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Part III Algebraic Topology // The small simplices theorem

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of subsets of X whose interiors cover X, and $C^{\mathcal{U}}_\bullet(X) \subset C_\bullet(X)$ be the sub-chain complex generated by those singular simplices $\sigma : \Delta^n \to X$ whose image lies entirely within some U_α. The goal of this note is to prove the following theorem.

Theorem 1. *The map* $H^{\mathcal{U}}_*(X) := H_*(C^{\mathcal{U}}_\bullet(X)) \to H_*(X)$ *is an isomorphism.*

1. Barycentric subdivision.

Definition 2. If $x = \{x_0, \ldots, x_n\}$ is a collection of points in $\mathbb{R}^N$ which span an n-simplex, we write $b_x = \frac{1}{n+1} \sum x_i$ for the *barycentre* of the simplex x. In particular, we write $b_n \in \Delta^n \subset \mathbb{R}^{n+1}$ for the barycentre of the standard n-simplex.

Let us write $\iota_n : \Delta^n \to \Delta^n$ for the identity map considered as a singular n-simplex, so as an element of $C_n(\Delta^n)$. If $\sigma : \Delta^i \to \Delta^n$ is a singular i-simplex, let

$$\begin{aligned} \mathrm{Cone}^{\Delta^n}_i(\sigma) : \Delta^{i+1} &\longrightarrow \Delta^n \\ (t_0, t_1, \ldots, t_{i+1}) &\longmapsto t_0 \cdot b_n + (1 - t_0) \cdot \sigma\left(\frac{(t_1, \ldots, t_{i+1})}{1 - t_0}\right), \end{aligned}$$

where we have used that Δ^n is convex to linearly interpolate between $b_n, \sigma(t_1, \ldots, t_{i+1}) \in \Delta^n$. This construction extended linearly gives a homomorphism $\mathrm{Cone}^{\Delta^n}_i : C_i(\Delta^n) \to C_{i+1}(\Delta^n)$, which satisfies

$$d(\mathrm{Cone}^{\Delta^n}_i(\sigma)) = \begin{cases} \sigma - \mathrm{Cone}^{\Delta^n}_{i-1}(d\sigma) & i > 0 \\ \sigma - \epsilon(\sigma) \cdot b_n & i = 0. \end{cases}$$

Therefore, if we let $c_\bullet : C_\bullet(\Delta^n) \to C_\bullet(\Delta^n)$ be the chain map given by $c_0(\sigma) = \epsilon(\sigma) \cdot b_n$ on a 0-simplex σ, and by $c_i(\sigma) = 0$ on simplices of higher dimension, then

$$d\mathrm{Cone}^{\Delta^n} + \mathrm{Cone}^{\Delta^n} d = \mathrm{id}_{C_\bullet(\Delta^n)} - c_\bullet.$$

Remark 3. This in particular shows that $H_i(\Delta^n) = 0$ for $i > 0$.

Definition 4. If $p^X_\bullet : C_\bullet(X) \to C_\bullet(X)$ is a collection of chain maps, one for each space X, we say they are *natural* if for each map $f : X \to Y$ of spaces we have $f_n \circ p^X_n = p^Y_n \circ f_n$. We make the analogous definition for a collection of chain homotopies $F^X_\bullet : C_\bullet(X) \to C_{\bullet+1}(X)$.

Definition 5. Define homomorphisms $\rho^X_n : C_n(X) \to C_n(X)$ inductively by:

(i) Let $\rho^X_0 = \mathrm{id}_{C_0(X)}$ for all spaces X.

