# Algebraic Topology, Examples 3

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Michaelmas 2013

**1.** An *abstract simplicial complex* consists of a finite set $V_X$ (called the *vertices*) and a collection $X$ (called the *simplices*) of subsets of $V_X$ such that if $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$. A map $f : (V_X, X) \to (V_Y, Y)$ of abstract simplicial complexes is a function $f : V_X \to V_Y$ such that $f(\sigma) \in Y$ for all $\sigma \in X$.

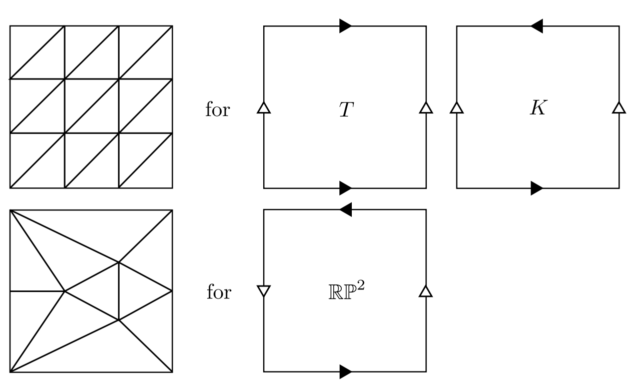
(i) For a simplicial complex $K$ in $\mathbb{R}^m$, show that the *abstraction* of $K$,

$$V_X = \{0\text{-simplices of } K\} \qquad X = \{\{a_0, \ldots, a_n\} \subset V_X \mid \langle a_0, \ldots, a_n \rangle \in K\}$$

is an abstract simplicial complex. Show that any abstract simplicial complex arises in this way (up to isomorphism), and that if simplicial complexes $K$ and $L$ have isomorphic abstractions, then $|K|$ and $|L|$ are homeomorphic.

(ii) Show that there exists an infinite sequence of points $(x_1, x_2, \ldots) \in \mathbb{R}^m$ such that any $(m+1)$ of them are affinely independent. Hence show that if $(V_X, X)$ is an abstract simplicial complex with all simplices of dimension $\leq n$, then there is a simplicial complex $K$ in $\mathbb{R}^{2n+1}$ with abstraction isomorphic to $(V_X, X)$.

**2.** Show that there is a triangulation of the torus, Klein bottle, and projective plane as follows:




How many vertices, edges and faces does each triangulation have? What is the number $\chi = \text{vertices} - \text{edges} + \text{faces}$ for each triangulation?

**3.** Use the simplicial approximation theorem to show that

(i) if $K$ and $L$ are simplicial complexes, there are at most countably many homotopy classes of continuous maps $f : |K| \to |L|$,

(ii) if $m < n$ then any continuous map $S^m \to S^n$ is homotopic to a constant map.

**4.** Let $K$ be a simplicial complex, and suppose that $\sigma \in K$ is not a proper face of any simplex. Show that $L = K \setminus \{\sigma\}$ is again a simplicial complex, and that the inclusion map $V_L \to V_K$ gives a simplicial map $i : L \to K$.

If $\sigma$ has dimension $n$, note that $d_n(\sigma)$ is a $(n-1)$-cycle and consists of simplices of $L$, so represents a class $[d_n(\sigma)] \in H_{n-1}(L)$; this defines a homomorphism $\varphi : \mathbb{Z} \to H_{n-1}(L)$ by $1 \mapsto [d_n(\sigma)]$. Construct a homomorphism $\phi : H_n(K) \to \mathbb{Z}$ such that

$$0 \longrightarrow H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\varphi} H_{n-1}(L) \xrightarrow{i_*} H_{n-1}(K) \longrightarrow 0$$

is exact (i.e. the image of one map is *precisely* the kernel of the next), and show that $i_* : H_j(L) \to H_j(K)$ is an isomorphism for $j \neq n-1, n$.

**5.** Let $K$ be a simplicial complex, and suppose that $\sigma \in K$ is not a proper face of any simplex, and that $\tau \leq \sigma$ is a face of one dimension lower which is not a face of any other simplex. Show that $L = K \setminus \{\sigma, \tau\}$ is again a simplicial complex, and that the inclusion map $V_L \to V_K$ gives a simplicial map $i : L \to K$.

(i) By constructing a chain homotopy inverse to $i_\bullet : C_\bullet(L) \to C_\bullet(K)$, show that $i_* : H_j(L) \to H_j(K)$ is an isomorphism for all $j$.

(ii) * Prove the same thing using the previous question (twice) instead.

**6.** Using the two previous questions, compute the homology groups of the simplicial complexes described in Q2, and describe generators for each homology group.

**7.** Using the simplicial approximation theorem, show that for any vertex $v$ of $K$ the based map $(|K_{(2)}|, v) \to (|K|, v)$ (i.e. the inclusion of the 2-skeleton) induces an isomorphism on fundamental groups. Show how to prove the same result using the Seifert–van Kampen theorem.

**8.** Let $K$ be an $n$-dimensional simplicial complex such that

(i) every $(n-1)$-simplex is a face of precisely two $n$-simplices, and

(ii) every pair of $n$-simplices can be connected by a sequence of $n$-simplices such that adjacent terms share an $(n-1)$-dimensional face.

Show that $H_n(K)$ is either $\mathbb{Z}$ or trivial. In the first case show $H_n(K)$ is generated by a cycle which is a sum of all $n$-simplices with suitable orientations.

**9.** * For simplicial complexes $K$ and $L$ inside $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, show that $|K| \times |L| \subset \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is the polyhedron of a simplicial complex. [Prove it first in the case where both $K$ and $L$ consist of a single simplex (plus all its faces).]