Algebraic Topology, Examples 2

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1. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = e$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite).

1. Show that the quotient map $X \to X/G$ is a covering map.
2. Deduce that if X is simply-connected and locally path-connected then for any point $[x] \in X/G$ we have $\pi_1(X/G, [x]) \cong G$.

2. If $p : \widetilde{X} \to X$ is a covering map such that $\widetilde{X}$ is path-connected, and $x_0 \in X$ and $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ are basepoints, show that the subgroups $p_*\pi_1(\widetilde{X}, \tilde{x}_0)$ and $p_*\pi_1(\widetilde{X}, \tilde{x}_1)$ of $\pi_1(X, x_0)$ are conjugate. Hence, using the correspondence proved in lectures, show that for a based space (X, x_0) which is path-connected, locally path-connected, and semilocally simply-connected, there is a bijection

$$\left\{ \begin{array}{l} \text{path-connected} \\ \text{covering spaces} \\ p : \widetilde{X} \to X \end{array} \right\} \Big/ \left\{ \begin{array}{l} \text{homeomorphisms} \\ h : \widetilde{X}_1 \to \widetilde{X}_2 \\ \text{s.t. } p_1 = p_2 \circ h \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{subgroups of} \\ \pi_1(X, x_0) \end{array} \right\}.$$

3. Show that the Klein bottle has a cell structure with a single 0-cell, two 1-cells, and a single 2-cell. Deduce that its fundamental group has a presentation $\langle a, b \mid baba^{-1} \rangle$, and show this is isomorphic to the group in Q12 of Sheet 1.

4. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b \rangle$ where a and b are given by the two characteristic loops. Describe covering spaces associated to

1. $\langle\langle a \rangle\rangle$, the normal subgroup generated by a,
2. $\langle a \rangle$, the subgroup generated by a,
3. the kernel of the homomorphism $\phi : \langle a, b \rangle \to \mathbb{Z}/4$ given by $\phi(a) = [1]$ and $\phi(b) = [3] = [-1]$.

Show that the free group on two letters contains a copy of itself as a proper subgroup.

5. Consider the 2-dimensional cell complex Y obtained from X in the previous question by attaching 2-cells along loops in the homotopy classes a^2 and b^2, so that

$$\pi_1(Y, x_0) \cong \langle a, b \,|\, a^2, b^2 \rangle.$$

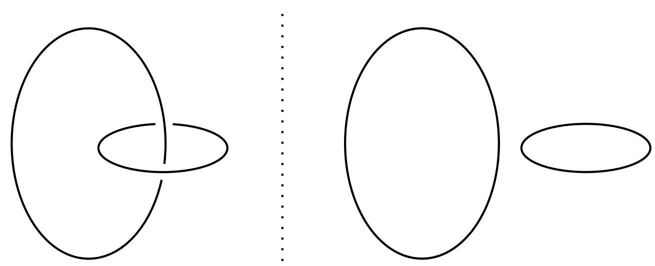
1. Construct, as a cell complex (in pictures), the covering space of Y corresponding to the subgroup $\langle a \,|\, a^2 \rangle$.
2. Construct, as a cell complex (in pictures), the covering space of Y corresponding to the kernel of the homomorphism $\phi : \langle a, b \,|\, a^2, b^2 \rangle \to \mathbb{Z}/2$ given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\mathrm{Ker}(\phi)$ is isomorphic to $\langle a, b \,|\, a^2, b^2 \rangle$.

6. Show that the groups

$$G = \langle a, b \,|\, a^3 b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \,|\, xyxy^{-1}x^{-1}y^{-1} \rangle$$

are isomorphic. Show that this group is non-abelian and infinite. [Construct surjective homomorphisms to S_3 and $\mathbb{Z}$.]

7. Consider the following configurations of pairs of circles in S^3 (we have drawn them in $\mathbb{R}^3$; add a point at infinity).



By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.

8. A *graph* is a cell complex which only has cells of dimension 0 and 1. A *tree* is a graph which is contractible. A tree T inside a graph G is *maximal* if no strictly larger subgraph is a tree.

1. If $T \subset G$ is a tree, show that the quotient map $G \to G/T$ is a homotopy equivalence, and that G/T has the structure of a cell complex.
2. Hence show that every connected graph is homotopy equivalent to a graph with a single vertex, and so show that the fundamental group of a connected graph is a free group.
3. Show that any free group occurs as the fundamental group of a graph, and deduce that a subgroup of a free group is again a free group.