Part III Characteristic classes and K-theory // Example Sheet 2

Hand in work to questions marked $*$ to my pigeon hole at CMS by 09:00 on Thursday 15 March if you would like it marked.

1. If $\pi : E \to X$ is a d-dimensional complex vector bundle over a finite CW-complex of dimension n, show that if $n < 2d$ then it has a nowhere vanishing section. *[Go by induction over cells.]*

   Similarly, if $\pi_i : E_i \to X$, $i = 1, 2$, are d-dimensional complex vector bundles over a finite CW-complex of dimension n and $E_1 \oplus \underline{\mathbb{C}^1} \cong E_2 \oplus \underline{\mathbb{C}^1}$, show that if $n + 1 < 2(d + 1)$ then $E_1 \cong E_2$. *[Translate from isomorphisms to vector bundles over $X \times [0, 1]$.]*

2. * Using Q1 and the clutching description of vector bundles, compute $K^0(S^2)$.

3. * Compute $K^*(S^1 \times S^1)$ and $K^*(\mathbb{RP}^2)$ as abelian groups, and hence compute $K^*(S)$ for every compact closed surface S.

4. Compute the graded ring structure on $K^*(S^1 \times S^1)$.

5. * If $\pi : E \to X$ is a vector bundle over a compact Hausdorff space, show there is a finite cover of X by *closed* sets $A_1, \ldots, A_n$ over each of which E is trivial. Hence, elaborating on Example 3.3.7, show that every element of $\tilde{K}^0(X)$ is nilpotent.

6. If Y is a finite CW complex only having cells of even dimension, show that
$$K^0(Y) \cong \mathbb{Z}^{\#\text{cells of } Y} \quad \text{and} \quad K^{-1}(Y) = 0.$$
   Hence show that for any X the external product $- \boxtimes - : K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$ is an isomorphism. *[Proceed by induction on the number of cells of Y.]*

7. Show that defining $c_i(E - F)$ by $c(E - F) = \frac{c(E)}{c(F)}$ gives well-defined (nonlinear!) functions $c_i : K^0(X) \to H^{2i}(X; \mathbb{Z})$. Using this, compute the ring structure on $K^0(\mathbb{CP}^2)$. *[You should use the splitting principle to find a formula for $c_1(E \otimes F)$ and $c_2(E \otimes F)$.]*

   Hence compute the ring structure of $K^0(\mathbb{CP}^2 \# \mathbb{CP}^2)$ and of $K^0(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)$, and show they are not isomorphic as rings.

8. * If $p : Y \to X$ is an n-fold covering space and $\pi : E \to Y$ is a vector bundle, show that there is a vector bundle $F \to X$ with $F_x = \bigoplus_{y \in p^{-1}(x)} E_y$. Show that this construction induces a homomorphism
$$p_! : K^0(Y) \longrightarrow K^0(X)$$
   and that this satisfies $p_!(p^*(x) \cdot y) = x \cdot p_!(y)$.

   Give an example for which $p_!(1) \neq n \in K^0(X)$. Nonetheless, using Q3 show that $p_!(1) \in K^0(X)$ becomes invertible in $K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}]$ and hence show that $p^* : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}] \to K^0(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}]$ is split injective.

Additional Questions

9. This question leads you through the proof of the *Constant Rank Theorem*: If $\pi_i : E_i \to X$, $i = 1, 2$, are vector bundles and $f : E_1 \to E_2$ is a morphism of vector bundles such that the rank of the linear map $f_x : (E_1)_x \to (E_2)_x$ is independent of $x \in X$, then
$$\begin{aligned} Ker(f) &:= \{v \in E_1 \mid f(v) = 0 \in (E_2)_{\pi_2(v)}\} \\ Im(f) &:= \{w \in E_2 \mid w = f(v), v \in E_1\} \end{aligned}$$
   are subbundles of E_1 and E_2 respectively.

(i) Prove that it is enough to consider the case where both vector bundles are trivial, so the morphism has the form $f : X \times \mathbb{F}^n \to X \times \mathbb{F}^m$ with $f(x, v) = (x, \phi(x)(v))$ where $\phi : X \to M_{n,m}(\mathbb{F})$ is a map taking values of constant rank r.

(ii) For a fixed $y \in X$ show that with respect to $\mathbb{F}^n = \mathbb{F}^r \oplus \mathbb{F}^{n-r}$ and $\mathbb{F}^m = \mathbb{F}^r \oplus \mathbb{F}^{m-r}$ one may suppose that

$$\phi(x) = \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix}$$

with $A(y) = I_r$, $B(y) = C(y) = D(y) = 0$. Hence show that $A(x)$ is invertible for all x in some open neighbourhood U of y.

(iii) Deduce that the composition $Ker(f)|_U \subset U \times \mathbb{F}^n \overset{proj}{\to} U \times \mathbb{F}^{n-r}$ is a continuous bijection and that the composition $U \times \mathbb{F}^r \subset U \times \mathbb{F}^n \overset{f}{\to} Im(f)|_U$ is a continuous bijection.

(iv) By relating kernels and images of f and its adjoint f^*, with respect to the standard inner products on $\mathbb{F}^n$ and $\mathbb{F}^m$, show that the compositions in (iii) are both homeomorphisms.

10. If A^a and B^b are smooth submanifolds of M^m meeting transversely in a manifold $N^n = A \cap B$, and for $X \in \{A, B, N\}$ denote by ν_X the normal bundle of X in M, show (using the previous question) that $\nu_N \cong \nu_A \oplus \nu_B$.

If $\pi : E^{x+d} \to X^x$ is a map between smooth manifolds which also has the structure of a d-dimensional real vector bundle, and $s : X \to E$ is a smooth section which is transverse to the zero section, show that the normal bundle in X of the $(x - d)$-dimensional manifold $Z = s^{-1}(s_0(X))$ is isomorphic to $E|_Z$. If X is a closed compact manifold and everything in sight is R-oriented, show that the Poincaré dual of $[Z] \in H_{x-d}(X; R)$ is $e(E) \in H^d(X; R)$. *[Recall that the Poincaré dual to a submanifold is given by the Thom class of its normal bundle, extended by zero]*

11. Show that a degree d homogeneous polynomial $p(z_0, \ldots, z_n)$ defines a section of the complex vector bundle $((\gamma_{\mathbb{C}}^{1,n+1})^\vee)^{\otimes d} \to \mathbb{CP}^n$. Assuming this section is transverse to the zero section, show that the subset $Z \subset \mathbb{CP}^n$ of solutions to $p(z) = 0$ is a manifold of dimension $2(n-1)$, has a canonical orientation, and is Poincaré dual to $d \cdot (-x)$. Using Chern classes show that its Euler characteristic is $(-1)^{n+1} d$ times the coefficient of x^{n-1} in $\frac{(1-x)^{n+1}}{1-d \cdot x}$.

It is a fact that Z must be connected: when $n = 2$ show that it is a surface of genus $\frac{1}{2}(d-1)(d-2)$.

Comments or corrections to `or257@cam.ac.uk`