Homotopy Theory, Examples 1

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1. Show that if $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map then $p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$. Describe the $\mathbb{Z}[\pi_1(X, x_0)]$-module structure on $\pi_n(X, x_0)$ in these terms. Hence

(i) for $X = S^1 \vee S^n$, with basepoint x_0 the wedge point, show that the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is non-trivial.

(ii) for $X = \mathbb{RP}^2$, with any basepoint x_0, show that the action of $\pi_1(X, x_0)$ on $\pi_2(X, x_0)$ is non-trivial.

[You will need to show that certain maps are not homotopic to each other: remember that homotopic maps induce equal maps on homology.]

2. (Homotopy equivalences are weak homotopy equivalences) Show that if $\varphi : X \to Y$ is a homotopy equivalence, and $x_0 \in X$, then $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, \varphi(x_0))$ is a bijection for all $n \geq 0$.

3. Let (X, A) be a pair of spaces having the homotopy extension property.

(i) If A is contractible, show that the quotient map $q : X \to X/A$ is a homotopy equivalence.

(ii) If (Y, A) is another pair which has the homotopy extension property, and $f : X \to Y$ satisfies $f|_A = \mathrm{Id}_A$ and is a homotopy equivalence, show that it is also a homotopy equivalence relative to A.

4. If $f : X \to Y$ is a continuous map from a compact space to a CW complex, then show that there is a finite sub-CW complex $Y' \subset Y$ such that f lands in Y'. [Hint: You might first show that f lands in some skeleton Y^n.]

5. (Homology and cohomology of infinite CW complexes) Show that if $Y_0 \subset Y_1 \subset \cdots \subset Y$ is a collection of nested sub-CW complexes which exhaust Y, then $H_n(Y; A)$ is the direct limit of

$$H_n(Y_0; A) \to H_n(Y_1; A) \to H_n(Y_2; A) \to \cdots .$$

[This is easiest using cellular homology, or else the previous question.] Give an example showing it is *not* true that $H^n(Y; A)$ is the inverse limit of

$$H^n(Y_0; A) \leftarrow H^n(Y_1; A) \leftarrow H^n(Y_2; A) \leftarrow \cdots .$$

[Hint: The direct limit of $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \cdots$ is $\mathbb{Z}[\frac{1}{2}]$, and the inverse limit of $\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow \cdots$ is zero.]

6. (Cellular Approximation Theorem) Prove that if $f : X \to Y$ is a map between CW complexes, then it is homotopic to a map f' which is *cellular* i.e. satisfies $f'(X^n) \subset Y^n$ for all n. [Hint: Consider the connectivity of (Y, Y^n).]

7. If $\cdots \to Z_2 \to Z_1 \to Z_0 \to X$ is the Whitehead tower of a space X based at $x_0 \in X$, show that Z_0 is weakly homotopy equivalent to the path component $X_0 \subset X$ containing x_0, and that Z_1 is weakly homotopy equivalent to the universal cover of X_0. [You may assume that X_0 is nice enough to have a universal cover.]

8. For a based space (X, x_0), let $\pi_1(X, x_0)^{ab} = \pi_1(X, x_0)/\pi_1(X, x_0)'$ be the abelianisation of the fundamental group. Show that the Hurewicz map $h : \pi_1(X, x_0) \to H_1(X; \mathbb{Z})$ factors as

$$h : \pi_1(X, x_0) \to \pi_1(X, x_0)^{ab} \xrightarrow{h^{ab}} H_1(X; \mathbb{Z}),$$

and that if X is path connected then h^{ab} is an isomorphism. [Hint: Prove it first for $X = \vee_I S^1$, then study how $\pi_1(X, x_0)^{ab}$ and $H_1(X; \mathbb{Z})$ change when cells are attached to X.]