Michaelmas Term 2025 O. Randal-Williams

Part III Algebraic Topology // Example Sheet 4

1. If $\{C_\bullet(a), \rho_{ab}\}_{a\in I}$ is a direct system of chain complexes, show that $H_k(\varinjlim C_\bullet(a)) = \varinjlim H_k(C_\bullet(a))$. Deduce that a direct limit of exact sequences is exact.

2.

(i) Which of the following are $\mathbb{Z}$-orientable? (a) $\mathbb{RP}^3$, (b) $\mathbb{RP}^2 \times \mathbb{CP}^2$, (c) $K\#T^2$, where K is the Klein bottle (# denotes connect sum).

(ii) Prove that any manifold has a $\mathbb{Z}$-orientable double cover.

3.* If M is a connected compact d-manifold and $x \in M$, show that $H_d(M \setminus x; \mathbb{F}_2) = 0$.

If in addition $H_d(M;\mathbb{Z}) \cong \mathbb{Z}$, deduce that the restriction map $\mathrm{res}_x : \mathbb{Z} \cong H_d(M;\mathbb{Z}) \to H_d(M|x;\mathbb{Z}) \cong \mathbb{Z}$ is injective, and that the index of its image is independent of x. [*Hint: Show it is locally constant as a function of x.*] Hence show that M is $\mathbb{Z}$-orientable.

4.

(i) Let M be a compact connected $\mathbb{Z}$-oriented d-manifold. Show that there is a degree one map $M \to S^d$.

(ii) If M and N are compact connected $\mathbb{Z}$-oriented manifolds of the same dimension and $f : M \to N$ is a map of non-zero degree, is $f^* : H^*(N;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ necessarily injective?

(iii) Prove that if a finite group G acts freely on S^n then some G-orbit is not contained in any open hemisphere. [*Hint: Construct a map $S^n/G \to S^n$.*]

5. Show that the only non-trivial cup-products in $(S^2\times S^8)\#(S^4\times S^6)$ are those forced by Poincaré duality. Give an example of a space in which that conclusion would not be true.

6. Let $f : \mathbb{CP}^n \to \mathbb{CP}^n$ be a map of degree 8. What can you say about n ?

7. Show that there is no map from $\mathbb{CP}^2$ to itself of degree -1. Show that there is no map from $\mathbb{CP}^2 \times \mathbb{CP}^2$ to itself of degree -1.

8.

(i) If M is a smooth manifold, show that it is equivalent to give an R-orientation of the manifold M and an R-orientation of the vector bundle TM.

(ii) Let V be a real n-dimensional vector space. Show that a $\mathbb{Z}$-orientation of V, meaning a choice of generator of $H^n(V, V - \{0\}) \cong \mathbb{Z}$, is equivalent to an orientation in the sense of linear algebra, i.e. a choice of ordered basis, where bases differing by a positive determinant matrix are equivalent.

(iii) If M is R-oriented and $Y \subset M$ is a compact submanifold, show an R-orientation of Y determines an R-co-orientation of Y (i.e. an R-orientation of its normal bundle).

(iv) If M is R-oriented and $Y, Z \subset M$ are compact R-oriented submanifolds which meet transversely, show that an ordering of Y and Z defines a R-co-orientation of $Y \cap Z$.

9.* Consider the manifold $S^m \times \mathbb{CP}^1$ with the free involution τ defined by $\tau(x, [z_0, z_1]) := (-x, [\bar{z}_0, \bar{z}_1])$. Let $P(m)$ be the quotient space under this involution. Compute the groups $H^*(P(m); \mathbb{Z})$ and the ring $H^*(P(m); \mathbb{F}_2)$. [*Hint: find a cell structure to compute the cohomology groups, and use the intersection product to compute the cohomology ring.*]

10.

(i) Suppose $Y \subset X$ is a smooth compact submanifold of a smooth compact manifold. Using the tubular neighbourhood theorem, prove $H^*_c(X \backslash Y) \cong H^*(X, Y)$.

(ii) Suppose $M \subset S^d$ is a compact $(d-1)$-dimensional smooth submanifold. Show that the complement $S^d \backslash M$ has one more path component than M does.

(iii) Suppose $M \subset \mathbb{R}^d$ is a compact $(d-1)$-dimensional smooth submanifold. Show that $\mathbb{R}^d \setminus M$ consists of a bounded and an unbounded region, and hence that the 1-dimensional normal bundle of $M \subset \mathbb{R}^d$ is trivial. Describe the degree of the map $\nu : M \to S^{d-1}$ which assigns to each point its unit outward-pointing normal vector. [*Hint: relate the degree of ν to a vector field on M.*]

11.

(i) Show by induction on the dimension that a non-degenerate skew-symmetric bilinear form over $\mathbb{R}$ is equivalent to a direct sum of copies of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Hence show that any oriented closed 6-manifold M has $\dim_{\mathbb{Q}} H_3(M; \mathbb{Q})$ even.

(ii) Let V be a vector space with a non-degenerate skew form as above. If $W \subset V$ is *isotropic*, meaning $\langle \cdot, \cdot \rangle|_{W \times W} \equiv 0$, show that $\dim(W) \leq \frac{\dim(V)}{2}$. What does this say about the cohomology classes defined by a collection of pairwise disjoint 3-dimensional submanifolds of a closed oriented 6-manifold?

12. Let M be a compact $\mathbb{Z}$-oriented smooth d-manifold.

(i) If $f : M \to M$ be an orientation-preserving smooth map such that $f^p = \mathrm{Id}_M$, and the fixed-points of f form a discrete set, show that

$$\#\{\text{fixed points of } f\} = \sum_{k=0}^{d} (-1)^k \operatorname{Tr}(f^* : H^k(M; \mathbb{Q}) \to H^k(M; \mathbb{Q}))$$

and if p is prime show that $\#\{\text{fixed points of } f\} \equiv \chi(M) \mod p$. [*Hint: Rational canonical form.*]

(ii) If the circle group S^1 acts smoothly on M with discrete fixed set M^{S^1}, show $\# M^{S^1} = \chi(M)$.

13. Let $n > 1$. For a continuous map $\phi : S^{2n-1} \to S^n$, let Y_ϕ be the space obtained by attaching a $(2n)$-cell to S^n via ϕ. Compute $H^*(Y_\phi)$. Fixing $\alpha_i \in H^i(Y_\phi)$ to be generators for $i \in \{n, 2n\}$, define $h(\phi)$ by $\alpha_n^2 = h(\phi)\alpha_{2n}$.

(i) If ϕ is homotopic to a constant map, then show that $h(\phi) = 0$.

(ii) Let n be even. Fix a base-point $e \in S^n$. By considering the attaching map of the $2n$-cell of $S^n \times S^n$ show that there is a map $\phi : S^{2n-1} \to S^n$ having $h(\phi) = \pm 2$. Hence show that there are infinitely-many non-homotopic maps $S^{2n-1} \to S^n$.

Comments or corrections to `or257@cam.ac.uk`