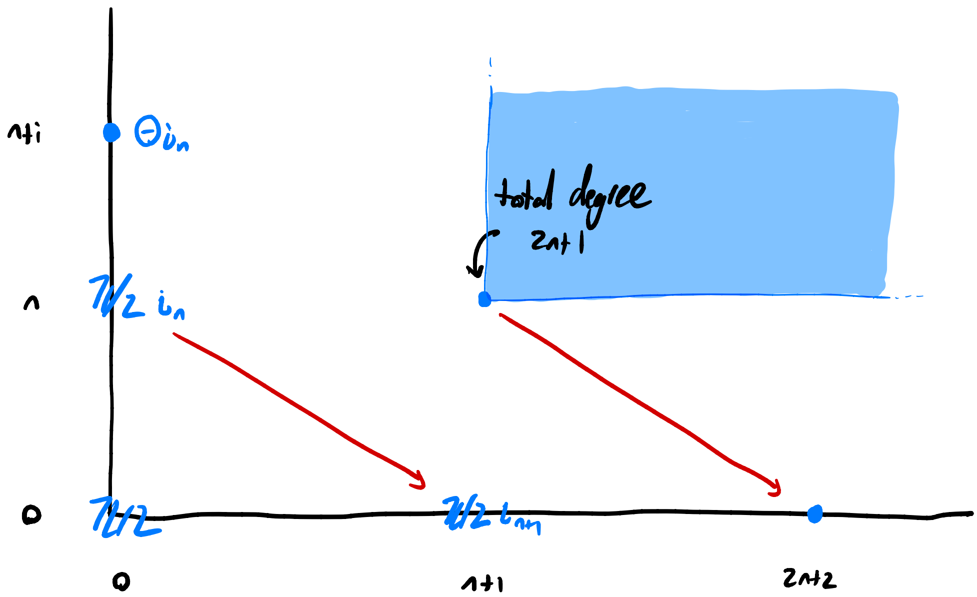


Figure 3.3 The Serre spectral sequence for the path fibration over K_{n+1} with $\mathbb{Z}/2$-coefficients.

We have

$$\bar{d}^{n+i+1}\mu_n^*(\theta\iota_n) = \bar{d}^{n+i+1}(\theta\iota_n \otimes 1 + 1 \otimes \theta\iota_n) = \theta\iota_{n+1} \otimes 1 + 1 \otimes \theta\iota_{n+1}$$

and $d^{n+i+1}\theta\iota_n = \theta\iota_{n+1}$, so

$$\mu_{n+1}^*(\theta\iota_{n+1}) = \mu_{n+1}^*(d^{n+i+1}\theta\iota_n) = \theta\iota_{n+1} \otimes 1 + 1 \otimes \theta\iota_{n+1}$$

as required. □

Lecture 23

Corollary 3.4.6. *In the situation above, if* $\theta : \widetilde{H}^n(-) \to \widetilde{H}^{n+i}(-)$ *is the function defined by* $\theta\iota_n$*, and* $\theta : \widetilde{H}^{n+1}(-) \to \widetilde{H}^{n+i+1}(-)$ *is the function defined by* $\theta\iota_{n+1}$*, then the square*

$$\begin{tikzcd} \widetilde{H}^n(X) \arrow[r, "\sigma"] \arrow[d, "\theta"] & \widetilde{H}^{n+1}(\Sigma X) \arrow[d, "\theta"] \\ \widetilde{H}^{n+i}(X) \arrow[r, "\sigma"] & \widetilde{H}^{n+i+1}(\Sigma X) \end{tikzcd}$$

commutes.

Proof. By naturality, it suffices to show it commutes when $X = K_n$ on the class ι_n. Choosing a nullhomotopy of the inclusion $K_n = \Omega K_{n+1} \to P_*K_{n+1}$ gives maps of pairs

$$\begin{tikzcd} (CK_n, K_n) \arrow[r, "\simeq"] \arrow[d] & (P_*K_{n+1}, K_n) \arrow[d, "\pi"] \\ (\Sigma K_n, *) \arrow[r, "f"] & (K_{n+1}, *) \end{tikzcd}$$

and because $\pi^* : H^{n+1}(K_{n+1}, *) \to H^{n+1}(P_* K_{n+1}, K_n)$ is an isomorphism, as it is part of the transgression of ι_n to ι_{n+1}, it follows that