(ii) If ρ^X_{n-1} has been defined for all spaces X, let

$$\begin{aligned} \rho^X_n : C_n(X) &\longrightarrow C_n(X) \\ \sigma &\longmapsto \sigma_\#(\mathrm{Cone}^{\Delta^n}_{n-1}(\rho^{\Delta^n}_{n-1}(d\iota_n))). \end{aligned}$$

Lemma 6. $\rho^X_\bullet : C_\bullet(X) \to C_\bullet(X)$ *is a natural chain map.*

Proof. If $f : X \to Y$ then

$$f_\#(\rho_n^X(\sigma)) = f_\#\sigma_\#(\mathrm{Cone}_{n-1}^{\Delta^n}(\rho_{n-1}^{\Delta^n}(d\iota_n))) = (f\circ\sigma)_\#(\mathrm{Cone}_{n-1}^{\Delta^n}(\rho_{n-1}^{\Delta^n}(d\iota_n)))$$

which is $\rho_n^Y(f\circ\sigma) = \rho_n^Y(f_\#(\sigma))$, so this is natural.

Let us suppose for an induction that $d\rho_{n-1}^X = \rho_{n-2}^X d$ for all spaces X, which is certainly satisfied when $n-1=0$. Then for $n \geq 1$ calculate

$$\begin{aligned} d\rho_n^X(\sigma) &= \sigma_\#(d(\mathrm{Cone}_{n-1}^{\Delta^n}(\rho_{n-1}^{\Delta^n}(d\iota_n)))) \\ &= \sigma_\#(\rho_{n-1}^{\Delta^n}(d\iota_n) - \mathrm{Cone}_{n-2}^{\Delta^n}(d\rho_{n-1}^{\Delta^n}(d\iota_n))) \\ &= \sigma_\#(\rho_{n-1}^{\Delta^n}(d\iota_n) - \mathrm{Cone}_{n-2}^{\Delta^n}(\rho_{n-1}^{\Delta^n}(dd\iota_n))) \\ &= \sigma_\#(\rho_{n-1}^{\Delta^n}(d\iota_n)) = \rho_{n-1}^X(d\sigma) \end{aligned}$$

as required, where at the end we have used the naturality property $\sigma_\# \circ \rho_{n-1}^{\Delta^n} = \rho_{n-1}^X \circ \sigma_\#$. □

We now wish to show that $\rho_\bullet^X$ is naturally chain homotopic to the identity.

Definition 7. Define homomorphisms $T_n^X : C_n(X) \to C_{n+1}(X)$ inductively by:

(i) Let $T_0^X = 0$ for all spaces X.

(ii) If T_{n-1}^X has been defined for all spaces X, let

$$\begin{aligned} T_n^X : C_n(X) &\longrightarrow C_{n+1}(X) \\ \sigma &\longmapsto \sigma_\#(\mathrm{Cone}_n^{\Delta^n}(\rho_n^{\Delta^n}(\iota_n) - \iota_n - T_{n-1}^{\Delta^n}(d\iota_n))). \end{aligned}$$

Lemma 8. $T_\bullet^X : C_\bullet(X) \to C_{\bullet+1}(X)$ *is a natural chain homotopy from* $\rho_\bullet^X$ *to the identity.*

Proof. It is natural for the same reason ρ_n^X was: $T_n^X(\sigma)$ is obtained by applying $\sigma_\#$ to an element of $C_{n+1}(\Delta^n)$.

Suppose for an induction that $dT_{n-1}^X + T_{n-2}^X d = \rho_{n-1}^X - \mathrm{id}_{C_{n-1}(X)}$ for all spaces X, which is certainly satisfied for $n-1=0$. Then for $n \geq 1$ calculate

$$\begin{aligned} dT_n^X(\sigma) &= \sigma_\#(d\mathrm{Cone}_n^{\Delta^n}(\rho_n^{\Delta^n}(\iota_n) - \iota_n - T_{n-1}^{\Delta^n}(d\iota_n))) \\ &= \sigma_\#((\mathrm{id}_{C_n(\Delta^n)} - \mathrm{Cone}_{n-1}^{\Delta^n}d)(\rho_n^{\Delta^n}(\iota_n) - \iota_n - T_{n-1}^{\Delta^n}(d\iota_n))). \end{aligned}$$

