Part III Algebraic Topology // Example Sheet 1

1. (i) Prove that homotopy equivalence is an equivalence relation on topological spaces.
   (ii) Which of the following are homotopy equivalent to S^1? (a) the annulus $\{1 < |z| < r\}$, (b) a bagel, (c) a genus two surface with a disc sewn across one of the holes, (d) the complement of a point in the real projective plane $\mathbb{RP}^2$.

2. Compute $H^0(X)$ for a topological space X. Give an example of a space X for which $H_0(X)$ and $H^0(X)$ are not isomorphic.

3. What can you say about the group G and/or the homomorphism α in an exact sequence of the shape
   (i) $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z} \to 0$;
   (ii) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2 \to 0$;
   (iii) $0 \to \mathbb{Z}/4 \xrightarrow{\alpha} G \oplus \mathbb{Z}/2 \to \mathbb{Z}/4 \to 0$?

4. (i) The *suspension* ΣX of a space X is the quotient of $X \times [0,1]$ by the map which collapses each end of the cylinder to a point: $X \times \{0\} \sim p_0$ and $X \times \{1\} \sim p_1$. Observe $\Sigma S^n \cong S^{n+1}$. Hence or otherwise prove there are maps $f : S^n \to S^n$ of any degree, for any $n > 0$.
   (ii) Suppose A is a closed manifold. Is ΣA necessarily homeomorphic to a closed manifold? Justify your answer.

5. If $f : S^n \to S^n$ has no fixed points, show that it is homotopic to the antipodal map. Hence show that if a group G acts freely on S^{2n} then $|G| \leq 2$.

6. (i) Finish the proof (begun in lectures) of the theorem that a short exact sequence of chain complexes has an associated long exact sequence on homology, by showing that the sequence obtained really is exact.
   (ii) Finish the proof (begun in lectures) of the 5-lemma.

7. If $X \subset \mathbb{R}^n$ is convex, show (*without using homotopy invariance!*) that $H_i(X) = 0$ for $i > 0$.

8. (i) Compute the homology groups of the closed orientable surface Σ_g of genus g.
   (ii) Compute $H_*(\Sigma_2, A)$ where A is a simple closed curve which: (a) separates Σ_2 into two genus one pieces with one boundary component each; (b) is a non-separating simple closed curve cutting along which gives a genus one surface with two holes, and (c) bounds an embedded disc.

9. Using Mayer–Vietoris, compute the cohomology groups of complex projective space $\mathbb{CP}^k$. For each n, construct a closed connected four-dimensional manifold X_n with $H^1(X_n) = 0$ and $H^2(X_n) \cong \mathbb{Z}^n$. [*Hint: look up the "connect sum".*]

10. (i) Define relative cohomology $H^*(X, A)$ in such a way that there is a long exact sequence
$$\cdots \to H^i(X, A) \to H^i(X) \to H^i(A) \to H^{i+1}(X, A) \to \cdots$$
    (ii) Compute the relative cohomology $H^*(D, \{p_1, \ldots, p_k\})$ of the closed disc in $\mathbb{C}$ relative to k points.

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