Homotopy Theory, Examples 4

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All cohomology is with $\mathbb{Z}/2$-coefficients unless otherwise specified.

1. We have $H^*(K(\mathbb{Z}/2,1)^n) = \mathbb{Z}/2[x_1, x_2, \ldots, x_n]$ where x_i is the cohomology class represented by projection to the ith factor. Let σ_i be the ith elementary symmetric polynomial in the x_i (i.e. $1 + \sum_{i=1}^n \sigma_i = \prod_{i=1}^n (1 + x_i)$). Show that $\mathrm{Sq}^i(\sigma_n) = \sigma_i \cdot \sigma_n$, and deduce that $\mathrm{Sq}^i \iota_n \neq 0 \in H^{n+i}(K(\mathbb{Z}/2, n))$ for all $0 \leq i \leq n$.

2. Using the fact that Steenrod operations commute with transgression, show that $H^{n+1}(K(\mathbb{Z}/2, n))$ is 1-dimensional with generator $\mathrm{Sq}^1 \iota_n$. Deduce that Sq^1 agrees with the Bockstein associated to $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ on every cohomology group of every space.

3. If M is a closed connected n-manifold, show that $w_1 = 0 \in H^1(M; \mathbb{Z}/2)$ if and only if M is orientable. *[Hint: Recall from the proof of Poincaré duality that* $\mathrm{tors} H_{n-1}(M; \mathbb{Z})$ *is 0 if M is orientable and $\mathbb{Z}/2$ if M is not orientable.]*

4.* If X is a CW-complex for which there is an isomorphism $H^*(X) \cong H^*(\mathbb{RP}^5/\mathbb{RP}^2)$ respecting Steenrod operations, show that there is a map $f : \mathbb{RP}^5/\mathbb{RP}^2 \to X$ inducing the isomorphism. *[Hint: How close is X to $K(\mathbb{Z}/2, 3)$?]*

5.* Compute $H^*(K(\mathbb{Z}/2, 4))$ for $* \leq 6$ (or further if you can). Hence show that $\pi_5(S^2) = \pi_5(S^3) = \mathbb{Z}/2$.

6. Suppose a connected n-dimensional manifold M embeds smoothly into S^{n+1}, decomposing it into two regions A and B with common boundary M (and inclusions $i_A : M \hookrightarrow A$ and $i_B : M \hookrightarrow B$).

(i) Show that $\mathrm{Sq}^i : H^{n-i}(M) \to H^n(M) = \mathbb{Z}/2$ is zero for all $i > 0$.

(ii) Show that $i_A^* \oplus i_B^* : \tilde{H}^*(A) \oplus \tilde{H}^*(B) \to \tilde{H}^*(M - \{*\})$ is an isomorphism.

(iii) Show that the map

$$H^*(A) \xrightarrow{i_A^*} H^*(M) \xrightarrow{-\cap[M]} H_{n-*}(M) \xrightarrow{(i_B)_*} H_{n-*}(B)$$

gives an isomorphism $\tilde{H}^*(A) \cong \tilde{H}_{n-*}(B)$.

(iv) Deduce that $\mathbb{RP}^n$ does not embed in $\mathbb{R}^{n+1}$ for $n > 1$.

7. If $E \to B$ is a real n-dimensional vector bundle, with Thom space $\mathrm{Th}(E)$ and $\mathbb{Z}/2$-Thom class $u \in \tilde{H}^n(\mathrm{Th}(E))$, define $w_i(E) \in H^i(B)$ to be the unique cohomology class which corresponds to $\mathrm{Sq}^i(u) \in \tilde{H}^{n+i}(\mathrm{Th}(E))$ under the Thom isomorphism.

(i) Show that $w_i(E) = 0$ for $i > n$.

(ii) Writing $w(E) = 1 + w_1(E) + w_2(E) + \cdots$ (which is a finite sum by (i)), show that $w(E \oplus F) = w(E) \cdot w(F)$, *[Hint: sum of vector bundles is given by pulling back* $E \times F \to B \times B$ *along the diagonal; relate* $\mathrm{Th}(E \times F)$ *to* $\mathrm{Th}(E)$ *and* $\mathrm{Th}(F)$*]*

(iii) If $L \to \mathbb{RP}^n$ is the canonical 1-dimensional vector bundle, show that $w(L) = 1 + x$ for $x \in H^1(\mathbb{RP}^n)$ the standard generator. *[Hint: show that* $\mathrm{Th}(L) \simeq \mathbb{RP}^{n+1}$*]*

(iv) Show that the tangent bundle $T\mathbb{RP}^n$ of $\mathbb{RP}^n$ satisfies $T\mathbb{RP}^n \oplus \epsilon^1 \cong L^{\oplus n+1}$, where ϵ^k is the trivial k-dimensional bundle *[Hint: produce an isomorphism* $TS^n \oplus \epsilon^1 \cong \epsilon^{n+1}$ *with an involution covering the antipodal map]*, so $w(T\mathbb{RP}^n) = (1+x)^{n+1}$. Similarly, show that $w(T\mathbb{CP}^n) = (1 + y)^{n+1}$ for $y \in H^2(\mathbb{CP}^n)$ the standard generator.

(v) If an n-dimensional manifold M is the boundary of a compact $(n + 1)$-dimensional manifold W, show that the $w_i(TM)$ are in the image of the restriction map $H^*(W) \to H^*(M)$. Deduce that if $\sum_{i=1}^{k} n_i = n$ then $\langle w_{n_1}(TM)w_{n_2}(TM) \cdots w_{n_k}(TM), [M]\rangle = 0$. Hence show that $\mathbb{RP}^{2k}$ is not the boundary of any compact $(2k + 1)$-manifold.

(vi) Show that the standard embedding $\mathbb{RP}^n \hookrightarrow \mathbb{CP}^n$ is not homotopic to an embedding (or even an immersion) into $\mathbb{CP}^{n-1}$, although it is homotopic to a map into $\mathbb{CP}^{n-1}$.