**Part IA Groups // Example Sheet 4**

1. Show that if $H \leq A_5$ then $|A_5/H| > 4$. [Consider an action on the set $A_5/H$.]

2. Let $G \subseteq SL_3(\mathbb{R})$ be the subset of all matrices of the form
$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$
Prove that $G$ is a subgroup. Let $H \subset G$ be the subset of those matrices with $a = c = 0$. Show that $H$ is a normal subgroup of $G$, and determine the quotient group $G/H$.

3. Let $G \subseteq SL_3(\mathbb{R})$ be the subset of all matrices of the form
$$\begin{bmatrix} a & 0 & 0 \\ b & c & d \\ e & f & g \end{bmatrix}.$$
Prove that $G$ is a subgroup. Construct a surjective homomorphism $\phi : G \to GL_2(\mathbb{R})$, and find its kernel.

4. Show that matrices $A, B \in SL_2(\mathbb{C})$ are conjugate in $SL_2(\mathbb{C})$ if and only if they are conjugate in $GL_2(\mathbb{C})$. With a few exceptions—which you should find—show that matrices in $SL_2(\mathbb{C})$ are conjugate if and only if they have the same trace.

5. Let $SL_2(\mathbb{R})$ act on $\hat{\mathbb{C}}$ by Möbius transformations. Find the orbit and stabiliser of $i$ and $\infty$. By considering the orbit of $i$ under the action of the stabiliser of $\infty$, show that every $g \in SL_2(\mathbb{R})$ can be written as $g = hk$ with $h$ upper triangular and $k \in SO(2)$. In how many ways can this be done?

6. Suppose that $N$ is a normal subgroup of $O(2)$. Show that if $N$ contains a reflection then $N = O(2)$.

7. Which pairs of elements of $SO(3)$ commute?

8. If $A \in M_{n\times n}(\mathbb{C})$ with entries $A_{ij}$, let $A^\dagger \in M_{n\times n}(\mathbb{C})$ have entries $\overline{A_{ji}}$. A matrix is called *unitary* if $AA^\dagger = I_n$. Show that the set $U(n)$ of unitary matrices is a subgroup of $GL_n(\mathbb{C})$. Show that
$$SU(n) = \{A \in U(n)\, s.t.\ \det A = 1\}$$
is a normal subgroup of $U(n)$ and that $U(n)/SU(n) \cong S^1$. Show that $Q_8$ is isomorphic to a subgroup of $SU(2)$.

9. Let $K$ be a normal subgroup of order 2 in a group $G$. Show that $K$ is a subgroup of the centre $Z(G)$ of $G$. Show that if $n$ is odd then $O(n) \cong SO(n) \times C_2$. Why doesn't a similar argument work if $n$ is even?

10. Let $X = \{B \in M_{2\times 2}(\mathbb{R}) \mid \mathrm{Tr}(B) = 0\}$. Show that $A * B = ABA^{-1}$ defines an action of $SL_2(\mathbb{R})$ on $X$. Find the orbit and stabiliser of
$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Show that the set of matrices in $X$ with determinant 0 is the union of three orbits.

11. * Prove that $S_n$ has a subgroup isomorphic to $Q_8$ if and only if $n \geq 8$. Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to $Q_8$?

12. * Let $G$ be a finite non-trivial subgroup of $SO(3)$. Let
$$X = \{v \in \mathbb{R}^3\, s.t.\ |v| = 1 \text{ and there exists a } g \in G \setminus \{e\} \text{ with } g * v = v\}.$$
Show that $G$ acts on $X$ and that there are either 2 or 3 orbits. What is $G$ if there are 2 orbits?

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