Lent Term 2018 O. Randal-Williams

Part III Characteristic classes and K-theory // Example Sheet 1

Hand in work to questions marked $*$ to my pigeon hole at CMS by 09:00 on 5th February if you would like it marked.

1. (i) If $\pi : E \to X$ and $\pi' : E' \to X$ are vector bundles, show that there is an isomorphism $Hom(E, E') \cong E^\vee \otimes E'$ of vector bundles over X.
   (ii) If $\pi : E \to X$ is a real vector bundle, show that the underlying real vector bundle of the complex vector bundle $E \otimes_\mathbb{R} \mathbb{C}$ is isomorphic to $E \oplus E$.
   (iii) Show that a complex vector bundle $\pi : E \to X$ is the complexification of a real vector bundle if and only if there is an isomorphism $\phi : E \to \overline{E}$ of complex vector bundles such that $\overline{\phi} \circ \phi = Id_E$.
   (iv) If $\pi : E \to X$ is a real vector bundle, show that it is the realification of a complex vector bundle if and only if there is a bundle map $J : E \to E$ satisfying $J^2 = -\mathrm{Id}$.
   (v) If a vector bundle $\pi : E \to X$ has an inner product and $E_0 \subset E$ is a subbundle, show that $E_0^\perp \subset E$ is also a subbundle.
   (vi) If $\pi : E \to X$ is a vector bundle and $E_0 \subset E$ is a subbundle, construct a vector bundle E/E_0 whose fibre over $x \in X$ is $E_x/(E_0 \cap E_x)$. If E is given an inner product show that $E/E_0 \cong E_0^\perp$.

2. If $\phi_1, \phi_2 : X \to Gr_n(\mathbb{F}^N)$ are maps such that $\phi_1^* \gamma_\mathbb{F}^{n,N}$ and $\phi_2^* \gamma_\mathbb{F}^{n,N}$ are isomorphic, show that ϕ_1 and ϕ_2 are homotopic after composing with the inclusion $i : Gr_n(\mathbb{F}^N) \to Gr_n(\mathbb{F}^{2N})$ using the first N coordinates. *[Hint: First show that i is homotopic to the inclusion i' using the last N coordinates, then show that $i \circ \phi_1 \simeq i' \circ \phi_2$.]*

3. If $\pi : E \to X$ is a $\mathbb{Z}$-oriented real vector bundle, and $-E$ denotes E with the opposite orientation, show that the Euler class satisfies $e(-E) = -e(E)$. If $\dim E$ is odd show that $2e(E) = 0$.

4. * Show that a real line bundle $\pi : L \to X$ is trivial if and only if $w_1(L) = 0 \in H^1(X; \mathbb{F}_2)$. Hence show that a real vector bundle $\pi : E \to X$ is orientable if and only if $w_1(E) = 0 \in H^1(X; \mathbb{F}_2)$. *[Hint: Associate a determinant line bundle $\det E \to X$, which is trivial if and only if E is orientable.]*

5. * If $\pi : E \to X$ is a complex vector bundle and $\pi_\mathbb{R} : E_\mathbb{R} \to X$ denotes its underlying real vector bundle, show that
$$w(E_\mathbb{R}) = c(E) \in H^*(X; \mathbb{F}_2)$$
and that
$$p_k(E_\mathbb{R}) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \cdots \pm 2c_1(E)c_{2k-1}(E) \mp c_{2k}(E) \in H^{4k}(X; R).$$

6. If $\pi : E \to X$ is a real vector bundle, show that
$$p_i(E) = w_{2i}(E)^2 \in H^{4i}(X; \mathbb{F}_2).$$

7. If $\pi : E \to X$ is a real vector bundle, show that $2c_{2i+1}(E \otimes_\mathbb{R} \mathbb{C}) = 0 \in H^{4i+2}(X; \mathbb{Z})$ for any $i \geq 0$. Hence show that if $\pi' : E' \to X$ is another real vector bundle then
$$2\left(p_k(E \oplus E') - \sum_{a+b=k} p_a(E) \cdot p_b(E')\right) = 0 \in H^{4k}(X; R).$$

