The Acyclic Carrier Theorem

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Definition 1. Let K and L be simplicial complexes, and $f_\bullet : C_\bullet(K) \to C_\bullet(L)$ be a chain map between their associated chain complexes. A *carrier* for $f_\bullet$ is a function

$$\Phi : K \longrightarrow \{\text{sub simplicial complexes of } L\}$$

such that

1. $f_n([a_0, \ldots, a_n])$ may be represented by a sum of simplices in $\Phi(\langle a_0, \ldots, a_n\rangle)$,
2. if $\tau \leq \sigma$ then $\Phi(\tau) \subseteq \Phi(\sigma)$.

A carrier Φ is called *acyclic* if $\Phi(\sigma)$ has the homology of the one point simplicial complex for every simplex $\sigma \in K$.

Definition 2. For a simplicial complex K define $\epsilon_K : C_0(K) \to \mathbb{Z}$ by $\epsilon_K([a_0]) = 1$ (and extending linearly). Say a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$ is *augmentation-preserving* if $\epsilon_K = \epsilon_L \circ f_0$.

Note that $\epsilon_K \circ \partial_1 = 0$ (as $\epsilon_K \circ \partial_1([a_0, a_1]) = \epsilon_K([a_1] - [a_0]) = 1 - 1 = 0$) so ϵ_K induces a well-defined map $(\epsilon_K)_* : H_0(K) \to \mathbb{Z}$. If $H_0(K) \cong \mathbb{Z}$ then it is generated by any vertex of K, so $(\epsilon_K)_*$ is an isomorphism.

Theorem 3. *Let $f_\bullet, g_\bullet : C_\bullet(K) \to C_\bullet(L)$ be augmentation-preserving chain maps and Φ be a carrier for both of them which is acyclic. Then $f_\bullet \simeq g_\bullet$.*

Proof. Choose a total order $\prec$ on V_K, so that a basis for $C_n(K)$ is given by

$$\{[a_0, \ldots, a_n] \,|\, \langle a_0, \ldots, a_n\rangle \in K \text{ and } a_0 \prec \cdots \prec a_n\}.$$

We will construct maps $H_n : C_n(K) \to C_{n+1}(L)$ by induction. For each vertex a_0 of K the element $x = (f_0 - g_0)([a_0])$ is a sum of 0-simplices of $\Phi(\langle a_0\rangle)$, so represents a class $[x] \in H_0(\Phi(\langle a_0\rangle)) \cong \mathbb{Z}$. But

$$(\epsilon_{\Phi(\langle a_0\rangle)})_*([x]) = \epsilon_L((f_0 - g_0)([a_0])) = \epsilon_L(f_0([a_0])) - \epsilon_L(g_0([a_0])) = 1 - 1 = 0$$

and $(\epsilon_{\Phi(\langle a_0\rangle)})_*$ is an isomorphism, so $[x] = 0$. Thus there is a $y \in C_1(\Phi(\langle a_0\rangle))$ so that $x = \partial_1 y$, and we define $H_0([a_0]) = y$. By construction this satisfies

$$(\partial_1 \circ H_0 + H_{-1} \circ \partial_0)([a_0]) = \partial_1 y = x = (f_0 - g_0)([a_0]).$$

Now let $n > 0$ and suppose that for each $i < n$ we have defined homomorphisms $H_i : C_i(K) \to C_{i+1}(L)$ such that

1. $\partial_{i+1} \circ H_i + H_{i-1} \circ \partial_i = f_i - g_i$,
2. $H_i(\sigma)$ may be represented by a sum of simplices in $\Phi(\sigma)$.

We wish to define $H_n([a_0, \dots, a_n])$, so consider

$$x = (f_n - g_n - H_{n-1} \circ \partial_n)([a_0, \dots, a_n]).$$

Both $f_n([a_0, \dots, a_n])$ and $g_n([a_0, \dots, a_n])$ are sums of simplices in $\Phi(\langle a_0, \dots, a_n \rangle)$ as both maps are carried by Φ. The chain $H_{n-1} \circ \partial_n([a_0, \dots, a_n])$ is a sum of simplices each in $\Phi(\tau)$ for some $\tau \leq \langle a_0, \dots, a_n \rangle$, but $\Phi(\tau) \subseteq \Phi(\langle a_0, \dots, a_n \rangle)$ by definition of a carrier, so $H_{n-1} \circ \partial_n([a_0, \dots, a_n])$ is also a sum of simplices in $\Phi(\langle a_0, \dots, a_n \rangle)$. Thus x lies in $C_n(\Phi(\langle a_0, \dots, a_n \rangle))$. We compute

$$\begin{aligned}
\partial_n(x) &= (\partial_n \circ f_n - \partial_n \circ g_n - \partial_n \circ H_{n-1} \circ \partial_n)([a_0, \dots, a_n]) \\
&= (f_{n-1} \circ \partial_n - g_{n-1} \circ \partial_n - \partial_n \circ H_{n-1} \circ \partial_n)([a_0, \dots, a_n]) \\
&= (f_{n-1} \circ \partial_n - g_{n-1} \circ \partial_n - (f_{n-1} - g_{n-1} - H_{n-2} \circ \partial_{n-1}) \circ \partial_n)([a_0, \dots, a_n]) \\
&= H_{n-1} \circ \partial_{n-1} \circ \partial_n([a_0, \dots, a_n]) \\
&= 0.
\end{aligned}$$

Thus, as Φ is acyclic (i.e. $\Phi(\langle a_0, \dots, a_n \rangle)$ has the homology of a point) we have $H_n(\Phi(\langle a_0, \dots, a_n \rangle)) = 0$ and so $x = \partial_{n+1}(y)$ for some $y \in C_{n+1}(\Phi(\langle a_0, \dots, a_n \rangle))$. We define $H_n([a_0, \dots, a_n]) = y$. By construction this satisfies

$$\partial_{n+1} \circ H_n + H_{n-1} \circ \partial_n = f_n - g_n,$$

so the H_n give a chain homotopy between $f_\bullet$ and $g_\bullet$. □