**Part IA Groups // Example Sheet 1**

1. Let $G$ be any group. Show that the identity $e$ is the unique solution of the equation $x^2 = x$ in $G$.

2. Let $H_1$ and $H_2$ be two subgroups of a group $G$. Show that the intersection $H_1 \cap H_2$ is a subgroup of $G$. Show that the union $H_1 \cup H_2$ is a subgroup of $G$ if and only if one of the $H_i$ contains the other.

3. Let $G = \{x \in \mathbb{R} \,|\, x \neq -1\}$, and let $x * y = x + y + xy$, where $xy$ denotes the usual product of two real numbers. Show that $(G, *, 0)$ is a group. What is the inverse $2^{-1}$ of 2 in this group? Solve the equation $2 * x * 5 = 6$.

4. Let $G$ be a finite group. Show that every element of $G$ has finite order. Show that there exists a positive integer $N$ such that for all $g \in G$ we have $g^N = e$.

5. Show that the set $G$ of complex numbers of the form $\exp(i\pi t)$ with $t$ rational is a group under multiplication (with identity 1). Show that $G$ is infinite, but that every element $a$ of $G$ has finite order.

6. Let $f : G \to H$ be a group homomorphism, and $a \in G$ have finite order. Show that the order of $f(a)$ is finite and divides the order of $a$.

7. Let $C_n$ be the cyclic group with $n$ elements and $D_{2n}$ the group of symmetries of the regular $n$-gon. If $n$ is odd and $\theta \colon D_{2n} \to C_n$ is a homomorphism, show that $\theta(g) = e$ for all $g \in D_{2n}$. Can you find all homomorphisms $D_{2n} \to C_n$ if $n$ is even? Find all homomorphisms $C_n \to C_m$.

8. Show that any subgroup of a cyclic group is cyclic.

9. Consider the Möbius transformations $f(z) = e^{2\pi i/n} z$ and $g(z) = 1/z$. Show that the subgroup $G$ of the Möbius group $\mathcal{M}$ generated by $f$ and $g$ is isomorphic to $D_{2n}$.

10. Express the Möbius transformation $f(z) = \frac{2z+3}{z-4}$ as the composition of tranformations of the form $z \mapsto az$, $z \mapsto z + b$ and $z \mapsto 1/z$. Hence show that $f$ sends the circle described by $|z - 2i| = 2$ onto the circle described by $|8z + (6 + 11i)| = 11$.

11. Let $G$ be the subgroup of Möbius transformations that send the set $\{0, 1, \infty\}$ to itself. What are the elements of $G$? Which standard group is isomorphic to $G$? What is the group of Möbius transformations that send the set $\{0, 2, \infty\}$ to itself.

12. For each of the following statements, give a proof or counterexample.
    (i) The Möbius group is generated by Möbius transformations of the form $z \mapsto az$ and $z \mapsto z + b$.
    (ii) The Möbius group is generated by Möbius transformations of the form $z \mapsto az$ and $z \mapsto 1/z$.
    (iii) The Möbius group is generated by Möbius transformations of the form $z \mapsto z+b$ and $z \mapsto 1/z$.

13. Show that any invertible function $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that preserves the cross-ratio, i.e. such that
$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)] \text{ for all distinct } z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}},$$
is a Möbius transformation.

14. Let $G$ be a group in which every element other than the identity has order two. Show that $G$ is abelian. *Show also that if $G$ is finite, then the order of $G$ is a power of 2.

15. Let $G$ be a finite group of even order. Show that $G$ contains an element of order two.

16. Show that every isometry of $\mathbb{C}$ is either of the form $z \mapsto az + b$ or the form $z \mapsto a\bar{z} + b$ with $a, b \in \mathbb{C}$ and $|a| = 1$ in either case. *Describe the finite subgroups of the group of isometries of $\mathbb{C}$.

Comments or corrections to `or257@cam.ac.uk`