Homotopy Theory, Examples 2

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Lent 2021

1.* Let (X, A) be a CW pair and $p : E \to B$ be a Serre fibration. Show that a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow & \overset{G}{\nearrow} & \downarrow p \\ X & \xrightarrow{F} & B \end{array}$$

admits a dashed map G making both triangles commute, if either

(i) $A \to X$ is a weak homotopy equivalence, or

(ii) $p : E \to B$ is a weak homotopy equivalence

2. Let $h : S^3 \to S^2$ be the Hopf bundle.

(i) Show that $S^2 \cup_h D^4 \simeq \mathbb{CP}^2$. Hence compute the cup product structure of $X(n) := S^2 \cup_{n \cdot h} D^4$ for $n \in \mathbb{Z}$, and show that $X(n)$ is not homotopy equivalent to a compact smooth manifold unless $n = \pm 1$.

(ii) If $c : T^3 = S^1 \times S^1 \times S^1 \to S^3$ is the map which collapses the complement of a ball to a point, prove that $h \circ c : T^3 \to S^2$ induces the trivial map on homology and homotopy, but is not homotopic to a constant map.

3. Let (X, x_0) be a $(n-1)$-connected CW complex. Show that there is a map $f : X \to K(\pi_n(X, x_0), n)$ which is an isomorphism on $\pi_n(-)$, and deduce that its homotopy fibre is n-connected.

4.* For an odd prime number p, by considering the (free) action of $\mathbb{Z}/p \subset S^1$ on $S^{2n-1} \subset \mathbb{C}^n$, construct an Eilenberg–MacLane space of type $(\mathbb{Z}/p, 1)$ and hence compute $H^*(K(\mathbb{Z}/p, 1); \mathbb{F}_p)$ as a ring. *[Hint: Use Poincaré duality for the manifolds $S^{2n-1}/\mathbb{Z}/p$ to deduce cup products.]*

Hence show that $\mathbb{Z}/p \times \mathbb{Z}/p$ cannot act freely on S^n for any $n \geq 2$. *[Hint: If it did, with quotient the n-manifold M, try to build a $K(\mathbb{Z}/p \times \mathbb{Z}/p, 1)$ by attaching cells to M, and consider its cohomology.]*[1]

5. If G is a compact Lie group and $H \leq G$ is a closed subgroup, the quotient map $p : G \to G/H$ can be shown to be a fibre bundle with fibre H. *[Assume this, or prove it if you have a passion for Lie groups.]* Let $O(n)$ be the group of $n \times n$ orthogonal matrices.

[1] It is a conjecture of G. Carlsson that if $(\mathbb{Z}/p)^r$ acts freely on $S^{n_1} \times \cdots \times S^{n_k}$ then $r \leq k$. Some special cases are known, but in this generality it remains open.

(i) By identifying $O(n)/O(n-1) \cong S^{n-1}$, show that the inclusion $O(n-1) \to O(n)$ is $(n-2)$-connected.

(ii) Show that $V_k(\mathbb{R}^n) := \{(v_1, v_2 \ldots, v_k) \in (\mathbb{R}^n)^k \mid v_i \text{ orthonormal}\}$ is homeomorphic to $O(n)/O(n-k)$, and hence deduce that it is $(n-k-1)$-connected.

(iii) Show that $O(n)/(O(k) \times O(n-k))$ is in bijection with the set of k-planes in $\mathbb{R}^n$, denoted $\mathrm{Gr}_k(\mathbb{R}^n)$ (the "Grassmannian"), and hence show that $\pi_i(\mathrm{Gr}_k(\mathbb{R}^n)) \cong \pi_{i-1}(O(k))$ for $i \leq n-k-1$.

6. (Homology Whitehead theorem) Show that a map $f : X \to Y$ between simply-connected CW complexes which induces an isomorphism on homology is a homotopy equivalence. *[Hint: Study the homology of the homotopy fibre of f, using the Serre spectral sequence.]*

7. Show that a closed simply-connected 3-manifold M is homotopy equivalent to S^3. *[Use the previous question.]*

By considering the group $G = \langle a, b \mid (ab)^2 a^{-3}, b^5 a^{-3} \rangle$, construct a space having the homology of a point but not being weakly homotopy equivalent to a point. *[You can assume G is a nontrivial group if you like, or prove it is nontrivial by producing a homomorphism onto A_5.]*

8. Show that the Moore–Postnikov tower of a map is unique up to homotopy equivalence.

9. If $\cdots \to Z_2 \to Z_1 \to Z_0 \to X$ is the Whitehead tower of a space X based at $x_0 \in X$, show that Z_0 is weakly homotopy equivalent to the path component $X_0 \subset X$ containing x_0, and that Z_1 is weakly homotopy equivalent to the universal cover of X_0. *[You may assume that X_0 is nice enough to have a universal cover.]*

10. Tensoring the short exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ with $C_*(X)$ gives a short exact sequence of chain complexes $0 \to C_*(X) \xrightarrow{n} C_*(X) \to C_*(X) \otimes \mathbb{Z}/n \to 0$ and so a long exact sequence on homology. The connecting homomorphism

$$\tilde{\beta} : H_k(X; \mathbb{Z}/n) \longrightarrow H_{k-1}(X; \mathbb{Z})$$

is called the *integral Bockstein homomorphism*.

Show that this gives a (singly-graded!) exact couple and hence that for each prime p there is a (singly-graded!) spectral sequence $\{E^r_s(p), d^r\}$ with $E^1_s(p) = H_s(X; \mathbb{Z}/p)$ and $d^r : E^r_s(p) \to E^r_{s-1}(p)$. Assuming that $H_s(X; \mathbb{Z})$ is a finitely-generated abelian group for each s, show that $E^r_s(p)$ is independent of r for all $r \gg 0$, and that this stable value is $(H_s(X; \mathbb{Z})/\mathrm{torsion}) \otimes \mathbb{Z}/p$.

Construct examples showing that d^r can be nontrivial for arbitrarily large r.