Lent Term 2023 O. Randal-Williams

Part III Characteristic classes and K-theory // Example Sheet 1

Hand in work to questions marked $*$ to my pigeon hole at CMS by 09:00 on Wednesday 8th February if you would like it marked.

1. (i) If $\pi : E \to X$ and $\pi' : E' \to X$ are vector bundles, show that there is an isomorphism $Hom(E, E') \cong E^\vee \otimes E'$ of vector bundles over X.

   (ii) If $\pi : E \to X$ is a real vector bundle, show that the underlying real vector bundle of the complex vector bundle $E \otimes_\mathbb{R} \mathbb{C}$ is isomorphic to $E \oplus E$.

   (iii) Show that a complex vector bundle $\pi : E \to X$ is the complexification of a real vector bundle if and only if there is an isomorphism $\phi : E \to \overline{E}$ of complex vector bundles such that $\overline{\phi} \circ \phi = Id_E$.

   (iv) If $\pi : E \to X$ is a real vector bundle, show that it is the realification of a complex vector bundle if and only if there is a bundle map $J : E \to E$ satisfying $J^2 = -\mathrm{Id}$.

   (v) If a vector bundle $\pi : E \to X$ has an inner product and $E_0 \subset E$ is a subbundle, show that $E_0^\perp \subset E$ is also a subbundle.

   (vi) If $\pi : E \to X$ is a vector bundle and $E_0 \subset E$ is a subbundle, construct a vector bundle E/E_0 whose fibre over $x \in X$ is $E_x/(E_0 \cap E_x)$. If E is given an inner product show that $E/E_0 \cong E_0^\perp$.

2. If $\pi : E \to X$ is a $\mathbb{Z}$-oriented real vector bundle, and $-E$ denotes E with the opposite orientation, show that the Euler class satisfies $e(-E) = -e(E)$. If $\dim E$ is odd show that $2e(E) = 0$.

3. * Show that a real line bundle $\pi : L \to X$ is trivial if and only if $w_1(L) = 0 \in H^1(X; \mathbb{F}_2)$. Hence show that a real vector bundle $\pi : E \to X$ is orientable if and only if $w_1(E) = 0 \in H^1(X; \mathbb{F}_2)$. [*Hint: Associate a determinant line bundle* $\det E \to X$*, which is trivial if and only if* E *is orientable.*]

4. * If $\pi : E \to X$ is a complex vector bundle and $\pi_\mathbb{R} : E_\mathbb{R} \to X$ denotes its underlying real vector bundle, show that
$$w(E_\mathbb{R}) = c(E) \in H^*(X; \mathbb{F}_2)$$
and that
$$p_k(E_\mathbb{R}) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \cdots \pm 2c_1(E)c_{2k-1}(E) \mp 2c_{2k}(E) \in H^{4k}(X; R).$$

5. Let $\pi : E \to X$ be a real vector bundle.

   (i) Show that $p_i(E) = w_{2i}(E)^2 \in H^{4i}(X; \mathbb{F}_2)$.

   (ii) If $\pi : E \to X$ is oriented and of dimension $2k$, show that $p_k(E) = e(E)^2 \in H^{4k}(X; \mathbb{Z})$.

6. If $\pi : E \to X$ is a real vector bundle, show that $2c_{2i+1}(E \otimes_\mathbb{R} \mathbb{C}) = 0 \in H^{4i+2}(X; \mathbb{Z})$ for any $i \geq 0$. Hence show that if $\pi' : E' \to X$ is another real vector bundle then
$$2\left(p_k(E \oplus E') - \sum_{a+b=k} p_a(E) \cdot p_b(E')\right) = 0 \in H^{4k}(X; R).$$

7. * Recall from Algebraic Topology that $H^*(\mathbb{RP}^{2n}; \mathbb{Z}) = \mathbb{Z}[t]/(2t, t^{n+1})$, with $t \in H^2(\mathbb{RP}^{2n}; \mathbb{Z})$. For the tautological bundle $\gamma_\mathbb{R}^{1,n+1} \to \mathbb{RP}^{2n}$, prove that
$$c_1(\gamma_\mathbb{R}^{1,n+1} \otimes_\mathbb{R} \mathbb{C}) = t \in H^2(\mathbb{RP}^{2n}; \mathbb{Z}) = \mathbb{Z}/2\{t\}.$$
[*Hint: Use Q4 and reduction modulo 2.*] Use this to show that the identity in the previous question does not hold without the "2".