$$f^* : H^{n+1}(K_{n+1}, *) \longrightarrow H^{n+1}(\Sigma K_n, *)$$

is an isomorphism too, and so f represents the cohomology class $\sigma\iota_n \in \widetilde{H}^{n+1}(\Sigma K_n)$. The definition of transgression gives the diagram

$$\begin{tikzcd}
H^{n+i+1}(CK_n, K_n) & H^{n+i+1}(P_* K_{n+1}, K_n) \arrow[l, "\simeq"'] & \widetilde{H}^{n+i}(K_n) \arrow[l, "\delta"'] \arrow[dll, bend left, "\sigma"] \\
\widetilde{H}^{n+i+1}(\Sigma K_n) \arrow[u, "\simeq"] & \widetilde{H}^{n+i+1}(K_{n+1}) \arrow[l, "f^*"'] \arrow[u, "\pi^*"'] &
\end{tikzcd}$$

in which the square and outer boundary commutes. We have $\delta(\theta\iota_n) = \pi^*(\theta\iota_{n+1})$ by definition of transgression, and so $\sigma(\theta\iota_n) = f^*(\theta\iota_{n+1}) = \theta(\sigma\iota_n)$ as required. $\square$

Corollary 3.4.7. *In the situation above, if we define*

$$\theta : H^n(X, A) \longrightarrow H^{n+i}(X, A)$$

for a CW-pair (X, A) using $\widetilde{H}^*(X/A) \xrightarrow{\simeq} H^*(X, A)$, then the square

$$\begin{tikzcd}
H^n(A) \arrow[r, "\delta"] \arrow[d, "\theta"] & H^{n+1}(X, A) \arrow[d, "\theta"] \\
H^{n+i}(A) \arrow[r, "\delta"] & H^{n+i+1}(X, A)
\end{tikzcd}$$

commutes.

Proof. We can extend the inclusion $A \to CA \simeq *$ to a map $f : X \to CA$, and hence get a map of pairs $f : (X, A) \to (CA, A)$, and an induced map $\hat{f} : X/A \to CA/A = \Sigma A$. By naturality the diagram

$$\begin{tikzcd}
H^n(A) \arrow[r, "\delta"] \arrow[dr, "\delta"] \arrow[drr, bend right, "\sigma"'] & H^{n+1}(X, A) & \widetilde{H}^{n+1}(X/A) \arrow[l, "\simeq"'] \\
& H^{n+1}(CA, A) \arrow[u, "f^*"] & \widetilde{H}^{n+1}(\Sigma A) \arrow[l, "\simeq"'] \arrow[u, "\hat{f}^*"']
\end{tikzcd}$$

commutes, so as θ commutes with σ and with $\hat{f}^*$ the claim follows. $\square$

So far the discussion has been completely general; we now construct the Sq^i.

Theorem 3.4.8. *There are natural homomorphisms* Sq^i *of degree* i *such that*

(i) $\mathrm{Sq}^0 = \mathrm{Id}$,

(ii) $\mathrm{Sq}^i(x) = x^2$ *if* $i = |x|$,

(iii) $\mathrm{Sq}^i(x) = 0$ *if* $i > |x|$,

(iv) $\delta \circ \mathrm{Sq}^i = \mathrm{Sq}^i \circ \delta$ *for the connecting map* $\delta : H^n(A) \to H^{n+1}(X, A)$,

(v) $\sigma \circ \mathrm{Sq}^i = \mathrm{Sq}^i \circ \sigma$ *for the suspension isomorphism* $\sigma : \widetilde{H}^n(X) \to \widetilde{H}^{n+1}(\Sigma X)$.

Proof. Define $\mathrm{Sq}^0 = \mathrm{Id}$. For $n > 0$, define $\mathrm{Sq}^n \iota_n = \iota_n^2 \in H^{2n}(K_n)$. Now

$$\mu_n^*(\iota_n^2) = (\mu_n(\iota_n))^2 = (\iota_n \otimes 1 + 1 \otimes \iota_n)^2 = \iota_n^2 \otimes 1 + 1 \otimes \iota_n^2$$

so $\mathrm{Sq}^n \iota_n$ is primitive. In the Serre spectral sequence for

$$K_n \simeq \Omega K_{n+1} \longrightarrow P_* K_{n+1} \longrightarrow K_{n+1}$$

we have $d^{n+1}(\iota_n^2) = 2\iota_n \otimes \iota_{n+1} = 0$. Thus ι_n^2 is transgressive: by Theorem 3.4.5 it transgresses to a unique element $\mathrm{Sq}^n \iota_{n+1} \in H^{2n+1}(K_{n+1})$ which is again primitive.

