Free Groups

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In lectures I gave the definition of a free group, described its multiplication, but didn't prove that it is indeed a group. This handout gives an alternative definition which is clearly a group, and show it is equivalent to the definition given in lectures.

Let $S = \{s_\alpha\}_{\alpha \in I}$ be a set, the *alphabet*, and let $S^{-1} = \{s_\alpha^{-1}\}_{\alpha \in I}$; suppose that $S \cap S^{-1} = \emptyset$. A *word* in the alphabet S is a (possibly empty) finite sequence

$$(x_1, x_2, \ldots, x_n)$$

of elements of $S \cup S^{-1}$. A word is *reduced* if it contains no subwords

$$(s_\alpha, s_\alpha^{-1}) \quad \text{or} \quad (s_\alpha^{-1}, s_\alpha).$$

We let W be the set of reduced words, and $P(W)$ be the group of permutations of the set W.

Definition 1. For each $\alpha \in I$, define a function $L_\alpha : W \to W$ by the formula

$$L_\alpha(x_1, x_2, \ldots, x_n) = \begin{cases} (s_\alpha, x_1, x_2, \ldots, x_n) & \text{if } x_1 \neq s_\alpha^{-1} \\ (x_2, \ldots, x_n) & \text{if } x_1 = s_\alpha^{-1}. \end{cases}$$

Note that in the second case $x_2 \neq s_\alpha$, otherwise $(x_1, x_2, \ldots, x_n)$ would not be reduced.

Lemma 2. *L_α is a bijection, so represents an element of $P(W)$.*

Proof. Let $(x_1, \ldots, x_n) \in W$. If $x_1 = s_\alpha$ then $x_2 \neq s_\alpha^{-1}$, so $L_\alpha(x_2, \ldots, x_n) = (x_1, \ldots, x_n)$. If $x_1 \neq s_\alpha$ then $(s_\alpha^{-1}, x_1, \ldots, x_n)$ is a reduced word, and $L_\alpha(s_\alpha^{-1}, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Thus L_α is surjective.

If $L_\alpha(x_1, x_2, \ldots, x_n) = L_\alpha(y_1, y_2, \ldots, y_m)$ and this reduced word starts with s_α then $x_1 \neq s_\alpha^{-1}$ and $y_1 \neq s_\alpha^{-1}$, and so $x_i = y_i$ for each i. If this reduced word does not start with s_α then $x_1 = y_1 = s_\alpha^{-1}$, and

$$(x_2, \ldots, x_n) = L_\alpha(x_1, x_2, \ldots, x_n) = L_\alpha(y_1, y_2, \ldots, y_m) = (y_2, \ldots, y_m),$$

so $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_m)$. □

Definition 3. The free group $F(S)$ is the subgroup of $P(W)$ generated by the elements $\{L_\alpha\}_{\alpha \in I}$.

Lemma 4. *The function $\phi : F(S) \to W$ given by $\sigma \mapsto \sigma \cdot ()$ is a bijection.*

This identifies $F(S)$ with the set of reduced words W in the alphabet S, and shows that the group operation is given by concatenation of words followed by word reduction. Thus the definition given in lectures is indeed a group.

Proof. If $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ is a reduced word, with $\epsilon_i \in \{\pm 1\}$, then

$$(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n}) = L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n} \cdot () = \phi(L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}),$$

and so ϕ is surjective.

As the $\{L_\alpha\}_{\alpha \in I}$ generate $F(S)$, any element σ may be represented by a concatenation

$$\sigma = L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n} \in P(W).$$

As $L_\alpha \cdot L_\alpha^{-1} = \mathrm{Id}_W$ and $L_\alpha^{-1} \cdot L_\alpha = \mathrm{Id}_W$, if the word $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ is not reduced then we can simplify $L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}$ while giving the same element $\sigma \in P(W)$. Thus we may suppose that any σ is represented by $L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}$ such that the associated word $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ is reduced. But then

$$\phi(\sigma) = \sigma \cdot () = (s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n}),$$

from which we can recover $\sigma = L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}$, which shows that ϕ is injective. □

Note that there is a function $\iota : S \to F(S)$ given by sending s_α to the word (s_α).

Lemma 5. *For any group H, the function*

$$\left\{ \begin{array}{l} \textit{group homomorphisms} \\ \varphi : F(S) \to H \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \textit{functions} \\ \phi : S \to H \end{array} \right\},$$

given by precomposing with ι, is a bijection.

Proof. Given a function $\phi : S \to H$, we want a homomorphism φ such that $\varphi((s_\alpha)) = \phi(s_\alpha)$. But there is a unique way to do this, by defining, on not necessarily reduced words,

$$\varphi((s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})) = \phi(s_{\alpha_1})^{\epsilon_1} \cdots \phi(s_{\alpha_n})^{\epsilon_n}.$$

Note that if $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ is not reduced, so contains for example $(s_\alpha, s_\alpha^{-1})$, then the product $\phi(s_{\alpha_1})^{\epsilon_1} \cdots \phi(s_{\alpha_n})^{\epsilon_n}$ contains $\phi(s_\alpha) \cdot \phi(s_\alpha)^{-1} = 1$ and so we may reduce the word $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ without changing the value of φ on it. As the group operation in $F(S)$ is given by concatenation and reduction of words, this shows that φ is a homomorphism. □