8. If a collection of characteristic classes $\{\pi : E \to X\} \mapsto c_i'(E) \in H^{2i}(X; R)$ of complex vector bundles satisfy the properties of Theorem 2.3.2 in the notes, show that they are equal to the Chern classes up to a scalar factor.

9. Show that there is no map $f : \mathbb{RP}^n \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $f(\ell) \in \ell^\perp$ for each line $\ell \in \mathbb{RP}^n$.

Additional Questions

10. This question leads you through the proof of the Constant Rank Theorem: If $\pi_i : E_i \to X$, $i = 1, 2$, are vector bundles and $f : E_1 \to E_2$ is a morphism of vector bundles such that the rank of the linear map $f_x : (E_1)_x \to (E_2)_x$ is independent of $x \in X$, then

$$\begin{aligned} Ker(f) &:= \{v \in E_1 \,|\, f(v) = 0 \in (E_2)_{\pi_2(v)}\} \\ Im(f) &:= \{w \in E_2 \,|\, w = f(v), v \in E_1\} \end{aligned}$$

are subbundles of E_1 and E_2 respectively.

(i) Prove that it is enough to consider the case where both vector bundles are trivial, so the morphism has the form $f : X \times \mathbb{F}^n \to X \times \mathbb{F}^m$ with $f(x, v) = (x, \phi(x)(v))$ where $\phi : X \to M_{n,m}(\mathbb{F})$ is a map taking values of constant rank r.

(ii) For a fixed $y \in X$ show that with respect to $\mathbb{F}^n = \mathbb{F}^r \oplus \mathbb{F}^{n-r}$ and $\mathbb{F}^m = \mathbb{F}^r \oplus \mathbb{F}^{m-r}$ one may suppose that

$$\phi(x) = \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix}$$

with $A(y) = I_r$, $B(y) = C(y) = D(y) = 0$. Hence show that $A(x)$ is invertible for all x in some open neighbourhood U of y.

(iii) Deduce that the composition $Ker(f)|_U \subset U \times \mathbb{F}^n \stackrel{proj}{\to} U \times \mathbb{F}^{n-r}$ is a continuous bijection and that the composition $U \times \mathbb{F}^r \subset U \times \mathbb{F}^n \stackrel{f}{\to} Im(f)|_U$ is a continuous bijection.

(iv) By relating kernels and images of f and its adjoint f^*, with respect to the standard inner products on $\mathbb{F}^n$ and $\mathbb{F}^m$, show that the compositions in (iii) are both homeomorphisms.

11. If A^a and B^b are smooth submanifolds of M^m meeting transversely in a manifold $N^n = A \cap B$, and ν_X denotes the normal bundle of X in M for $X \in \{A, B, N\}$, show (using Q10) that $\nu_N \cong \nu_A \oplus \nu_B$.

If $\pi : E^{x+d} \to X^x$ is a map between smooth manifolds which also has the structure of a d-dimensional real vector bundle, and $s : X \to E$ is a smooth section which is transverse to the zero section, show that the normal bundle in X of the $(x-d)$-dimensional manifold $Z = s^{-1}(0)$ is isomorphic to $E|_Z$. If X is a closed compact manifold and everything in sight is R-oriented, show that the Poincaré dual of $[Z] \in H_{x-d}(X; R)$ is $e(E) \in H^d(X; R)$. [*Recall that the Poincaré dual to a submanifold is given by the Thom class of its normal bundle, extended by zero.*]

12. Show that a degree d homogeneous polynomial $p(z_0, \dots, z_n)$ defines a section of the complex vector bundle $((\gamma_{\mathbb{C}}^{1,n+1})^\vee)^{\otimes d} \to \mathbb{CP}^n$. Assuming this section is transverse to the zero section, show that the subset $Z \subset \mathbb{CP}^n$ of solutions to $p(z) = 0$ is a manifold of dimension $2(n-1)$, has a canonical orientation, and is Poincaré dual to $d \cdot (-x)$. Using Chern classes show that its Euler characteristic is $(-1)^{n+1}d$ times the coefficient of x^{n-1} in $\frac{(1-x)^{n+1}}{1-d\cdot x}$.

It is a fact that Z must be connected: when $n = 2$ show that Z is a surface of genus $\frac{1}{2}(d-1)(d-2)$.

Comments or corrections to `or257@cam.ac.uk`