Now consider the Serre spectral sequence for

$$K_{n+1} \simeq \Omega K_{n+2} \longrightarrow P_* K_{n+2} \longrightarrow K_{n+2}.$$

The class not lying on an edge of lowest total degree is $\iota_{n+2} \otimes \iota_{n+1}$ of degree $2n+3$, so all classes of degree $\leq 2n+1$ transgress. In particular $\mathrm{Sq}^n \iota_{n+1}$ does, so by Theorem 3.4.5 it transgresses to a unique element $\mathrm{Sq}^n \iota_{n+2} \in H^{2n+2}(K_{n+2})$ which is again primitive. Continuing in this way we define $\mathrm{Sq}^{n+k} \iota_n$ for all $k \geq 0$.

Defining $\mathrm{Sq}^n \iota_j = 0$ for $j < n$, we obtain an operation Sq^n defined on cohomology classes of every degree, which satisfy all the required properties, except possibly

$$\begin{tikzcd}
\widetilde{H}^{n-1}(X) \arrow[r, "\sigma"] \arrow[d, "\mathrm{Sq}^n=0"] & \widetilde{H}^n(\Sigma X) \arrow[d, "\mathrm{Sq}^n=\text{square}"] \\
\widetilde{H}^{2n-1}(X) \arrow[r, "\sigma"] & \widetilde{H}^{2n}(\Sigma X)
\end{tikzcd}$$

commuting: but this does in fact commute, as cup products vanish on any suspension (the analogous naturality for δ follows from this as in Corollary 3.4.7). $\square$

To study how the Sq^i interact with cup products, we must first discuss products of based spaces. If (X, x_0) and (Y, y_0) are based spaces, their **smash product** is

$$X \wedge Y := \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}.$$

Note that $\Sigma X = S^1 \wedge X$. The usual Künneth theorem shows that

$$- \wedge - : H^*(X, x_0) \otimes H^*(Y, y_0) \xrightarrow{-\smile-} H^*(X \times Y, X \times \{y_0\} \cup \{x_0\} \times Y) = \widetilde{H}^*(X \wedge Y)$$

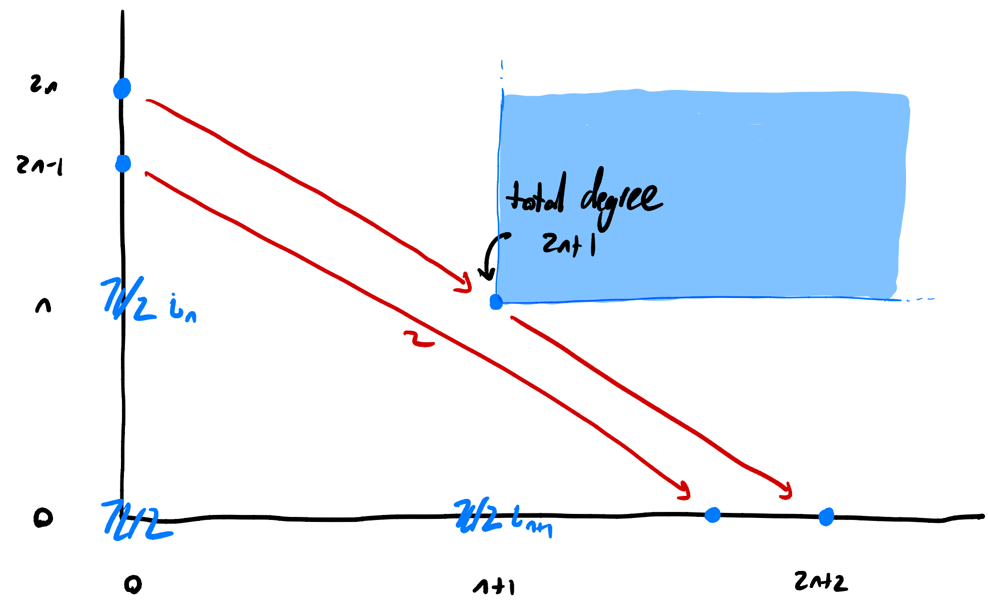
is an isomorphism (as we are working with coefficients in the field $\mathbb{Z}/2$). Lecture 24

Lemma 3.4.9. *The map* $\sigma\iota_n : \Sigma K_n \to K_{n+1}$ *is injective on cohomology in degrees* $* \leq 2n+1$.

