Michaelmas Term 2016 O. Randal-Williams

Part III Algebraic Topology // Example Sheet 3

1. If X is a finite cell complex, by showing that $C_\bullet^{cell}(X)$ is (unnaturally) isomorphic to a direct sum of chain complexes of the form $0 \to B_n(X) \overset{A_n}{\to} Z_n(X) \to 0$, show that
$$H^n(X) \cong \frac{H_n(X)}{\mathrm{Tors}(H_n(X))} \oplus \mathrm{Tors}(H_{n-1}(X)),$$
where $\mathrm{Tors}(A) \leq A$ denotes the subgroup of elements of finite order.

2. Let $E \to X$ be a vector bundle with inner product $\langle \cdot, \cdot \rangle$. Let $F \subset E$ be a subbundle. Prove that the orthogonal complement bundle $F^\perp$ is locally trivial.

3. (i) Explain how to view an open Möbius band as a 1-dimensional real bundle over S^1. Show that it is a non-trivial bundle.

   (ii) Show that a 1-dimensional real bundle over S^n with $n > 1$ is trivial. Hence show that 1-dimensional real bundles over a finite cell complex X up to isomorphism are naturally in 1-1 correspondence with elements of $H^1(X; \mathbb{Z}/2)$. [*Hint: Think about an associated double cover.*]

4. Show that a complex vector bundle has a canonical orientation.

5. If $\pi : E \to X$ is a d-dimensional real vector bundle which is not necessarily R-orientable, show that we still have $H^i(E, E^\#; R) = 0$ for $i < d$. If X is path-connected show that restriction to the fibre at $x \in X$ still gives an injective map $H^d(E, E^\#; R) \to H^d(E_x, E_x^\#; R) \cong R$.

   Give an example to show that $H^{i+d}(E, E^\#; R)$ need not be isomorphic to $H^i(X; R)$ in general.

6. (i) Show that any map $f : \mathbb{RP}^n \to \mathbb{RP}^m$ induces a trivial map on reduced cohomology if $n > m$. What about if $n < m$?

   (ii) Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$ although they have additively isomorphic (co)homology.

7. (i) If $f : S^n \to S^n$ is odd (i.e. $f(-x) = -f(x)$) show that it induces a map $\bar{f} : \mathbb{RP}^n \to \mathbb{RP}^n$. By considering the Gysin sequence show that f has odd degree.

   (ii) Show that any $g : S^n \to \mathbb{R}^n$ satisfies $g(x) = g(-x)$ for some $x \in S^n$.

8. (i) Let $L = \gamma_{1,n+1}^{\mathbb{C}} \to \mathbb{CP}^n$ be the canonical 1-dimensional complex bundle. By considering $\pi_1^* L \otimes_{\mathbb{C}} \pi_2^* L \to \mathbb{CP}^n \times \mathbb{CP}^n$, with the $\pi_i : \mathbb{CP}^n \times \mathbb{CP}^n \to \mathbb{CP}^n$ being projections to the factors, prove that the Euler class of $L \otimes_{\mathbb{C}} L$ is equal to twice the Euler class of L.

   (ii) Show that the unit circle bundle in $L \otimes_{\mathbb{C}} L$ is homeomorphic to $\mathbb{RP}^{2n+1}$. Hence, compute the cohomology of $\mathbb{RP}^{2n+1}$ from knowledge of the cohomology of $\mathbb{CP}^n$.

9. Let $V_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k$ be the subspace of k-tuples of orthonormal vectors in $\mathbb{C}^n$ (a *Stiefel manifold*). Show there is a vector bundle $E_k \to V_k(\mathbb{C}^n)$ with fibre over $(v_1, \ldots, v_k)$ given by the vector space $\mathrm{span}(v_1, \ldots, v_k) \leq \mathbb{C}^n$.

   Show that the forgetful map $(v_1, \ldots, v_k) \mapsto (v_1, \ldots, v_{k-1}) : V_k(\mathbb{C}^n) \to V_{k-1}(\mathbb{C}^n)$ exhibits $V_k(\mathbb{C}^n)$ as the sphere bundle of a certain vector bundle $F \to V_{k-1}(\mathbb{C}^n)$. Hence compute $H^*(V_k(\mathbb{C}^n); \mathbb{Z})$ as a ring.

   Deduce that the unitary group $U(n)$ has the same cohomology ring as $S^1 \times S^3 \times S^5 \times \cdots \times S^{2n-1}$, and hence that
$$\sum_{j \geq 0} \mathrm{rk}\, H^j(U(n); \mathbb{Z})\, t^j \;=\; \prod_{i=1}^{n} (1 + t^{2i-1}).$$

10. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let X be a compact Hausdorff space, and $Gr_k = Gr_k(\mathbb{F}^\infty) = \bigcup_n Gr_k(\mathbb{F}^n)$ be the infinite Grassmannian. The bundles $\gamma^{\mathbb{F}}_{k,n} \to Gr_k(\mathbb{R}^n)$ assemble to a bundle $\gamma^{\mathbb{F}}_k \to Gr_k$. To a map $f : X \to Gr_k$ we associate the pullback $f^*\gamma^{\mathbb{R}}_k$. Fix the standard inner product on $\mathbb{F}^\infty$ throughout.

