Homotopy Theory, Examples 1

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Lent 2021

1.* (Homotopy equivalences are weak homotopy equivalences) Show that if $\varphi : X \to Y$ is a homotopy equivalence, and $x_0 \in X$, then $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, \varphi(x_0))$ is a bijection for all $n \geq 0$.

2. Give an example of a weak homotopy equivalence $f : X \to Y$ for which there does not exist a weak homotopy equivalence $g : Y \to X$.

3. Let (X, A) be a pair of spaces having the homotopy extension property.

(i) If A is contractible, show that the quotient map $q : X \to X/A$ is a homotopy equivalence.

(ii) If (Y, A) is another pair which has the homotopy extension property, and $f : X \to Y$ satisfies $f|_A = \mathrm{Id}_A$ and is a homotopy equivalence, show that it is also a homotopy equivalence relative to A.

4. Recall that the mapping cylinder M_f of a map $f : X \to Y$ is $(X \times [0,1] \sqcup Y)/(x, 1) \in X \times [0,1] \sim f(x) \in Y$. Show that the pair (M_f, X) has the homotopy extension property.

5. If $f : X \to Y$ is a continuous map from a compact space to a CW complex, then show that there is a finite sub-CW complex $Y' \subset Y$ such that f lands in Y'. [Hint: You might first show that f lands in some skeleton Y^n.]

6. (Homology and cohomology of infinite CW complexes) Show that if $Y_0 \subset Y_1 \subset \cdots \subset Y$ is a collection of nested sub-CW complexes which exhaust Y, then $H_n(Y; A)$ is the direct limit of

$$H_n(Y_0; A) \to H_n(Y_1; A) \to H_n(Y_2; A) \to \cdots .$$

[This is easiest using cellular homology, or else the previous question.] Give an example showing it is *not* true that $H^n(Y; A)$ is the inverse limit of

$$H^n(Y_0; A) \leftarrow H^n(Y_1; A) \leftarrow H^n(Y_2; A) \leftarrow \cdots .$$

7. (Cellular Approximation Theorem) Prove that if $f : X \to Y$ is a map between CW complexes, then it is homotopic to a map f' which is *cellular* i.e. satisfies $f'(X^n) \subset Y^n$ for all n. [Hint: Consider the connectivity of (Y, Y^n).]

8. For a based space (X, x_0), let $\pi_1(X, x_0)^{ab} := \pi_1(X, x_0)/\pi_1(X, x_0)'$ be the abelianisation of the fundamental group. Show that the Hurewicz map $h : \pi_1(X, x_0) \to H_1(X; \mathbb{Z})$ factors as

$$h : \pi_1(X, x_0) \to \pi_1(X, x_0)^{ab} \stackrel{h^{ab}}{\to} H_1(X; \mathbb{Z}),$$

and that if X is path connected then h^{ab} is an isomorphism. [Hint: Prove it first for $X = \vee_I S^1$, then study how $\pi_1(X, x_0)^{ab}$ and $H_1(X;\mathbb{Z})$ change when cells are attached to X.]

9.* Show that if $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map then $p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$. Describe the $\mathbb{Z}[\pi_1(X, x_0)]$-module structure on $\pi_n(X, x_0)$ in these terms. There is a resulting homomorphism

$$\tilde{h} : \pi_n(X, x_0) \xleftarrow{\sim} \pi_n(\widetilde{X}, \tilde{x}_0) \xrightarrow{h} H_n(\widetilde{X};\mathbb{Z}).$$

(i) For $X = S^1 \vee S^n$, with basepoint x_0 the wedge point, calculate $\pi_n(X, x_0)$ as a $\pi_1(X, x_0)$-module.

(ii) For $X = \mathbb{RP}^2$, with any basepoint x_0, calculate $\pi_2(X, x_0)$ as a $\pi_1(X, x_0)$-module.

(iii) Let $f : S^2_\alpha \vee S^2_\beta \to S^2_\alpha \vee S^2_\beta$ be the map which is the identity on S^2_α and which on S^2_β is the sum of the identity map and a homeomorphism $S^2_\beta \to S^2_\alpha$. Let X be the mapping torus of f, i.e. the quotient space of $(S^2_\alpha \vee S^2_\beta) \times [0, 1]$ under the identifications $(x, 0) \sim (f(x), 1)$. The mapping torus of the restriction $f|_{S^2_\alpha}$ forms a subspace $A = S^1 \times S^2_\alpha \subset X$.

By considering the universal covers of A and X, show that the maps $\pi_2(A) \to \pi_2(X) \to \pi_2(X, A)$ form a short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$, and compute the action of $\pi_1(A)$ on these three groups. In particular, show the action of $\pi_1(A)$ is trivial on $\pi_2(A)$ and $\pi_2(X, A)$ but is nontrivial on $\pi_2(X)$.