Proof. Return to the diagram

$$\begin{tikzcd}
H^{n+i+1}(CK_n, K_n) & H^{n+i+1}(P_*K_{n+1}, K_n) \arrow[l, "\simeq"'] & \widetilde{H}^{n+i}(K_n) \arrow[l, "\delta"'] \arrow[dl, dashed, "\text{transgresson}"] \\
\widetilde{H}^{n+i+1}(\Sigma K_n) \arrow[u, "\simeq"] & \widetilde{H}^{n+i+1}(K_{n+1}) \arrow[l, "(\sigma\iota_n)^*"'] \arrow[u, "\pi^*"]
\end{tikzcd}$$

which shows that $(\sigma\iota_n)^*$ is injective in the range of degrees in which the transgression is single-valued. From the Serre spectral sequence



this is seen to be degrees $* \leq 2n+1$. □

Consider the maps

$$\sigma_L : \Sigma K_{n-1} \wedge K_m \xrightarrow{\sigma\iota_{n-1}\wedge \mathrm{Id}} K_n \wedge K_m$$
$$\sigma_R : K_n \wedge \Sigma K_{m-1} \xrightarrow{\mathrm{Id}\wedge\sigma\iota_{m-1}} K_n \wedge K_m.$$

It follows from the previous lemma that the maps

$$\sigma_L^* : \bigoplus_{\substack{i \leq 2n-1 \\ j}} \widetilde{H}^i(K_n) \otimes \widetilde{H}^j(K_m) \longrightarrow \widetilde{H}^*(\Sigma K_{n-1}) \otimes \widetilde{H}^*(K_m)$$

$$\sigma_R^* : \bigoplus_{\substack{i \\ j \leq 2m-1}} \widetilde{H}^i(K_n) \otimes \widetilde{H}^j(K_m) \longrightarrow \widetilde{H}^*(K_n) \otimes \widetilde{H}^*(\Sigma K_{m-1})$$

are both injective, so in total degrees $* \leq 2(n+m)-1$ the map

$$\sigma_L^* \oplus \sigma_R^* : \widetilde{H}^*(K_n) \otimes \widetilde{H}^*(K_m) \longrightarrow (\widetilde{H}^*(\Sigma K_{n-1}) \otimes \widetilde{H}^*(K_m)) \oplus (\widetilde{H}^*(K_n) \otimes \widetilde{H}^*(\Sigma K_{m-1}))$$

is injective.

Theorem 3.4.10. *We have*

$$\mathrm{Sq}^k(\iota_n \wedge \iota_m) = \sum_{i+j=k} \mathrm{Sq}^i(\iota_n) \wedge \mathrm{Sq}^j(\iota_m) \in \widetilde{H}^*(K_n \wedge K_m).$$

Proof. If $k = n+m$ this becomes the identity

$$(\iota_n \wedge \iota_m)^2 = \iota_n^2 \wedge \iota_m^2$$

which is certainly true. So we proceed by induction on the quantity $n+m-k$.

Let $z \in \widetilde{H}^{n+m+k}(K_n \wedge K_m)$ be the difference of the two terms. Under σ_L^* this becomes

$$\sigma_L^*(z) = \mathrm{Sq}^k(\sigma\iota_{n-1} \wedge \iota_m) + \sum_{i+j=k} \mathrm{Sq}^i(\sigma\iota_n) \wedge \mathrm{Sq}^k(\iota_m) \in \widetilde{H}^{n+m+k}(\Sigma K_{n-1} \wedge K_m)$$

which under the suspension isomorphism is

$$\mathrm{Sq}^k(\iota_{n-1} \wedge \iota_m) + \sum_{i+j=k} \mathrm{Sq}^i(\iota_n) \wedge \mathrm{Sq}^k(\iota_m) \in \widetilde{H}^{(n-1)+m+k}(K_{n-1} \wedge K_m).$$

As $(n-1)+m-k < n+m-k$ this vanishes by induction. Similarly $\sigma_R^*(z) = 0$, so $z = 0$. □

The final property of Steenrod squares, that Sq^1 agrees with Bockstein operation, is Example Sheet 4 Q2.

