Lent Term 2023 O. Randal-Williams

Part III Characteristic classes and K-theory // Example Sheet 3

Hand in work to questions marked $*$ to my pigeon hole at CMS by 09:00 on Tuesday 14th March if you would like it marked.

1. If $\pi : E \to X$ is a vector bundle over a compact Hausdorff space, show there is a finite cover of X by *closed* sets $A_1, \ldots, A_n$ over each of which E is trivial. Hence, elaborating on Example 3.3.7, show that every element of $\tilde{K}^0(X)$ is nilpotent.

2. If X is a compact Hausdorff space, show that

$$K^{-1}(X) \cong \{\text{maps } X \to GL_\infty(\mathbb{C})\}/\text{homotopy}$$
$$K^0(X) \cong \{\text{maps } X \to \mathbb{Z} \times Gr_\infty(\mathbb{C}^\infty)\}/\text{homotopy}$$

where $GL_\infty(\mathbb{F})$ is given by an appropriate union of the $GL_n(\mathbb{C})$'s and $Gr_\infty(\mathbb{C}^\infty)$ is given by an appropriate union of the $Gr_n(\mathbb{C}^\infty)$'s. *[There is a point-set topological subtlety that you should at least identify, and ideally resolve.]*

3. If Y is a finite CW complex only having cells of even dimension, show that

$$K^0(Y) \cong \mathbb{Z}^{\#\text{cells of } Y} \quad \text{and} \quad K^{-1}(Y) = 0.$$

Hence show that for any X the external product $- \boxtimes - : K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$ is an isomorphism. *[Proceed by induction on the number of cells of Y.]*

4. $*$ Show that defining $c_i(E - F)$ by $c(E - F) = \frac{c(E)}{c(F)}$ gives well-defined (nonlinear!) functions $c_i : K^0(X) \to H^{2i}(X; \mathbb{Z})$. Using this, compute the ring structure on $K^0(\mathbb{CP}^2)$. *[You should use the splitting principle to find a formula for $c_1(E \otimes F)$ and $c_2(E \otimes F)$.]*

   Hence compute the ring structure of $K^0(\mathbb{CP}^2 \# \mathbb{CP}^2)$ and of $K^0(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)$, and show they are not isomorphic as rings.

5. $*$ If $p : Y \to X$ is an n-fold covering space and $\pi : E \to Y$ is a vector bundle, show that there is a vector bundle $F \to X$ with $F_x = \bigoplus_{y \in p^{-1}(x)} E_y$. Show that this construction induces a homomorphism

$$p_! : K^0(Y) \longrightarrow K^0(X)$$

   and that this satisfies $p_!(p^*(x) \cdot y) = x \cdot p_!(y)$.

   Give an example for which $p_!(1) \neq n \in K^0(X)$. Nonetheless, using Q1 show that $p_!(1) \in K^0(X)$ becomes invertible in $K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}]$ and hence show that $p^* : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}] \to K^0(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}]$ is split injective.

6. Show that two n-dimensional complex vector bundles over $\mathbb{CP}^n$ having the same Chern classes are isomorphic.

7. (i) $*$ By considering $p_1(TS^4) \in H^4(S^4; \mathbb{Z})$, show that the vector bundle $TS^4 \to S^4$ does not admit a complex structure.

   (ii) $*$ By considering $ch_n(TS^{2n}) \in H^{2n}(S^{2n}; \mathbb{Q})$, show that the vector bundle $TS^{2n} \to S^{2n}$ does not admit a complex structure for $n \geq 4$.

   [Recall that if $\pi : E \to B$ is an n-dimensional complex vector bundle, then it is $\mathbb{Z}$-oriented and $c_n(E) = e(E) \in H^{2n}(B; \mathbb{Z})$.]

8. Write $Q = \gamma_{\mathbb{H}}^{1,n+1} \to \mathbb{HP}^n$ for the tautological quaternionic line bundle, and let $z = e(Q) \in H^4(\mathbb{HP}^n;\mathbb{Z})$. Show that $H^*(\mathbb{HP}^n;\mathbb{Z}) = \mathbb{Z}[z]/(z^{n+1})$. By analysing a suitable map $f : \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ show that $ch(Q) = 2\cosh(\sqrt{-z})$, and hence show that $K^0(\mathbb{HP}^n) = \mathbb{Z}[Q]/((Q-2)^{n+1})$.

9. The isomorphism $ch : K^0(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{2*}(X;\mathbb{Q})$ means that the vector space $H^{2*}(X;\mathbb{Q})$ contains two canonical integral lattices: $H^{2*}(X;\mathbb{Z})/\text{torsion}$ and $ch(K^0(X))$. Give an example to show that these need not be equal.

Additional Questions

10. Complex conjugation $E \mapsto \overline{E}$ induces an involution on each $K^i(X)$. Show that $K^i(X) \otimes \mathbb{Z}[\frac{1}{2}]$ decomposes into ± 1 eigenspaces for this involution, and that the $+1$ eigenspace of $K^0(X) \otimes \mathbb{Z}[\frac{1}{2}]$ agrees with

$$\{\text{Grothendieck group of } \textit{real} \text{ vector bundles on } X\} \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Use these eigenspaces to define a new 4-periodic theory $T^*(-)$, and establish a 12-term exact cycle relating $T^*(X)$, $T^*(A)$, and $\widetilde{T}^*(X/A)$ when A is a closed subspace of a compact Hausdorff space X.

11. Revisit Q4 using the Chern character.

Comments or corrections to `or257@cam.ac.uk`