Homotopy Theory, Examples 3

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1.* The unit quaternions $S^3 \subset \mathbb{H}$ act freely on the unit sphere $S^7 \subset \mathbb{H} \times \mathbb{H}$ with quotient $\mathbb{HP}^1 \cong S^4$, giving a fibre bundle $\nu : S^7 \to S^4$ with fibre S^3. Let $p : E \to S^2 \times S^2$ be the pullback of this fibre bundle along a map $S^2 \times S^2 \to S^4$ of degree 1, and let $\pi : E \xrightarrow{p} S^2 \times S^2 \xrightarrow{\pi_1} S^2$, a fibre bundle with fibre $S^3 \times S^2$.

Compute the homology Serre spectral sequences for the fibrations ν, p, and π.

2. Say that a space X has *rational Euler characteristic defined* if $\bigoplus_{n \geq 0} H_n(X; \mathbb{Q})$ is a finite-dimensional vector space, in which case set $\chi_{\mathbb{Q}}(X) := \sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{Q}} H_n(X; \mathbb{Q})$.

Let $p : E \to B$ be a Serre fibration over a path-connected base with fibre $F := p^{-1}(b_0)$, where $\pi_1(B, b_0)$ acts trivially on $H_*(F; \mathbb{Q})$. Show that if F and B have rational Euler characteristic defined then so does E, and that

$$\chi_{\mathbb{Q}}(E) = \chi_{\mathbb{Q}}(B) \cdot \chi_{\mathbb{Q}}(F).$$

3.* Let $f : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ be a map representing $2 \cdot \iota \in H^2(K(\mathbb{Z}, 2); \mathbb{Z})$, where ι is the canonical cohomology class of $K(\mathbb{Z}, 2)$. Show that the homotopy fibre of f is $K(\mathbb{Z}/2, 1) \simeq \mathbb{RP}^\infty$, then compute the Serre spectral sequence in both homology and cohomology for the map f (converted into a fibration).

4. By considering the fibrations $SU(n) \to S^{2n-1}$ with fibre $SU(n-1)$, and their cohomology Serre spectral sequence, prove inductively that for $n \geq 2$

$$H^*(SU(n); \mathbb{Z}) \cong \mathbb{Z}\langle x_3, x_5, \ldots, x_{2n-1} \rangle / (x_i^2, x_i x_j + x_j x_i)$$

as rings, where x_i has degree i.

5. Compute $H^*(K(\mathbb{Z}, k); \mathbb{Q})$ for all k. *[Hint: As usual use $P_* K(\mathbb{Z}, k) \to K(\mathbb{Z}, k)$ and the equivalence $\Omega K(\mathbb{Z}, k) \simeq K(\mathbb{Z}, k-1)$.]*

Hence show that

(i) $\pi_i(S^{2n+1})$ is finite for $i \neq 2n+1$,

(ii) $\pi_i(S^{2n})$ is finite for $i \neq 2n, 4n-1$, and $\pi_{4n-1}(S^{2n})$ has rank 1.

[Hint: Compute the rational (co)homology of the homotopy fibre of a map $S^k \to K(\mathbb{Z}, k)$ representing a generator of $H^k(S^k; \mathbb{Z})$.]

6. Let $p : E \to B$ be a Serre fibration, $e_0 \in E$, $b_0 := p(e_0)$, $F := p^{-1}(b_0)$. Show that

$$\begin{aligned} P_{e_0} E &\longrightarrow P_{b_0} B \\ \gamma &\longmapsto p \circ \gamma \end{aligned}$$

is a Serre fibration, and hence deduce that the homotopy fibre of the inclusion $i : F \to E$ over the point $e_0 \in E$ is weakly homotopy equivalent to $\Omega_{b_0} B$.

Using this, compute the integral (co)homology of the homotopy fibre X of a map $S^3 \to K(\mathbb{Z}, 3)$ representing a generator of $H^3(S^3; \mathbb{Z})$, and show that for every prime p the group $\pi_{2p}(S^3) \cong \pi_{2p}(S^2)$ contains a $\mathbb{Z}/p$-summand, and that this is the smallest degree containing p-torsion.

7.* Let $R \subset \mathbb{Q}$ be a ring. If X is a 1-connected space such that $H_*(X; R) \cong H_*(S^n; R)$, show that there is a map $f : S^n \to X$ inducing an isomorphism on $H_*(-; R)$, and hence that $\pi_i(X) \otimes R \cong \pi_i(S^n) \otimes R$ for all i. *[Hint: a) Show that such rings have* $\mathrm{Tor}(-, R) = 0$*; b) Consider the class of abelian groups A such that $A \otimes R = 0$.]*

8. (Edge homomorphisms) For a Serre fibration $p : E \to B$ with connected fibre F over b_0, and simply connected base, consider the maps of fibrations

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] & E \arrow[r, "p"] \arrow[d, "p"] & B \arrow[d, equal] \\
\{b_0\} \arrow[r] & B \arrow[r, equal] & B
\end{tikzcd}$$

and show using the naturality of the Serre spectral sequence that the compositions

$$H_q(F) = H_0(B; H_q(F)) = E^2_{0,q} \longrightarrow E^\infty_{0,q} = F^0 H_q(E) \longrightarrow H_q(E)$$
$$H_p(E) = F^p H_p(E) \longrightarrow E^\infty_{p,0} \longrightarrow E^2_{p,0} = H_p(B)$$

agree with the maps induced by i and p respectively.

9. For a Serre fibration $p : E \to B$ with fibre F, show that if $H^*(E; R) \to H^*(F; R)$ is surjective then $\pi_1(B, b_0)$ acts trivially on $H^*(F; R)$. Hence prove the *Leray–Hirsch Theorem*: under this condition, and the assumption that $H^*(F; R)$ is a free R-module, there is an isomorphism

$$H^*(E; R) \cong H^*(B; R) \otimes_R H^*(F; R).$$

If V is a complex vector space, let $\mathbb{P}(V) := (V \setminus 0)/\mathbb{C}^\times$ be its projectivisation. (If V has dimension n, choosing a basis for it gives a homeomorphism $\mathbb{P}(V) \cong \mathbb{CP}^{n-1}$.) If $p : E \to B$ is a complex vector bundle, let $\mathbb{P}(E) \to B$ be the fibre bundle with fibre $\mathbb{CP}^{n-1}$ obtained by applying the construction $\mathbb{P}(-)$ to each fibre of p.

Show that there is a canonical 1-dimensional vector bundle $L \to \mathbb{P}(E)$, and by considering its Euler class $e(L)$ show that $\pi : \mathbb{P}(E) \to B$ satisfies the assumptions of the Leray–Hirsch theorem for $R = \mathbb{Z}$. Hence show that there is a ring isomorphism

$$H^*(\mathbb{P}(E); \mathbb{Z}) \cong H^*(B; \mathbb{Z})[e(L)]/(p(e(L)))$$

for some monic polynomial $p(x) \in H^*(B; \mathbb{Z})[x]$.