Algebraic Topology, Examples 1

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1. Let $a : S^n \to S^n$ be the antipodal map, $a(x) = -x$. Show that a is homotopic to the identity map when n is odd.

2. Let $f : S^1 \to S^1$ be a map which is not homotopic to the identity map. Show that there exists an $x \in S^1$ such that $f(x) = x$, and a $y \in S^1$ so that $f(y) = -y$.

3. Suppose that $f : X \to Y$ is a map for which there exist maps $g, h : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ h \simeq \mathrm{Id}_Y$. Show that f, g, and h are homotopy equivalences.

4. Show that a retract of a contractible space is contractible.

5. Show that if a space X deformation retracts to a point $x_0 \in X$, then for every open neighbourhood $x_0 \in U$ there exists a smaller open neighbourhood $x_0 \in V \subset U$ such that the inclusion $(V, x_0) \hookrightarrow (U, x_0)$ is pointed nullhomotopic.

6. Construct a 2-dimensional cell complex which contains both the annulus $S^1 \times I$ and the Möbius band as deformation retracts.

7. For $m < n$, consider S^m as a subspace of S^n given by

$$\{(x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0) \mid \sum x_i^2 = 1\}.$$

Show that the complement $S^n - S^m$ is homotopy equivalent to S^{n-m-1}.

8. A space is called *locally path connected* if for every point $x \in X$ and every neighbourhood $U \ni x$, there exists a smaller neighbourhood V, i.e. $x \in V \subset U$, which is path connected. Show that a locally path connected space which is connected is also path connected.

9. Recall that for a map $f : S^{n-1} \to X$ we define the *space obtained by attaching an n-cell to X along f* to be

$$X \cup_f D^n := (X \amalg D^n)/\sim$$

where $\sim$ is the smallest equivalence relation containing $b \sim f(b)$ for every $b \in S^{n-1} \subset D^n$. Show that if f and f' are homotopic maps $S^{n-1} \to X$, then $X \cup_f D^n \simeq X \cup_{f'} D^n$.

10. Show that The Möbius band does not retract onto its boundary.

11. For based spaces (X, x_0) and (Y, y_0) show there is an isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Hence show that the inclusion $i : (S^1 \times \{1\}) \cup (\{1\} \times S^1) \hookrightarrow S^1 \times S^1$ does not admit a retraction. [Here we consider $S^1 \subset \mathbb{C}$ as the elements of unit modulus, so it contains the point 1.]

12. Construct a covering map $\pi : \mathbb{R}^2 \to K$ of the Klein bottle, and hence show that $\pi_1(K, k_0)$ is isomorphic to the group G with elements $(m, n) \in \mathbb{Z}^2$ and group operation

$$(m, n) * (p, q) = (m + (-1)^n \cdot p, n + q).$$

1. Let $a = (-1, 0)$ and $b = (0, 1)$. Show that $b^{-1} = (0, -1)$ and that $bab^{-1}a = (0, 0)$. Describe the centre $Z(G) \triangleleft G$ and commutator subgroup $G' \triangleleft G$ in terms of the generators a and b.

2. Describe the group $\mathrm{Aut}(G)$ of all self-isomorphisms of G. [You may use, and prove, that both the centre and commutator subgroups are *characteristic subgroups* i.e. are preserved by any automorphism.] Describe the normal subgroup $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$ of inner automorphisms (those induced by conjugation in G), and the quotient group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

3. Show that any self-isomorphism $\phi : G \to G$ is induced by a (pointed) homotopy equivalence of (K, k_0).

4. Let $H \leq G$ be the subgroup generated by a and b^2, and describe the covering space $p : C \to K$ which corresponds to this subgroup. Show that any automorphism of G preserves H, so there is an induced homomorphism

$$\mathrm{res} : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(H),$$

and describe the image and kernel of this homomorphism.

13.* Let G be a path-connected, locally path connected topological group, and $p : \hat{G} \to G$ be a covering map. Let $\epsilon \in p^{-1}(e)$ be a point in the fibre over the identity $e \in G$.

1. Show that $\hat{G}$ has a unique structure of a topological group with unit ϵ so that p is a homomorphism.

2. Show that $\mathrm{Ker}(p) \subset \hat{G}$ lies in the centre of $\hat{G}$.

3. Show that $SO(3)$, the group of rotations of $\mathbb{R}^3$ (or equivalently of orthogonal 3×3 matrices of determinant 1), is homeomorphic to the projective space $\mathbb{RP}^3$.

4. Together, 1. and 3. give a group $\widehat{SO(3)}$ homeomorphic to S^3. Identify this group with a well-known matrix group.