3.5 Outlook

The Steenrod squares satisfy the **Adem relations**: if $0 < i < 2j$ then the identity

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{i/2} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

holds. It is not hard to show using this that if a is not a power of 2 then Sq^a is decomposable (as we can write $a = i+j$ with $0 < i < 2j$ and $\binom{j-1}{i} \equiv 1 \mod 2$): the first few are

$$\begin{aligned}
\mathrm{Sq}^3 &= \mathrm{Sq}^1\mathrm{Sq}^2 \\
\mathrm{Sq}^5 &= \mathrm{Sq}^1\mathrm{Sq}^4 \\
\mathrm{Sq}^6 &= \mathrm{Sq}^2\mathrm{Sq}^4 + \mathrm{Sq}^5\mathrm{Sq}^1 = \mathrm{Sq}^2\mathrm{Sq}^4 + \mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^1 \\
\mathrm{Sq}^7 &= \mathrm{Sq}^1\mathrm{Sq}^6 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4 \quad (\text{using } \mathrm{Sq}^1\mathrm{Sq}^1 = 0) \\
\mathrm{Sq}^9 &= \mathrm{Sq}^1\mathrm{Sq}^8.
\end{aligned}$$

Example 3.5.1. Suppose X is a space having $H^*(X;\mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^3)$ with $|x| = n$ (so X is analogous to $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{HP}^2$). Then

$$0 \neq x^2 = \mathrm{Sq}^n(x).$$

If n is not a power of 2 then Sq^n is decomposable, but $\mathrm{Sq}^i(x) = 0$ for all $0 < i < n$ because the group it lies in is zero. Thus n must be a power of 2. △

In fact, J. F. Adams has showed that Sq^{2^i} is "decomposable in terms of higher-order operations" if $i \geq 4$, which implies that in the above example we must have $n = 1, 2, 4, 8$; the last example is provided by the octonionic projective plane $\mathbb{OP}^2$. This is the famous Hopf Invariant 1 Theorem.

In a different direction, given a word $\mathrm{Sq}^I := \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_r}$ in Steenrod squares, if $i_j < 2i_{j+1}$ then we can apply an Adem relation to write $\mathrm{Sq}^{i_j}\mathrm{Sq}^{i_{j+1}}$ as a linear combination of $\mathrm{Sq}^a\mathrm{Sq}^b$'s with $a \geq 2b$. Iterating this, we can write any word in the Steenrod squares as a linear combination of Sq^I's with $i_j \geq 2i_{j+1}$: such an Sq^I is called **admissible**.

Theorem 3.5.2. *$H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ is a polynomial ring over $\mathbb{Z}/2$ on the classes $\mathrm{Sq}^I\iota_n$ such that*

(i) Sq^I is admissible, and

*(ii) $I = (i_1, i_2, \ldots, i_r)$ has **excess** $e(I) := \sum_{j=1}^r (i_j - 2i_{j+1}) < n$.*

Example 3.5.3. Only $I = (0)$ has $e(I) = 0$. Only the admissible sequences

$$I = (1), (2,1), (4,2,1), (8,4,2,1), \ldots$$

have $e(I) = 1$. Thus

$$H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2) = \mathbb{Z}/2[\iota_2, \mathrm{Sq}^1\iota_2, \mathrm{Sq}^2\mathrm{Sq}^1\iota_2, \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1\iota_2, \ldots].$$

△

The proof of this theorem does not go beyond the methods of this course: it holds for $n = 1$ by observation, and can then be proved inductively using Kudo's transgression theorem and some spectral sequence yoga: in particular it does not use the Adem relations. To prove the Adem relations one can proceed as follows. As the Steenrod squares commute with the suspension isomorphism, it suffices to check the Adem relation for $\mathrm{Sq}^i\mathrm{Sq}^j$ on cohomology classes of degree $n \geq i + j$. From the theorem it is easy to check that the map

$$\iota_1 \times \cdots \times \iota_1 : \underbrace{K(\mathbb{Z}/2,1) \times \cdots \times K(\mathbb{Z}/2,1)}_{n \text{ times}} \longrightarrow K(\mathbb{Z}/2,n)$$

is injective on cohomology in degrees $\leq 2n$, and from this it is a matter of algebra to *verify* that the Adem relation for $\mathrm{Sq}^i\mathrm{Sq}^j\iota_n$ holds, as $n+i+j \leq 2n$ and we know how the Steenrod squares act on $H^*(K(\mathbb{Z}/2,1);\mathbb{Z}/2) = \mathbb{Z}/2[\iota_1]$. (Of course one has to get quite good at binomial coefficients modulo 2.)