Now by the inductive assumption we have

$$d(\rho_n^{\Delta^n}(\iota_n) - \iota_n - T_{n-1}^{\Delta^n}(d\iota_n)) = \rho_{n-1}^{\Delta^n}(d\iota_n) - d\iota_n - dT_{n-1}^{\Delta^n}(d\iota_n) = T_{n-2}^{\Delta^n}(dd\iota_n) = 0,$$

so the expression simplifies to

$$dT_n^X(\sigma) = \sigma_\#(\rho_n^{\Delta^n}(\iota_n) - \iota_n - T_{n-1}^{\Delta^n}(d\iota_n)) = \rho_n^X(\sigma) - \sigma - T_{n-1}^X(d\sigma)$$

as required, where we have again used naturality of ρ^X and T^X. □

2. Some geometry of simplices.

The standard simplex Δ^n is a metric space via the metric inherited from $\mathbb{R}^{n+1}$, so we may talk about the diameter of a subset of Δ^n. For points $v_0, v_1, \ldots, v_n \in \Delta^n$, we write $[v_0, v_1, \ldots, v_n] : \Delta^n \to \Delta^n$ for the map $(t_0, t_1, \ldots, t_n) \mapsto \sum_i t_i v_i$; when it is injective, let us also write $[v_0, v_1, \ldots, v_n]$ for the image of this map, which is the convex hull of the v_i.

Lemma 9. $\mathrm{diam}([v_0, v_1, \ldots, v_n]) = \max_{i,j}\{|v_i - v_j|\}$

Proof. For $v \in [v_0, v_1, \dots, v_n]$ we have

$$\begin{aligned}\left|v - \sum_i t_i v_i\right| &= \left|\sum_i t_i v - \sum_i t_i v_i\right| \\ &\le \sum_i t_i |v_i - v| \\ &\le \max_j \{|v - v_j|\}\end{aligned}$$

and by convexity the latter is maximised when v is a vertex. □

Lemma 10. *Each simplex of $\rho_n^{\Delta^n}([v_0, \dots, v_n])$ has diameter $\le \frac{n}{n+1}\mathrm{diam}([v_0, v_1, \dots, v_n])$.*

Proof. Let us prove this by induction on dimension; it clearly holds for 0-simplices.

Now $\rho_n^{\Delta^n}([v_0, \dots, v_n])$ is a signed sum of n-simplices $[b_v, x_1, \dots, x_n]$ where b_v is the barycentre of $[v_0, \dots, v_n]$ and the x_i lie in the boundary of $[v_0, \dots, v_n]$. If the maximal distance between two vertices of such a simplex is between two x_i's, then this takes place in a face of $[v_0, \dots, v_n]$, which has dimension $< n$ so the distance between them is $\le \frac{n}{n+1}\mathrm{diam}([v_0, v_1, \dots, v_n])$ by inductive assumption.

If the maximal distance is between b_v and some x_i, then x_i lies in some face $[v_0, \dots, \widehat{v_j}, \dots, v_n]$ so $|b_v - x_i| \le |b_v - v_k|$ for some k. But

$$\begin{aligned}|b_v - v_k| &= \left|\tfrac{1}{n+1}\sum_i v_i - \tfrac{n+1}{n+1} v_k\right| \\ &= \tfrac{1}{n+1}\left|\sum_i v_i - v_k\right| \\ &\le \sum_i \tfrac{1}{n+1}|v_i - v_k|\end{aligned}$$

and each $|v_i - v_k|$ is at most $\mathrm{diam}([v_0, \dots, v_n])$, though $|v_k - v_k| = 0$. This shows that $|b_v - v_k| \le \frac{n}{n+1}\mathrm{diam}([v_0, \dots, v_n])$ as required. □

Proposition 11. *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of subsets of X whose interiors cover.*

(i) If $c \in C_n^{\mathcal{U}}(X)$ then $\rho_n^X(c) \in C_n^{\mathcal{U}}(X)$ too.

(ii) If $c \in C_n(X)$ then there is a $k \gg 0$ such that $(\rho_n^X)^k(c) \in C_n^{\mathcal{U}}(X)$.

