Lent Term 2023 O. Randal-Williams

Part III Characteristic classes and K-theory // Example Sheet 4

Hand in work to questions marked $*$ to my pigeon hole at CMS by 09:00 on Wednesday 26th of April if you would like it marked.

1. On Example Sheet 2 Q1 you showed that if there is an n-dimensional real division algebra then the tangent bundle $T\mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$ is trivial, and using Stiefel–Whitney classes you showed that this may only happen if $n = 2^k$.

   By considering $[(T\mathbb{RP}^{n-1}) \otimes_{\mathbb{R}} \mathbb{C}] \in K^0(\mathbb{RP}^{n-1})$, show that in fact one must have $n = 1, 2, 4$, or 8.

2. $*$ Show that $ch_n(\psi^k(x)) = k^n \cdot ch_n(x)$. Hence show that the endomorphisms ψ^k of $K^0(X) \otimes \mathbb{Q}$ may be simultaneously diagonalised, and that the cohomology groups $H^{2n}(X;\mathbb{Q})$ may be recovered as eigenspaces of these endomorphisms.

3. Show that the action of the ψ^k on $\widetilde{K}^0(\mathbb{CP}^3/\mathbb{CP}^1)$ may be simultaneously diagonalised over $\mathbb{Z}$, but that their action on $\widetilde{K}^0(\mathbb{CP}^2)$ may not.

4. If $\pi : E \to X$ is a complex vector bundle of dimension k, show that $\Lambda_{\frac{t}{1-t}}(E - k) \in K^0(X)[[t]]$ is a polynomial (in t) of degree at most k.

   If there is an immersion $i : \mathbb{RP}^n \looparrowright \mathbb{R}^{n+k}$, by using the expansion $(\frac{1}{1+s})^{n+1} = \sum_{j=0}^{\infty}(-1)^j\binom{n+j}{j}s^j$ show that $2^{\lfloor n/2 \rfloor - j + 1}$ divides $\binom{n+j}{j}$ for all $j > k$. Investigate what this means for $n \leq 20$.

5. $*$ Show that the sphere bundle of $(\gamma_{\mathbb{C}}^{1,n+1})^{\otimes k} \to \mathbb{CP}^n$ is homeomorphic to the manifold $L_k^{2n+1} = S^{2n+1}/(\mathbb{Z}/k)$, where $\mathbb{Z}/k$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ as the k roots of unity.

   Hence show that $K^{-1}(L_k^{2n+1}) \cong \mathbb{Z}$ and that $\tilde{K}^0(L_k^{2n+1})[\frac{1}{k}] = 0$.

   Show that $\tilde{K}^0(L_k^5)$ is $\mathbb{Z}/k \oplus \mathbb{Z}/k$ if k is odd, and is $\mathbb{Z}/(k/2) \oplus \mathbb{Z}/(2k)$ if k is even.

6. Show that the normal bundle of $\mathbb{FP}^n \subset \mathbb{FP}^{n+k}$ is $(\gamma_{\mathbb{F}}^{1,n+1})^{\oplus k}$, and that there is an open tubular neighbourhood of $\mathbb{FP}^n$ whose complement deformation retracts to $\mathbb{FP}^{k-1} \subset \mathbb{FP}^{n+k}$. Hence show that there is a homotopy equivalence $Th((\gamma_{\mathbb{F}}^{1,n+1})^{\oplus k} \to \mathbb{FP}^n) \simeq \mathbb{FP}_k^{n+k} := \mathbb{FP}^{n+k}/\mathbb{FP}^{k-1}$.

7. $*$ Compute the K-theory of $\mathbb{RP}_k^{n+k}$, including the action of the Adams operations.

8. For $\nu := [(\gamma_{\mathbb{R}}^{1,n+1}) \otimes \mathbb{C}] - 1 \in K^0(\mathbb{RP}^n)$ show that

$$\rho^\ell(\nu) = \begin{cases} \frac{\ell-1}{2}\nu & \text{if } \ell \text{ is odd} \\ \frac{\ell}{2}\nu & \text{if } \ell \text{ is even} \end{cases} \in K^0(\mathbb{RP}^n).$$

9. If $\pi : E \to X$ is a d-dimensional real vector bundle, show that the mapping cone of $p : \mathbb{S}(E) \to X$ is homeomorphic to the Thom space $Th(E)$. If $\pi' : E' \to X$ is another such vector bundle and there is a map $f : \mathbb{S}(E) \to \mathbb{S}(E')$ which commutes with the projections to X and is a homotopy equivalence, show that there is a map $\phi : Th(E) \to Th(E')$ which induces an isomorphism on K-theory. If they are complex vector bundles show furthermore that $\phi^*(\lambda_{E'}) = U \cdot \lambda_E$ for some unit $U \in K^0(X)$, and hence that their cannibalistic classes satisfy $\rho^k(E') = \frac{\psi^k(U)}{U}\rho^k(E)$ for each k.

10. Using Q8 and Q9 show that if the real vector bundle $(\gamma_{\mathbb{R}}^{1,n+1})^{\oplus k} \to \mathbb{RP}^n$ is trivial then k is even *[use Stiefel–Whitney classes]*, and $(\ell + \frac{\ell-1}{2}\nu)^{k/2} = \ell^{k/2} \in K^0(\mathbb{RP}^n)$ for all odd $\ell \in \mathbb{N}$. Deduce from this that $2^{\lfloor n/2 \rfloor + 1}$ divides $\ell^{k/2} - 1$ for all odd ℓ.

Additional Questions

11. (i) If $f : M \to N$ is a map of closed manifolds equipped with a complex orientation, show that the Gysin map $f_!^K : K^0(M) \to K^0(N)$ satisfies $f_!^K(f^*(x) \cdot y) = x \cdot f_!^K(y)$.

    (ii) If $\pi : E \to N$ is a complex vector bundle over a smooth manifold, $s : N \to E$ is a smooth section transverse to the zero section, and $M := s^{-1}(0)$, show that the inclusion $i : M \to N$ has a complex orientation, and that $i_!^K(1) = e^K(E) \in K^0(N)$.

    (iii) When k is even show that the inclusion $i : \mathbb{RP}^n \to \mathbb{RP}^{n+k}$ has a complex orientation. Determine the map $i_!^K : K^0(\mathbb{RP}^n) \to K^0(\mathbb{RP}^{n+k})$.

12. If a compact 8-manifold M is given a complex structure on its tangent bundle TM, with Chern classes $c_i = c_i(TM)$, show that the integers

$$\langle [M], c_1^4 - 4c_1^2 c_2 + 4c_2^2 \rangle \qquad \langle [M], c_2^2 - 2c_1 c_3 \rangle$$

are independent of the choice of complex structure, and that

$$\begin{aligned} \langle [M], -c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4 \rangle &\equiv 0 \mod 720 \\ \langle [M], 2c_1^4 + c_1^2 c_2 \rangle &\equiv 0 \mod 12 \\ \langle [M], c_1 c_3 - 2c_4 \rangle &\equiv 0 \mod 4 \end{aligned}$$

[You should certainly use a computer to do the second part, and don't expect to easily get the full answer.]

Very Additional Question

13. Taking inspiration from Example Sheet 2 Q10, calculate $K^*(Gr_2(\mathbb{C}^4))$ as a ring. *[This is really very involved, especially getting the multiplicative structure. Probably you will need to use something like* `Sage` *to do some commutative algebra calculations.]*

Comments or corrections to `or257@cam.ac.uk`