Part IA Groups // Example Sheet 2

1. Determine under what conditions on $\lambda, \mu \in \mathbb{C}$ the Möbius transformations $f(z) = \lambda z$ and $f(z) = \mu z$ are conjugate in $\mathcal{M}$.

2. What is the order of the Möbius transformation $f(z) = iz$? What are its fixed points? If h is another Möbius transformation what can you say about the order and the fixed points of hfh^{-1}? Construct a Möbius transformation of order 4 that fixes 1 and -1.

3. Show that $t * (x, y) := (e^t x, e^{-t} y)$ defines an action of the group $(\mathbb{R}, +, 0)$ on the set $\mathbb{R}^2$. What are the orbits and stabilisers of this action? There is a differential equation that is satisfied by each of the orbits. What is it?

4. Suppose that Q is a quadrilateral in $\mathbb{R}^2$. Show that its group of symmetries $G(Q)$ has order at most 8. For which n is there a $G(Q)$ of order n? *Which groups can arise as a $G(Q)$ (up to isomorphism)?

5. Let G be the group of all symmetries of a cube. Show that G acts on the set of 4 lines joining diagonally opposite pairs of vertices. Show that if ℓ is one of these lines then $G_\ell \cong D_6 \times C_2$.

6. Let $S^1 := \{t \in \mathbb{C} \ s.t. \ |t| = 1\}$, which is a group under multiplication, and let

$$S^3 = \{(w_1, w_2) \in \mathbb{C}^2 \ s.t. \ |w_1|^2 + |w_2|^2 = 1\}.$$

Show that $(t_1, t_2) * (w_1, w_2) := (t_1 w_1, t_2 w_2)$ defines an action of the group $S^1 \times S^1$ on the set S^3. Describe the orbits of this action and find all stabilisers.

7. Let H be a subgroup of a group G. Show that there is a (natural) bijection between the set of left cosets of H in G and the set of right cosets of H in G.

8. If G is a finite group, H is a subgroup of G, and K is a subgroup of H, show that $|G/K| = |G/H| \cdot |H/K|$.

9. Show that if a group G contains an element of order 6, and an element of order 10, then G has order at least 30.

10. Show that D_{2n} has one conjugacy class of reflections if n is odd and two conjugacy classes of reflections if n is even.

11. Let G be a finite group and let $\mathrm{Sub}(G)$ be the set of all its subgroups. Show that $g * H := gHg^{-1}$ defines an action of G on $\mathrm{Sub}(G)$. Show that for $H \in \mathrm{Sub}(G)$ the size of the orbit of H under this action is at most $|G/H|$. Deduce that if $H \neq G$ then G is not the union of all conjugates of H.

12. Suppose that G acts on X and that $y = g \cdot x$ for some $x, y \in X$ and $g \in G$. Show that $G_y = gG_xg^{-1}$.

13. Let G be a finite abelian group acting faithfully on a set X. Show that if the action is transitive then $|G| = |X|$.

14. Show that every group of order 10 is cyclic or dihedral. *Can you extend your proof to groups of order $2p$, where p is any odd prime number?

15. Let p be a prime. By considering the conjugation action show that every group of order p^2 is abelian. Deduce that there are precisely two groups of order p^2 up to isomorphism.

16. Show that the set $\{1, 3, 5, 7\}$ forms a group under multiplication modulo 8. Is it isomorphic to $C_2 \times C_2$ or C_4.

Comments or corrections to `or257@cam.ac.uk`