Michaelmas Term 2016 O. Randal-Williams

Part III Algebraic Topology // Example Sheet 2

1. Say a map $f : X \to Y$ between cell complexes is *cellular* if $f(X^n) \subset Y^n$ for every n. Show how to associate to such an f a chain map $f_{\#}^{cell} : C_{\bullet}^{cell}(X) \to C_{\bullet}^{cell}(Y)$ and show that the induced map $f_*^{cell} : H_*^{cell}(X) \to H_*^{cell}(Y)$ agrees with $f_* : H_*(X) \to H_*(Y)$ under a suitable identification of the homology groups.

2. Let $X = S^n \cup_f D^{n+1}$ be given by gluing an $(n+1)$-cell to S^n by a map $f : S^n \to S^n$ of degree $m > 1$. Show that the natural map $X \to X/S^n \cong S^{n+1}$ is trivial on homology $H_{*>0}$, but is non-trivial on cohomology $H^{*>0}$. What happens if we instead consider the inclusion map $S^n \hookrightarrow X$?

3. (i) Let X be a cell complex and $A \subset X$ be a subcomplex. Prove that the pair (X, A) is good.
   (ii) Let X be a cell complex and $K \subset X$ a compact subspace. Prove that K intersects only finitely many open cells in X. Hence show that any element of $H_i(X)$ lies in the image of $H_i(X^m) \to H_i(X)$ for some $m \gg 0$.

4. Show that for each $m \in \mathbb{Z}$ and any space X there are short exact sequences of chain complexes

$$0 \longrightarrow C^{\bullet}(X) \longrightarrow C^{\bullet}(X) \longrightarrow C^{\bullet}(X; \mathbb{Z}/m) \longrightarrow 0$$

$$0 \longrightarrow C^{\bullet}(X; \mathbb{Z}/m) \longrightarrow C^{\bullet}(X; \mathbb{Z}/m^2) \longrightarrow C^{\bullet}(X; \mathbb{Z}/m) \longrightarrow 0$$

and hence describe "Bockstein operations"

$$\tilde{\beta} : H^i(X; \mathbb{Z}/m) \longrightarrow H^{i+1}(X) \qquad \text{and} \qquad \beta : H^i(X; \mathbb{Z}/m) \longrightarrow H^{i+1}(X; \mathbb{Z}/m).$$

How are these two operations related? Show that $\beta(x \smile y) = \beta(x) \smile y + (-1)^{|x|} x \smile \beta(y)$. Compute the effect of β and $\tilde{\beta}$ for $m = 2$ and $X = \mathbb{RP}^n$, and hence compare $H^*(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}/4)$ with $H^*(\mathbb{RP}^2; \mathbb{Z}/4) \otimes H^*(\mathbb{RP}^2; \mathbb{Z}/4)$.

5. Compute the cohomology ring of $S^1 \times S^1$. Hence compute the cohomology ring of the closed oriented surface Σ_g of genus g.

6. Recall that $H^2(\Sigma_g) \cong \mathbb{Z}$ for every $g \geq 0$, and define the degree of a map of oriented surfaces to be the induced map on H^2. For which g is there a map $\Sigma_g \to \Sigma_1$ of positive degree? For which g is there a map $\Sigma_1 \to \Sigma_g$ of positive degree?

7. A map $\pi : E \to B$ is called a *covering map* if there is an open cover $\{U_\alpha\}$ of B such that $\pi^{-1}(U_\alpha)$ is a disjoint union $\coprod V_{\alpha,\beta}$ with each $\pi|_{V_{\alpha,\beta}} : V_{\alpha,\beta} \to U_\alpha$ a homeomorphism.
   (i) If $\pi : E \to B$ is a covering map with finite fibres of cardinality N, show how to construct a map $\pi^! : H_*(B) \to H_*(E)$ such that $\pi_* \circ \pi^!$ is multiplication by N.
   (ii) In the same situation, show that $\chi(E) = N \cdot \chi(B)$.
   (iii) Show there is a covering map $\Sigma_g \to \Sigma_h$ if and only if $g = kh - k + 1$ for some $k \in \mathbb{N}$.

8. If $f : \mathbb{C}^n \to \mathbb{C}^n$ has components the elementary symmetric functions

$$(z_1, \ldots, z_n) \mapsto (\sigma_i(\underline{z})) \qquad \sigma_1 = \sum_j z_j \qquad \sigma_2 = \sum_{i<j} z_i z_j \quad \cdots \quad \sigma_n = \prod_j z_j$$

then prove that f extends to a map $\psi : S^{2n} \to S^{2n}$ of degree $n!$.

Hence construct a map $\phi : (\mathbb{CP}^1)^n \to \mathbb{CP}^n$ of degree $n!$, and compute the effect of the map $\phi^* : H^2(\mathbb{CP}^n) \to H^2((\mathbb{CP}^1)^n)$. Deduce that there is a $x \in H^2(\mathbb{CP}^n)$ such that x^n is a generator of the abelian group $H^{2n}(\mathbb{CP}^n)$, and hence that $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ as a ring.

[*Hint: relate* $\mathbb{CP}^k$ *to the space of degree* k *homogeneous polynomials in two variables.*]

9. By considering a map to the wedge (one-point-union) of two copies of $\mathbb{CP}^2$, or otherwise, compute $H^*(\mathbb{CP}^2 \# \mathbb{CP}^2)$ as a ring. Deduce that $\mathbb{CP}^2 \# \mathbb{CP}^2$ is not homotopy equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$, even though they have the same (co)homology groups additively.

10. Show that there is a *relative cup product*

$$\smile : H^i(X, A) \times H^j(X, B) \to H^{i+j}(X, A \cup B)$$

[*Hint: it may be helpful to consider a cochain complex $C^*_{A+B}(X)$ of cochains vanishing on simplices lying wholly in A or B, and use the Small Simplices Theorem.*] Using this, show that if X has a cover by n contractible (i.e. homotopy equivalent to a point) open sets, then the *cup-length*

$$\max \left\{ k \mid \exists\, a_1, \ldots, a_k \in H^{*>0}(X),\ a_1 \smile \ldots \smile a_k \neq 0 \right\}$$

is strictly smaller than n. What does this say about the ring $H^*(\Sigma X)$, where Σ is the suspension operation?

11. (i) Let $e : [0,1]^k \to S^n$ be a map which is a homeomorphism onto its image $D \subset S^n$. By considering the open sets

$$A = S^n \setminus e([0,1]^{k-1} \times [0, 1/2]) \qquad B = S^n \setminus e([0,1]^{k-1} \times [1/2, 1])$$

in S^n, show by induction on k that $\widetilde{H}_i(S^n \setminus D) = 0$.

(ii) If $e : S^k \to S^n$ is a map which is a homeomorphism onto its image $S \subset S^n$, compute $\widetilde{H}_i(S^n \setminus S)$. Think about the consequence of this in the case $(n, k) = (2, 1)$.

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