(i) Suppose $f_0, f_1 : X \to Gr_k$ are maps with image in $Gr_k(\mathbb{F}^N)$ for some N. Let $U \subset Gr_k(\mathbb{F}^N) \times Gr_k(\mathbb{F}^N)$ be the following open neighbourhood of the diagonal:

$$U = \{(v_1, v_2) \mid v_1 \cap v_2^\perp = \{0\}\}.$$

Show that if $f_0(x)$ and $f_1(x)$ belong to U for every $x \in X$ then $f_0^*\gamma^{\mathbb{F}}_k \cong f_1^*\gamma^{\mathbb{F}}_k$.

(ii) By splitting the homotopy into many small intervals, deduce that if $f_0, f_1 : X \to Gr_k$ are homotopic then $f_0^*\gamma^{\mathbb{F}}_k$ and $f_1^*\gamma^{\mathbb{F}}_k$ are isomorphic.

(iii) Let $i_j : V_j \hookrightarrow \mathbb{F}^N$ be the inclusion of k-dimensional subspaces V_j, for $j = 0, 1$, and let $\alpha : V_0 \to V_1$ be a linear isomorphism. Show that

$$\gamma : t \longmapsto (t \cdot (i_0 \oplus \{0\}^n) + (1 - t) \cdot (\{0\}^n \oplus i_1 \circ \alpha))(V_0)$$

is a continuous path from $V_0 \oplus \{0\}$ to $\{0\} \oplus V_1$ in $Gr_k(\mathbb{F}^{2N})$.

(iv) Let $f_0, f_1 : X \to Gr_k$ have image in $Gr_k(\mathbb{F}^N)$ and $f_0^*\gamma^{\mathbb{F}}_k \cong f_1^*\gamma^{\mathbb{F}}_k$. Let $T : \mathbb{F}^N \oplus \mathbb{F}^N \to \mathbb{F}^N \oplus \mathbb{F}^N$ be the map $(\xi, \eta) \mapsto (-\eta, \xi)$. Show that f_0 and $T \circ f_1$ are homotopic as maps from X to $Gr_k(\mathbb{F}^{2N})$, and deduce that $f_0 \simeq f_1 : X \to Gr_k$.

Conclude that the set of isomorphism classes $\mathrm{Vect}_k(X)$ of k-dimensional vector bundles over X is in bijection with the set $[X, Gr_k]$ of homotopy classes of maps.

11. (i) Show that $Gr_k(\mathbb{C}^n)$ is a smooth manifold of dimension $2k(n-k)$. Show that the map $j : Gr_{k-1}(\mathbb{C}^n) \to Gr_k(\mathbb{C}^{n+1})$, which adds on the last coordinate direction to a $(k-1)$-dimensional subspace of $\mathbb{C}^n$, is the inclusion of a submanifold. Show that the complement U of the image of j is homotopy equivalent to the subspace $Gr_k(\mathbb{C}^n) \subset Gr_k(\mathbb{C}^{n+1})$, and hence deduce that $H^i(Gr_k(\mathbb{C}^{n+1}), Gr_k(\mathbb{C}^n); \mathbb{Z}) = 0$ for $i < 2(n+1-k)$. [*Hint: Tubular neighbourhood theorem.*]

(ii) For the canonical bundle $\gamma^{\mathbb{C}}_{n,k} \to Gr_k(\mathbb{C}^n)$, show that $S(\gamma^{\mathbb{C}}_{k,n}) \cong S((\gamma^{\mathbb{C}}_{k-1,n})^\perp)$, and hence deduce that there is an exact sequence

$$\cdots \longrightarrow H^{i-2k}(Gr_k(\mathbb{C}^n)) \xrightarrow{-\cdot e(\gamma^{\mathbb{C}}_k(\mathbb{C}^n))} H^i(Gr_{k,n}) \longrightarrow H^i(Gr_{k-1}(\mathbb{C}^n)) \longrightarrow H^{i-2k+1}(Gr_k(\mathbb{C}^n)) \longrightarrow \cdots$$

defined for $i \leq 2(n-k)$. Hence show by induction on k that the infinite complex Grassmannian Gr_k has cohomology ring $H^*(Gr_k; \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_k]$ for certain classes c_i of degree $2i$ (the *Chern classes*).

Comments or corrections to `or257@cam.ac.uk`