Homotopy Theory, Examples 4

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All cohomology is with $\mathbb{F}_2$ coefficients unless otherwise specified.

1. We have $H^*(K(\mathbb{Z}/2,1)^n) = \mathbb{F}_2[x_1, x_2, \ldots, x_n]$ where x_i is the cohomology class represented by projection to the ith factor. Let σ_i be the ith elementary symmetric polynomial in the x_i (i.e. $1 + \sum_{i=1}^n \sigma_i = \prod_{i=1}^n (1 + x_i)$). Show that $Sq^i(\sigma_n) = \sigma_i \cdot \sigma_n$, and deduce that $Sq^i \iota_n \neq 0 \in H^{n+i}(K(\mathbb{Z}/2, n))$ for all $0 \leq i \leq n$.

2. Using the fact that Steenrod operations commute with transgression, show that $H^{n+1}(K(\mathbb{Z}/2, n))$ is one-dimensional with generator $Sq^1 \iota_n$. Show that Sq^1 agrees with the Bockstein associated to $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ on every cohomology group of every space.

3. Using the fact that Steenrod operations commute with transgression, show that $H^*(K(\mathbb{Z}/2, 2))$ is a polynomial algebra on generators $\iota_2, Sq^1\iota_2, Sq^2Sq^1\iota_2, Sq^4Sq^2Sq^1\iota_2, \ldots$. Hence show that the map $K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,1) \to K(\mathbb{Z}/2,2)$ representing $\iota_1 \otimes \iota_1$ is injective on cohomology.

4. If X is a space for which there is an isomorphism $H^*(X) \cong H^*(\mathbb{RP}^5/\mathbb{RP}^2)$ respecting Steenrod operations, show that there is a map $f : X \to \mathbb{RP}^5/\mathbb{RP}^2$ inducing the isomorphism. [Hint: How close is $\mathbb{RP}^5/\mathbb{RP}^2$ to $K(\mathbb{Z}/2, 3)$?]

5. Compute $H^*(K(\mathbb{Z}/2, 4))$ for $* \leq 6$ (or further if you can). Hence show that $\pi_5(S^2) = \pi_5(S^3) = \mathbb{Z}/2$.

6. The operation $Sq = Sq^0 + Sq^1 + Sq^2 + \cdots$ may be formally inverted (as it starts with $Sq^0 = \mathrm{Id}$), with inverse $Sq^{-1} = Sq_0^{-1} + Sq_1^{-1} + \cdots$. For $x \in H^1(\mathbb{RP}^\infty)$ the standard generator, show that $Sq^{-1}(x) = x + x^2 + x^4 + x^8 + x^{16} + \cdots$.

If M is an n-dimensional manifold with total Wu class $v = 1 + v_1 + v_2 + \cdots$, show that $\langle Sq(x) \cdot y, [M] \rangle = \langle x \cdot Sq^{-1}(y) \cdot v, [M] \rangle$ for all $x, y \in H^*(M)$. Hence show that the Poincaré duality isomorphisms $\phi^k : H^k(M) \cong H_{n-k}(M) \cong H^{n-k}(M)^*$ satisfy

$$\phi^{k+i}(Sq^i(x))(-) = \phi^k(x) \left(\sum_{a+b=n-k} Sq_a^{-1}(-) \cdot v_b \right).$$

7. Suppose a connected n-dimensional manifold M embeds smoothly into S^{n+1}, decomposing it into two regions A and B with common boundary M (and inclusions $i_A : M \hookrightarrow A$ and $i_B : M \hookrightarrow B$).

(i) Show that $Sq^i : H^{n-i}(M) \to H^n(M) = \mathbb{F}_2$ is zero for all $i > 0$.

(ii) Show that $i_A^* \oplus i_B^* : \tilde{H}^*(A) \oplus \tilde{H}^*(B) \to \tilde{H}^*(M - \{*\})$ is an isomorphism.

(iii) Show that the map

$$H^*(A) \xrightarrow{i_A^*} H^*(M) \xrightarrow{-\cap[M]} H_{n-*}(M) \xrightarrow{(i_B)_*} H_{n-*}(B)$$

gives an isomorphism $\tilde{H}^*(A) \cong \tilde{H}_{n-*}(B)$.

(iv) Deduce that $\mathbb{RP}^n$ does not embed in $\mathbb{R}^{n+1}$ for $n > 1$.

8. If $E \to B$ is a real n-dimensional vector bundle, with Thom space $\mathrm{Th}(E)$ and $\mathbb{F}_2$-Thom class $u \in H^n(\mathrm{Th}(E))$, define $w_i(E) \in H^i(B)$ to be the unique cohomology class which corresponds to $Sq^i(u) \in H^{n+i}(\mathrm{Th}(E))$ under the Thom isomorphism.

(i) Show that $w_i(E) = 0$ for $i > n$.

(ii) Writing $w(E) = 1 + w_1(E) + w_2(E) + \cdots$ (which is a finite sum by (i)), show that $w(E \oplus F) = w(E) \cdot w(F)$, [Hint: sum of vector bundles is given by pulling back $E \times F \to B \times B$ along the diagonal; relate and $\mathrm{Th}(E \times F)$ to $\mathrm{Th}(E)$ and $\mathrm{Th}(F)$]

(iii) If $L \to \mathbb{RP}^n$ is the canonical 1-dimensional vector bundle, show that $w(L) = 1 + x$ for $x \in H^1(\mathbb{RP}^n)$ the standard generator. [Hint: show that $\mathrm{Th}(L) \simeq \mathbb{RP}^{n+1}$]

(iv) Show that the tangent bundle $T\mathbb{RP}^n$ of $\mathbb{RP}^n$ satisfies $T\mathbb{RP}^n \oplus \epsilon^1 \cong L^{\oplus n+1}$, where ϵ^k is the trivial k-dimensional bundle [Hint: produce an isomorphism $TS^n \oplus \epsilon^1 \cong \epsilon^{n+1}$ with an involution covering the antipodal map], so $w(T\mathbb{RP}^n) = (1+x)^{n+1}$. Similarly, show that $w(T\mathbb{CP}^n) = (1+y)^{n+1}$ for $y \in H^2(\mathbb{CP}^n)$ the standard generator.

(v) If an n-dimensional manifold M is the boundary of a compact $(n+1)$-dimensional manifold W, show that the $w_i(TM)$ are in the image of the restriction map $H^*(W) \to H^*(M)$. Deduce that if $\sum_{i=1}^k n_i = n$ then $\langle w_{n_1}(TM) w_{n_2}(TM) \cdots w_{n_k}(TM), [M] \rangle = 0$. Hence show that $\mathbb{RP}^{2k}$ is not the boundary of any compact $(2k+1)$-manifold.

(vi) Show that the standard embedding $\mathbb{RP}^n \hookrightarrow \mathbb{CP}^n$ is not homotopic to an embedding (or even an immersion) into $\mathbb{CP}^{n-1}$, although it is homotopic to a map into $\mathbb{CP}^{n-1}$.