Part III Algebraic Topology // Example Sheet 3

1.* (i) Compute the cohomology ring of the closed oriented surface Σ_g of genus g.

(ii) As $H^2(\Sigma_g) \cong \mathbb{Z}$ for every $g \geq 0$, define the degree of a map of oriented surfaces to be the induced map on H^2. For which g is there a map $\Sigma_g \to \Sigma_1$ of positive degree? For which g is there a map $\Sigma_1 \to \Sigma_g$ of positive degree?

2. If $f : \mathbb{C}^n \to \mathbb{C}^n$ has components the elementary symmetric functions

$$(z_1, \ldots, z_n) \mapsto (\sigma_i(\underline{z})) \qquad \sigma_1 = \sum_j z_j \qquad \sigma_2 = \sum_{i<j} z_i z_j \quad \cdots \quad \sigma_n = \prod_j z_j$$

then prove that f extends to a map $\psi : S^{2n} \to S^{2n}$ of degree $n!$.

Hence construct a map $\phi : (\mathbb{CP}^1)^n \to \mathbb{CP}^n$ of degree $n!$, and compute the effect of the map $\phi^* : H^2(\mathbb{CP}^n) \to H^2((\mathbb{CP}^1)^n)$. Deduce that there is a $x \in H^2(\mathbb{CP}^n)$ such that x^n is a generator of the abelian group $H^{2n}(\mathbb{CP}^n)$, and hence that $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ as a ring.

[*Hint: relate* $\mathbb{CP}^k$ *to the space of degree* k *homogeneous polynomials in two variables.*]

3. By considering a map to the wedge (one-point-union) of two copies of $\mathbb{CP}^2$, or otherwise, compute $H^*(\mathbb{CP}^2 \# \mathbb{CP}^2)$ as a ring. Deduce that $\mathbb{CP}^2 \# \mathbb{CP}^2$ is not homotopy equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$, even though they have the same (co)homology groups additively.

4. If X is a finite cell complex, by showing that $C_\bullet^{cell}(X)$ is (unnaturally) isomorphic to a direct sum of chain complexes of the form $0 \to B_n(X) \xrightarrow{A_n} Z_n(X) \to 0$, show that

$$H^n(X) \cong \frac{H_n(X)}{\mathrm{Tors}(H_n(X))} \oplus \mathrm{Tors}(H_{n-1}(X)),$$

where $\mathrm{Tors}(A) \leq A$ denotes the subgroup of elements of finite order.

5. Let $E \to X$ be a vector bundle with inner product $\langle \cdot, \cdot \rangle$. Let $F \subset E$ be a subbundle. Prove that the orthogonal complement bundle $F^\perp$ is locally trivial.

6. (i) Explain how to view an open Möbius band as a 1-dimensional real vector bundle over S^1. Show that it is a non-trivial bundle.

(ii) Show that a 1-dimensional real bundle over S^n with $n > 1$ is trivial. Hence show that 1-dimensional real bundles over a finite cell complex X up to isomorphism are naturally in 1-1 correspondence with elements of $H^1(X; \mathbb{Z}/2)$. [*Hint: Think about an associated double cover.*]

7. Show that a complex vector bundle has a canonical orientation.

8. If $\pi : E \to X$ is a d-dimensional real vector bundle which is not necessarily R-orientable, show that we still have $H^i(E, E^\#; R) = 0$ for $i < d$. If X is path-connected show that restriction to the fibre at $x \in X$ still gives an injective map $H^d(E, E^\#; R) \to H^d(E_x, E_x^\#; R) \cong R$.

Give an example to show that $H^{i+d}(E, E^\#; R)$ need not be isomorphic to $H^i(X; R)$ in general.

9. (i) Show that any map $f : \mathbb{RP}^n \to \mathbb{RP}^m$ induces a trivial map on reduced cohomology if $n > m$. What about if $n < m$?

(ii) Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$ although they have additively isomorphic (co)homology.

10. (i) If $f : S^n \to S^n$ satisfies $f(-x) = -f(x)$, show that it induces a map $\bar{f} : \mathbb{RP}^n \to \mathbb{RP}^n$. By considering the Gysin sequence show that f has odd degree.

(ii) Show that any $g : S^n \to \mathbb{R}^n$ satisfies $g(x) = g(-x)$ for some $x \in S^n$.

11.* (i) Let $L = \gamma^{\mathbb{C}}_{1,n+1} \to \mathbb{CP}^n$ be the canonical 1-dimensional complex bundle. By considering $\pi_1^* L \otimes_{\mathbb{C}} \pi_2^* L \to \mathbb{CP}^n \times \mathbb{CP}^n$, with the $\pi_i : \mathbb{CP}^n \times \mathbb{CP}^n \to \mathbb{CP}^n$ being projections to the factors, prove that the Euler class of $L \otimes_{\mathbb{C}} L$ is equal to twice the Euler class of L.

(ii) Show that the unit circle bundle in $L \otimes_{\mathbb{C}} L$ is homeomorphic to $\mathbb{RP}^{2n+1}$. Hence, compute the cohomology of $\mathbb{RP}^{2n+1}$ from knowledge of the cohomology of $\mathbb{CP}^n$.

12. Let $V_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k$ be the subspace of k-tuples of orthonormal vectors in $\mathbb{C}^n$ (a *Stiefel manifold*). Show there is a vector bundle $E_k \to V_k(\mathbb{C}^n)$ with fibre over $(v_1, \ldots, v_k)$ given by the vector space $\mathrm{span}(v_1, \ldots, v_k) \leq \mathbb{C}^n$.

Show that the forgetful map $(v_1, \ldots, v_k) \mapsto (v_1, \ldots, v_{k-1}) : V_k(\mathbb{C}^n) \to V_{k-1}(\mathbb{C}^n)$ exhibits $V_k(\mathbb{C}^n)$ as the sphere bundle of a certain vector bundle over $V_{k-1}(\mathbb{C}^n)$. Hence compute $H^*(V_k(\mathbb{C}^n); \mathbb{Z})$ as a ring.

Deduce that the unitary group $U(n)$ has the same cohomology ring as $S^1 \times S^3 \times S^5 \times \cdots \times S^{2n-1}$, and hence that

$$\sum_{j \geq 0} \mathrm{rk}\, H^j(U(n); \mathbb{Z})\, t^j = \prod_{i=1}^{n} (1 + t^{2i-1}).$$

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