Lent Term 2018 O. Randal-Williams

Part III Characteristic classes and K-theory // Example Sheet 3

Hand in work to questions marked $*$ to my pigeon hole at CMS by 09:00 on Thursday 26 April if you would like it marked.

1. (i) * By considering $p_1(TS^4) \in H^4(S^4;\mathbb{Z})$, show that the vector bundle $TS^4 \to S^4$ does not admit a complex structure.

   (ii) * By considering $ch_n(TS^{2n}) \in H^{2n}(S^{2n};\mathbb{Q})$, show that the vector bundle $TS^{2n} \to S^{2n}$ does not admit a complex structure for $n \geq 4$.

   [Recall that if $\pi : E \to B$ is an n-dimensional complex vector bundle, then it is $\mathbb{Z}$-oriented and $c_n(E) = e(E) \in H^{2n}(B;\mathbb{Z})$.]

2. Write $Q = \gamma_{\mathbb{H}}^{1,n+1} \to \mathbb{HP}^n$ for the tautological quaternionic line bundle, and let $z = e(Q) \in H^4(\mathbb{HP}^n;\mathbb{Z})$. Show that $H^*(\mathbb{HP}^n;\mathbb{Z}) = \mathbb{Z}[z]/(z^{n+1})$. By analysing a suitable map $f : \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ show that $ch(Q) = 2\cosh(\sqrt{-z})$, and hence show that $K^0(\mathbb{HP}^n) = \mathbb{Z}[Q]/((Q-2)^{n+1})$.

3. On Q11 of Sheet 1 you showed that if there is an n-dimensional real division algebra then the tangent bundle $T\mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$ is trivial, and using Stiefel–Whitney classes you showed that this may only happen if $n = 2^k$.

   By considering $[(T\mathbb{RP}^{n-1}) \otimes_{\mathbb{R}} \mathbb{C}] \in K^0(\mathbb{RP}^{n-1})$, show that in fact one must have $n = 1, 2, 4,$ or 8.

4. * Show that $ch_n(\psi^k(x)) = k^n \cdot ch_n(x)$. Hence show that the endomorphisms ψ^k of $K^0(X)\otimes\mathbb{Q}$ may be simultaneously diagonalised, and that the cohomology groups $H^{2n}(X;\mathbb{Q})$ may be recovered as eigenspaces of these endomorphisms.

5. Show that the action of the ψ^k on $\widetilde{K}^0(\mathbb{CP}^3/\mathbb{CP}^1)$ may be simultaneously diagonalised over $\mathbb{Z}$, but that the action on $\widetilde{K}^0(\mathbb{CP}^2)$ may not.

6. If $\pi : E \to X$ is a complex vector bundle of dimension k, show that $\Lambda_{\frac{t}{1-t}}(E-k) \in K^0(X)[[t]]$ is a polynomial (in t) of degree at most k.

   If there is an immersion $i : \mathbb{RP}^n \looparrowright \mathbb{R}^{n+k}$, by using the expansion $(\frac{1}{1+s})^{n+1} = \sum_{j=0}^{\infty}(-1)^j\binom{n+j}{j}s^j$ show that $2^{\lfloor n/2\rfloor - j+1}$ divides $\binom{n+j}{j}$ for all $j > k$. Investigate what this means for $n \leq 20$.

7. Show that the sphere bundle of $(\gamma_{\mathbb{C}}^{1,n+1})^{\otimes k} \to \mathbb{CP}^n$ is homeomorphic to the manifold $L_k^{2n+1} = S^{2n+1}/(\mathbb{Z}/k)$, where $\mathbb{Z}/k$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ as the k roots of unity.

   Hence show that $K^{-1}(L_k^{2n+1}) \cong \mathbb{Z}$ and that $\tilde{K}^0(L_k^{2n+1})[\frac{1}{k}] = 0$.

   Show that $\tilde{K}^0(L_k^5)$ is $\mathbb{Z}/k \oplus \mathbb{Z}/k$ if k is odd, and is $\mathbb{Z}/(k/2) \oplus \mathbb{Z}/(2k)$ if k is even.

8. Show that the normal bundle of $\mathbb{FP}^n \subset \mathbb{FP}^{n+k}$ is $(\gamma_{\mathbb{F}}^{1,n+1})^{\oplus k}$, and that there is an open tubular neighbourhood of $\mathbb{FP}^n$ whose complement deformation retracts to $\mathbb{FP}^{k-1} \subset \mathbb{FP}^{n+k}$. Hence show that there is a homotopy equivalence $Th((\gamma_{\mathbb{F}}^{1,n+1})^{\oplus k} \to \mathbb{FP}^n) \simeq \mathbb{FP}_k^{n+k} := \mathbb{FP}^{n+k}/\mathbb{FP}^{k-1}$.

9. * Compute the K-theory of $\mathbb{RP}_k^{n+k}$, including the action of the Adams operations.

10. For $L = (\gamma_{\mathbb{R}}^{1,n+1}) \otimes \mathbb{C} \to \mathbb{RP}^n$ show that

$$\rho^\ell(L) = \begin{cases} \ell + \frac{\ell-1}{2}x & \text{if } \ell \text{ is odd} \\ \ell + \frac{\ell}{2}x & \text{if } \ell \text{ is even} \end{cases} \in K^0(\mathbb{RP}^n).$$

Additional Questions

11. If $\pi : E \to X$ is a d-dimensional real vector bundle, show that the mapping cone of $p : \mathbb{S}(E) \to X$ is homeomorphic to the Thom space $Th(E)$. If $\pi' : E' \to X$ is another such vector bundle and there is a map $f : \mathbb{S}(E) \to \mathbb{S}(E')$ which commutes with the projections to X and is a homotopy equivalence, show that there is a map $\phi : Th(E) \to Th(E')$ which induces an isomorphism on K-theory. If they are complex vector bundles show furthermore that $\phi^*(\lambda_{E'}) = U \cdot \lambda_E$ for some unit $U \in K^0(X)$, and hence that their cannibalistic classes satisfy $\rho^k(E') = \frac{\psi^k(U)}{U}\rho^k(E)$.

12. Using Q10 and Q11 show that if the real vector bundle $(\gamma_{\mathbb{R}}^{1,n+1})^{\oplus k} \to \mathbb{RP}^n$ is trivial then k is even *[use Stiefel–Whitney classes]*, and $(\ell + \frac{\ell-1}{2}x)^{k/2} = \ell^{k/2}$ for all odd $\ell \in \mathbb{N}$. Deduce from this that $2^{\lfloor n/2 \rfloor + 1}$ divides $\ell^{k/2} - 1$ for all odd ℓ.

13. If a compact 8-manifold M is given a complex structure on its tangent bundle TM, with Chern classes $c_i = c_i(TM)$, show that the integers

$$\langle [M], c_1^4 - 4c_1^2c_2 + 4c_2^2 \rangle \qquad \langle [M], c_2^2 - 2c_1c_3 \rangle$$

are independent of the choice of complex structure, and that

$$\begin{aligned}\langle [M], -c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4 \rangle &\equiv 0 \mod 720\\ \langle [M], 2c_1^4 + c_1^2c_2 \rangle &\equiv 0 \mod 12\\ \langle [M], c_1c_3 - 2c_4 \rangle &\equiv 0 \mod 4\end{aligned}$$

[You should certainly use a computer to do the second part, and don't expect to easily get the full answer.]

Comments or corrections to `or257@cam.ac.uk`