8. * Recall from Algebraic Topology that $H^*(\mathbb{RP}^{2n}; \mathbb{Z}) = \mathbb{Z}[t]/(2t, t^{n+1})$, with $t \in H^2(\mathbb{RP}^{2n}; \mathbb{Z})$.
For the bundle $\gamma^1_{\mathbb{R}} \to \mathbb{RP}^{2n}$, prove that

$$c_1(\gamma^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = t \in H^2(\mathbb{RP}^{2n}; \mathbb{Z}) = \mathbb{Z}/2\{t\}.$$

[Hint: Use Q5 and reduction modulo 2.] Use this to show that the identity in the previous question does not hold without the "2".

9. If a collection of characteristic classes $\{\pi : E \to X\} \mapsto c'_i(E) \in H^{2i}(X; R)$ of complex vector bundles satisfy the properties of Theorem 2.3.2 in the notes, show that they are equal to the Chern classes up to a scalar factor.

10. Show that there is no map $f : \mathbb{RP}^n \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $f(\ell) \in \ell^\perp$ for each line $\ell \in \mathbb{RP}^n$.

11. A *division algebra* structure on a finite-dimensional real vector space V is a map $\cdot : V \otimes V \to V$ such that $- \cdot x : V \to V$ is an isomorphism for all $x \neq 0$. Given such a structure, show that the tangent bundle of $\mathbb{P}(V)$ is trivial as follows:

   (i) Choose a basis $e_1, \ldots, e_n$ of V, and define isomorphisms $v_i : V \to V$ intrinsically by $v_i(x \cdot e_1) = x \cdot e_i$. Show that $v_1(x) = x$ and that the $v_i(x)$ are linearly independent.

   (ii) For each $\ell \in \mathbb{P}(V)$ use $v_2, \ldots, v_n$ to define linear maps $\bar{v}_2, \ldots, \bar{v}_n : \ell \to \ell^\perp$ which are linearly independent. (Form $\perp$ with respect to an auxiliary inner product on V.) Show these define a trivialisation of $T\mathbb{P}(V)$.

   Deduce that if $(V, \cdot)$ is a division algebra then $\dim V$ is a power of 2. (The division algebras you know, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, indeed have this property.)

12. * If a compact n-manifold embeds into $\mathbb{R}^{n+1}$, show that all its Stiefel–Whitney classes are zero. Show that this need not be the case if it immerses into $\mathbb{R}^{n+1}$.

13. Let $\alpha(n)$ denote the number of 1's when n is written in binary. By computing Stiefel–Whitney classes, show that for each n there is an n-manifold which does not immerse into $\mathbb{R}^{2n-\alpha(n)-1}$.
[In 1985 R. L. Cohen proved that every compact n-manifold immerses into $\mathbb{R}^{2n-\alpha(n)}$.]

14. If $f : M^d \to \mathbb{R}^n$ is an embedding of a compact manifold, with $(n-d)$-dimensional normal bundle $\nu_f \to M$, then show that $e(\nu_f) = 0 \in H^{n-d}(M; \mathbb{F}_2)$. You will need to use a tubular neighbourhood, excision, and the commutativity of a diagram

$$\begin{array}{ccc} H^i(X,A) \otimes H^j(X,A) & \xrightarrow{\smile} & H^{i+j}(X,A) \\ \downarrow & \nearrow_{\smile} & \\ H^i(X) \otimes H^j(X,A) & & \end{array}$$

which you should construct.
Hence show that $\mathbb{RP}^{2^k}$ does not embed in $\mathbb{R}^{2^{k+1}-1}$.
[In 1944 H. Whitney proved that every compact n-manifold embeds into $\mathbb{R}^{2n}$.]

Comments or corrections to `or257@cam.ac.uk`