Proof. The first part follows from naturality of ρ_n^X: if $\sigma : \Delta^n \to U_\alpha$ and $i_\alpha : U_\alpha \to X$ is the inclusion of spaces, then $\rho_n^X((i_\alpha)_\#(\sigma)) = (i_\alpha)_\#(\rho_n^{U_\alpha}(\sigma))$ is a sum of simplices in U_α.

For the second part, by (i) and the fact that an n-chain is a *finite* sum of singular n-simplices, we may suppose that c is a single singular n-simplex $\sigma : \Delta^n \to X$. Then $\mathcal{V} = \{\sigma^{-1}\mathring{U}_\alpha\}_{\alpha \in I}$ is an open cover of Δ^n, which is a compact metric space. By the *Lesbegue Number Lemma* there is an $\epsilon > 0$ such that each ϵ-ball in Δ^n is contained in some $\sigma^{-1}\mathring{U}_\alpha$. By iterating Lemma 10, each simplex of $(\rho_n^{\Delta^n})^k(\iota_n)$ has diameter $\le (\frac{n}{n+1})^k\mathrm{diam}(\Delta^n)$, so by choosing $k \gg 0$ we may suppose that each simplex of $(\rho_n^{\Delta^n})^k(\iota_n)$ has diameter less than ϵ, and so lies in some $\sigma^{-1}\mathring{U}_\alpha$.

Hence $(\rho_n^{\Delta^n})^k(\iota_n) \in C_n^{\mathcal{V}}(\Delta^n)$, and so $(\rho_n^X)^k(\sigma) = \sigma_\#((\rho_n^{\Delta^n})^k(\iota_n)) \in C_n^{\mathcal{U}}(X)$, as required. □

3. Proof of Theorem 1.

We consider the chain map $C^{\mathcal{U}}_\bullet(X) \to C_\bullet(X)$ given by inclusion, and the induced map $U : H^{\mathcal{U}}_*(X) \to H_*(X)$ on homology.

Let $[c] \in H_n(X)$. By Proposition 11 there is a $k \gg 0$ such that $(\rho^X_n)^k(c) \in C^{\mathcal{U}}_n(X)$. As $\rho^X_\bullet$ is naturally chain homotopic to the identity, so is the composition $(\rho^X_\bullet)^k$. One could find a formula for such a chain homotopy in terms of $T^X_\bullet$, but the formula does not matter so let us simply write $F^k_\bullet$ for such a chain homotopy, satisfying $dF^k_n + F^k_{n-1}d = (\rho^X_n)^k - \mathrm{id}$. Then

$$(\rho^X_n)^k(c) - c = dF^k_n(c) + F^k_{n-1}d(c)$$

but the last term vanishes as c is a cycle (because it represents a homology class). Thus $(\rho^X_n)^k(c)$ is equivalent to c modulo boundaries, so $U : H^{\mathcal{U}}_n(X) \to H_n(X)$ is surjective.

Now let $[c] \in H^{\mathcal{U}}_n(X)$ be such that $U([c]) = 0 \in H_n(X)$. Thus there is a $z \in C_{n+1}(X)$ such that $d(z) = c \in C_n(X)$. By Proposition 11 there is a $k \gg 0$ such that $(\rho^X_{n+1})^k(z) \in C^{\mathcal{U}}_{n+1}(X)$, and we have

$$(\rho^X_{n+1})^k(z) - z = dF^k_{n+1}(z) + F^k_n d(z)$$

and so applying d we get

$$d((\rho^X_{n+1})^k(z) - F^k_n d(z)) = d(z) = c.$$

Now $(\rho^X_{n+1})^k(z) \in C^{\mathcal{U}}_{n+1}(X)$ by our choice of k, and as $d(z) = c \in C^{\mathcal{U}}_n(X)$ and the chain homotopy $F^k_\bullet$ is natural, $F^k_n d(z) \in C^{\mathcal{U}}_{n+1}(X)$ too. Thus c is a boundary in $C^{\mathcal{U}}_\bullet(X)$, so $[c] = 0 \in H^{\mathcal{U}}_n(X)$, and hence $U : H^{\mathcal{U}}_n(X) \to H_n(X)$ is injective.