Part IA Groups // Example Sheet 3

1. Show that every group of order 10 is cyclic or dihedral. *Can you extend your proof to groups of order $2p$, where p is any odd prime number?

2. Let p be a prime, and G be a group of order p^2. By considering the orbits of the action of G on itself by conjugation, show that G is abelian. Deduce that there are precisely two groups of order p^2 up to isomorphism.

3. Show that any subgroup of D_{2n} consisting of rotations is normal.

4. Show that a subgroup H of a group G is normal if and only if it is a union of conjugacy classes in G.

5. Suppose that G is a group in which every subgroup is normal. Must G be abelian?

6. Suppose that H is a subgroup of C_n. What is C_n/H?

7. Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite group in which every element has finite order.

8. Let K be a subgroup of a group G. Show that K is a normal subgroup if and only if it is the kernel of some group homomorphism $\phi : G \to H$.

9. Consider the subgroup Γ of $(\mathbb{C}, +, 0)$ consisting of elements $m + in$ with $m, n \in \mathbb{Z}$. By considering $x + iy \mapsto (e^{2\pi ix}, e^{2\pi iy})$, show that the group $\mathbb{C}/\Gamma$ is isomorphic to $S^1 \times S^1$ (see Sheet 1 Q15).

10. Suppose $a, b \in \mathbb{Z}$ and consider $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ given by $\phi(x, y) = ax + by$. Show that ϕ is a group homomorphism and describe $\mathrm{Im}(\phi)$ and $\mathrm{Ker}(\phi)$. What characterises the cosets of $\mathrm{Ker}(\phi)$ in $\mathbb{Z}^2$?

11. Let G be a finite group and H a proper subgroup. Let $k = |G/H|$ and suppose that $|G|$ does not divide $k!$. By considering the action of G on G/H, show that H contains a non-trivial normal subgroup of G. Deduce that a group of order 28 has a normal subgroup of order 7.

12. Show that if a group G of order 28 has a normal subgroup of order 4 then G is abelian.

13. Write the following permutations as compositions of disjoint cycles and hence compute their order:
    (a) $(12)(1234)(12)$,
    (b) $(123)(1234)(132)$,
    (c) $(123)(235)(345)(45)$.

14. Show that S_n is generated by each of the following sets of permutations:
    (a) $\{(j, j + 1) \mid 1 \leq j < n\}$,
    (b) $\{(1, k) \mid 1 < k \leq n\}$,
    (c) $\{(12), (123 \cdots n)\}$.

15. What is the largest possible order of an element of S_5? Of S_9?

16. Let $X = \mathbb{Z}/31\mathbb{Z}$, and $\sigma : X \to X$ be given by $\sigma(x + 31\mathbb{Z}) = 2x + 31\mathbb{Z}$. Show that σ is a permutation, and decompose it as a composition of disjoint cycles.

17. Prove that
$$\sigma * p(x_1, x_2, x_3, x_4) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$
defines an action of the group S_4 on the set of polynomials in variables x_1, x_2, x_3, x_4. Show that the stabiliser H of the polynomial $x_1x_2 + x_3x_4$ has order 8, and decide which of C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_8, or Q_8 it is isomorphic to.

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